JHEP HYPER VERSION

Received: *March 15, 1999*, Accepted: *April 25, 1999*

On the Yangian [$Y(\mathfrak{e}_8)$] quantum symmetry of maximal supergravity in two dimensions*

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Abstract: We present the algebraic framework for the quantization of the classical bosonic charge algebra of maximally extended ($N=16$) supergravity in two dimensions, thereby taking the first steps towards an exact quantization of this model. At the core of our construction is the Yangian algebra $Y(\mathfrak{e}_8)$ whose RTT presentation we discuss in detail. The full symmetry algebra is a centrally extended twisted version of the Yangian double $\mathcal{D}Y(\mathfrak{e}_8)_c$. We show that there exists only one special value of the central charge for which the quantum algebra admits an ideal by which the algebra can be divided so as to consistently reproduce the classical coset structure $\mathrm{E}_{8(8)}/\mathrm{SO}(16)$ in the limit $\hbar \to 0$.

Keywords: 2D Gravity, Supergravity Models, Quantum Groups, Integrable Field Theories.

*Research supported in part by EC under TMR contract ERBFMRX-CT96-0012.

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Contents

1. Introduction

Dimensionally reduced gravity and supergravity are well known to possess hidden symmetries [1, 2]. Of special interest in this context is the case of two dimensions, where these symmetries become infinite dimensional, generalizing the so-called Geroch group of general relativity [3, 4, 5]. The existence of infinite dimensional symmetries in these models is intimately linked to their integrability, which is borne out by the existence of linear systems for their classical equations of motion, both for the bosonic models [6, 7, 4] and their locally supersymmetric extensions [8, 9]. In this paper, we will focus attention on the maximally extended $N=16$ supergravity, whose scalar sector is governed by an $\mathrm{E}_{8(8)}/\mathrm{SO}(16)$ nonlinear σ-model, and whose equations of motion admit a rigid non-compact $\mathrm{E}_{9(9)}$ symmetry.[1]

The canonical structure of these models and the Lie-Poisson realization of the associated infinite dimensional symmetries were analyzed only quite recently [13, 14]. As shown there, the affine Lie algebra seen at the level of the classical equations of motion is converted into a quadratic algebra of Yangian type in the canonical formulation. One key feature of this result, which we exploit in this paper, is that the quadratic algebra, and therefore at least part of the model, can be quantized directly by replacing the Poisson algebra of charges by an exchange algebra involving a suitable R matrix, whereas a standard field theoretic quantization would appear to be prohibitively difficult. The relevant R-matrix based on the exceptional group E_8 has already been derived in [15]. The structure that appears upon quantization

[1] This symmetry acts as a solution generating "isometry group", or as a group of "dressing transformations" [4, 7, 10, 11, 12].

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is the Yangian $Y(\mathfrak{e}_8)$. As a consequence, the physical states of the quantized theory must belong to multiplets of $Y(\mathfrak{e}_8)$ rather than multiplets of the affine algebra $\mathfrak{e}_9$ as one might have naively expected.

The Yangian of the exceptional algebra $\mathfrak{e}_8$ is distinguished from the Yangians of the classical Lie algebras by the fact that its fundamental representation is reducible over $\mathfrak{e}_8$, namely decomposes into $\mathbf{249} = \mathbf{1} \oplus \mathbf{248}$. The R-matrix associated to this representation has been given by Chari and Pressley in [15]. Using their result and the general analysis of Drinfeld [16] we obtain the RTT presentation of $Y(\mathfrak{e}_8)$ which may be viewed [17] as the quantization of group-valued E_8 matrices endowed with the symplectic structure of dimensionally reduced gravity. The full quantum structure that appears upon quantization of the algebra of classical nonlocal charges is a centrally extended twisted version of the Yangian double, that reflects the $\mathrm{E}_{8(8)}/\mathrm{SO}(16)$ coset structure of the classical model.

The presence of this extra coset structure and its quantum consistency require further properties of the $Y(\mathfrak{e}_8)$ R-matrix beyond those discussed in [15]. We explain these in detail here. In particular, for a discrete set of values of the central extension, the algebra $\mathcal{D}Y(\mathfrak{e}_8)_c$ possesses nontrivial ideals which may be divided out to reduce the number of degrees of freedom. Remarkably there is only one among the altogether eight "exceptional" values of the central extension, which admits a non-trivial ideal for which the quantum monodromy matrix becomes symmetric in the limit $\hbar \to 0$ and the associated ideal can be consistently divided out to recover the classical coset space $\mathrm{E}_8/\mathrm{SO}(16)$ of $N=16$ supergravity. The relevant value of the central extension ($c = 1$ with our normalization) differs from the critical value $c = 15$ for which the quantum algebra admits an additional infinite-dimensional center [18].

The main open problem which remains is the compatibility of the local supersymmetry constraints with the $Y(\mathfrak{e}_8)$ charge algebra at the quantum level. In this paper we have concentrated on the direct quantization of the algebra of nonlocal charges which are classically invariant under supersymmetry, i.e. Poisson commute weakly with the supersymmetry generators. A complete treatment should in addition contain a quantum version of the supersymmetry constraint algebra (an $N=16$ superconformal algebra) which could serve to define the physical states as its kernel. The Yangian structure exhibited in this paper would then become a spectrum generating algebra for $N=16$ supergravity. Let us emphasize, however, that the interplay between canonical constraints and non-local conserved charges in integrable field theories has so far not been studied at all at the quantum level, as the existing literature deals exclusively with flat space models rather than the generally covariant and locally supersymmetric models we are concerned with here.

Our results underline the importance of quantum group structures for dimensionally reduced gravity and supergravity. The ultimate aim here is the identification of a "quantum Geroch group" which would act on the space of physical states in the same way as the classical Geroch group acts on the moduli space of classical solu-

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tions. The relevance of these structures for string and M-theory seems also obvious. After all, the resulting symmetries can be regarded as quantum deformations of the infinite dimensional U-duality symmetries that have been conjectured to appear in compactified string and M-theory [19]. However, our results also indicate that some widely held perceptions and expectations may need to be revised. In particular, the underlying symmetry of the full quantum theory may turn out to be related to some (hyperbolic?) extension of $Y(\mathfrak{e}_8)$, rather than just the arithmetic duality groups $\mathrm{E}_{9(9)}(\mathbb{Z})$ and $\mathrm{E}_{10(10)}(\mathbb{Z})$.

2. E_8 preliminaries

In this section we collect some basic facts on the exceptional algebra $\mathfrak{e}_8$ thereby fixing the notation for the following. In particular, we give very explicit expressions for the projectors onto the irreducible parts of the tensor product of two adjoint representations of $\mathfrak{e}_8$.

The generators of $\mathfrak{e}_8$ in the adjoint (and thus fundamental) representation are denoted by X^a. We are here interested its non-compact maximally split form with maximal compact subalgebra $\mathfrak{so}(16)$, giving rise to the coset space $\mathrm{E}_{8(8)}/\mathrm{SO}(16)$. Accordingly, we split E_8 indices $a, b, \ldots$ as $([IJ], A), \ldots$, with $I,\ J = 1, \ldots, 16$ and $A = 1, \ldots, 128$ corresponding to the decomposition $\mathbf{248} \to \mathbf{120} \oplus \mathbf{128}$ of the adjoint representation of $\mathfrak{e}_8$ into the adjoint and the fundamental spinor representation of $\mathfrak{so}(16)$. The generators satisfy the commutation relations

$$[X^a, X^b] = f^{ab}{}_c X^c\,, \tag{2.1}$$

with the convention that summation over antisymmetrized pairs of indices $[IJ]$ is always accompanied by a factor $\frac{1}{2}$, viz.

$$X^a Y_a \equiv X^A Y_A + \frac{1}{2} X^{IJ} Y_{IJ}\,.$$

The structure constants are most conveniently given in their fully antisymmetric form obtained by raising the index c with the help of the Cartan-Killing form η^{ab}. For the adjoint representation, the latter is defined by

$$\eta^{ab} := \frac{1}{60} \mathrm{tr}\,(X^a X^b) = \frac{1}{60} f^{ac}{}_d f^{bd}{}_c\,, \tag{2.2}$$

which yields

$$\eta^{AB} = \delta^{AB}\,, \qquad \eta^{IJ\,KL} = -2\,\delta^{IJ}_{KL}\,. \tag{2.3}$$

The $\mathfrak{e}_8$ structure constants are then completely characterized by

$$f^{IJ,\,KL,\,MN} = -8\,\delta^{\overline{I[K}\,\delta^{L]J}}_{MN}\,, \qquad f^{IJ,\,A,\,B} = -\frac{1}{2}\Gamma^{IJ}_{AB}\,, \tag{2.4}$$

where the matrices Γ^{IJ}_{AB} are obtained from the $\mathfrak{so}(16)$ Γ-matrices in the standard fashion

$$\Gamma^{I}_{A\dot{A}}\Gamma^{J}_{\dot{A}B} = \delta^{IJ}\delta_{AB} + \Gamma^{IJ}_{AB}\,. \tag{2.5}$$

The maximal compact subalgebra $\mathfrak{so}(16)$ can be characterized alternatively as the subalgebra invariant under the symmetric space involution

$$\tau(X^a) = -(X^a)^T\,. \tag{2.6}$$

For the formulation of the Yang Baxter equation we will need to deal with operators acting on the tensor product $\mathbf{248}\otimes\mathbf{248}$. The associated matrices will be denoted as $O_{ab}{}^{cd}$, where we refer to the indices ab as "incoming" and to the indices cd as "outgoing". The product of two such matrices O and P is consequently given by

$$(OP)_{ab}{}^{cd} := O_{ab}{}^{ef}P_{ef}{}^{cd}\,.$$

As with the generators above, the Cartan-Killing metric must be used whenever indices are raised or lowered from their "canonical" position on such matrices. We define

$$\overset{21}{O}_{ab}{}^{cd} := \overset{12}{O}_{ba}{}^{dc}\,. \tag{2.7}$$

To write down the projectors we need the operators $\mathbb{1}$, Π (i.e. the identity and the exchange operator, respectively), and $\tilde{\Pi}$, which are given by

$$\mathbb{1}_{ab}{}^{cd} = \delta^c_a\,\delta^d_b\,, \qquad \Pi_{ab}{}^{cd} = \delta^d_a\,\delta^c_b\,, \qquad \tilde{\Pi}_{ab}{}^{cd} = \eta_{ab}\,\eta^{cd}\,. \tag{2.8}$$

A further important operator is the symmetric Casimir element defined in the adjoint representation by

$$\begin{aligned}\Omega_{\mathfrak{e}_8} &\equiv \eta_{ab}\, X^a\otimes X^b \\ &= -\frac{1}{2}\,X^{IJ}\otimes X^{IJ} + X^A\otimes X^A \;\in\; \mathfrak{so}(16)\otimes\mathfrak{so}(16) + \mathfrak{k}\otimes\mathfrak{k}\,.\end{aligned} \tag{2.9}$$

In terms of the structure constants of E_8, the Casimir element can be alternatively expressed as

$$(\Omega_{\mathfrak{e}_8})_{ab}{}^{cd} = f^e{}_a{}^c f_{eb}{}^d\,. \tag{2.10}$$

We will also need the twisted Casimir element $\Omega^\tau_{\mathfrak{e}_8}$, defined by

$$\begin{aligned}\Omega^\tau_{\mathfrak{e}_8} &\equiv \eta_{ab}X^a\otimes\tau(X^b) \\ &= -\frac{1}{2}\,X^{IJ}\otimes X^{IJ} - X^A\otimes X^A\,,\end{aligned} \tag{2.11}$$

i.e. in indices:

$$(\Omega^\tau_{\mathfrak{e}_8})_{ab}{}^{cd} = -(\Omega_{\mathfrak{e}_8})_{cb}{}^{ad} = -f_{ea}{}^c f_{eb}{}^d\,. \tag{2.12}$$

The twisted Casimir element is obviously not E_8 but only SO(16) invariant.

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The tensor product of two adjoint representations of E_8 splits into its irreducible components according to $\mathbf{248}\otimes\mathbf{248} = \mathbf{1}\oplus\mathbf{248}\oplus\mathbf{3875}\oplus\mathbf{27000}\oplus\mathbf{30380}$. The corresponding projectors are given by:

$$
\begin{aligned}
\mathcal{P}_1 &= \frac{1}{248}\,\tilde{\Pi}, \\
\mathcal{P}_{248} &= \frac{1}{60}\left(\Omega_{\mathfrak{e}_8}\Pi - \Omega_{\mathfrak{e}_8}\right), \\
\mathcal{P}_{3875} &= \frac{1}{14}\left(\mathbb{1} - \frac{1}{4}\,\tilde{\Pi} + \Pi - \frac{1}{2}(\Omega_{\mathfrak{e}_8}\Pi + \Omega_{\mathfrak{e}_8})\right), \\
\mathcal{P}_{27000} &= \frac{1}{7}\left(3\,\mathbb{1} + \frac{3}{31}\,\tilde{\Pi} + 3\,\Pi + \frac{1}{4}(\Omega_{\mathfrak{e}_8}\Pi + \Omega_{\mathfrak{e}_8})\right), \\
\mathcal{P}_{30380} &= \frac{1}{2}\,\mathbb{1} - \frac{1}{2}\,\Pi + \frac{1}{60}\left(\Omega_{\mathfrak{e}_8} - \Omega_{\mathfrak{e}_8}\Pi\right).
\end{aligned}
\tag{2.13}
$$

To verify that these operators indeed satisfy orthogonal projection relations, one needs the following relation

$$
\Omega_{\mathfrak{e}_8}^2 = 12\,\mathbb{1} + 12\,\Pi + 12\,\tilde{\Pi} - 20\,\Omega_{\mathfrak{e}_8} + 10\,\Omega_{\mathfrak{e}_8}\Pi\,,
\tag{2.14}
$$

whose validity we have established with the help of a computer. In terms of the $\mathfrak{e}_8$ structure constants this relation becomes

$$
f^{e}{}_{ag} f_{beh} f^{gic} f_i{}^{hd} = 24\delta^{\,c}_{(a}\delta^{d}_{b)} + 12\eta_{ab}\eta^{cd} - 20 f^{e}{}_{a}{}^{c} f_{eb}{}^{d} + 10 f^{e}{}_{a}{}^{d} f_{eb}{}^{c}\,.
$$

In indices, the projectors read:

$$
\begin{aligned}
(\mathcal{P}_1)_{ab}{}^{cd} &= \frac{1}{248}\,\eta_{ab}\eta^{cd}, \\
(\mathcal{P}_{248})_{ab}{}^{cd} &= -\frac{1}{60}\,f^{e}{}_{ab} f_{e}{}^{cd}, \\
(\mathcal{P}_{3875})_{ab}{}^{cd} &= \frac{1}{7}\,\delta^{\,c}_{(a}\delta^{d}_{b)} - \frac{1}{56}\,\eta_{ab}\eta^{cd} - \frac{1}{14}\,f^{e}{}_{a}{}^{(c} f_{eb}{}^{d)}, \\
(\mathcal{P}_{27000})_{ab}{}^{cd} &= \frac{6}{7}\,\delta^{\,c}_{(a}\delta^{d}_{b)} + \frac{3}{217}\,\eta_{ab}\eta^{cd} + \frac{1}{14}\,f^{e}{}_{a}{}^{(c} f_{eb}{}^{d)}, \\
(\mathcal{P}_{30380})_{ab}{}^{cd} &= \delta^{c}_{[a}\delta^{d}_{b]} + \frac{1}{60}\,f^{e}{}_{ab} f_{e}{}^{cd}.
\end{aligned}
\tag{2.15}
$$

All these projectors are manifestly symmetric w.r.t. interchange of the two subspaces, i.e. $\overset{12}{\mathcal{P}}_j = \overset{21}{\mathcal{P}}_j$. Furthermore, any E_8 matrix $\mathcal{V}$ obeys

$$
\mathcal{P}_j\,\mathcal{V}\otimes\mathcal{V} = \mathcal{V}\otimes\mathcal{V}\,\mathcal{P}_j\,,
\tag{2.16}
$$

which together with the normalization $\det\mathcal{V} = 1$ can be taken as defining relations for the group elements of E_8.

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3. R-matrix and the Yangian $Y(\mathfrak{e}_8)$

Here, we review the Yangian algebra $Y(\mathfrak{e}_8)$ and the R-matrix associated to its fundamental representation **249** [15]. The Yangian $Y(\mathfrak{e}_8)$ [16] is recursively defined as the associative algebra with generators $\mathcal{X}^a$ and $\mathcal{Y}^a$ $(a = 1, \ldots, 248)$ and relation

$$\begin{aligned}
&\left[\mathcal{X}^a, \mathcal{X}^b\right] = \mathrm{i}\hbar f^{ab}{}_c\, \mathcal{X}^c \,, \qquad \left[\mathcal{X}^a, \mathcal{Y}^b\right] = \mathrm{i}\hbar f^{ab}{}_c\, \mathcal{Y}^c \,, \\
&\left[\mathcal{Y}^a \left[\mathcal{Y}^b, \mathcal{X}^c\right]\right] - \left[\mathcal{X}^a \left[\mathcal{Y}^b, \mathcal{Y}^c\right]\right] = -\hbar^2 L^{abc} \,, \\
&\text{with } \; L^{abc} = \frac{1}{24} f^{ad}{}_g f^{be}{}_h f^{cf}{}_i f^{ghi} \{\mathcal{X}^d, \mathcal{X}^e, \mathcal{X}^f\} \,, \\
&\text{and } \; \{\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3\} = \sum_\sigma \mathcal{X}^{\sigma(1)} \mathcal{X}^{\sigma(2)} \mathcal{X}^{\sigma(3)} \,.
\end{aligned} \tag{3.1}$$

It admits a nontrivial coproduct and antipode structure whose explicit form will not be needed here, see e.g. Thm. 12.1.1 of [20] for details.

Due to the fact that L^{abc} does not vanish when the $\mathcal{X}^a$ are evaluated in the fundamental representation of $\mathfrak{e}_8$, it is not possible to lift this representation of $\mathfrak{e}_8$ to a representation of $Y(\mathfrak{e}_8)$. Rather, the minimal representation of $Y(\mathfrak{e}_8)$ is reducible over $\mathfrak{e}_8$ and contains an additional trivial representation of $\mathfrak{e}_8$ [16]. With respect to $\mathfrak{so}(16)$ we thus have the decomposition

$$\mathbf{249} \to \mathbf{1} \oplus \mathbf{120} \oplus \mathbf{128} \,. \tag{3.2}$$

For compactness of notation, we will label the extra singlet by 0 and use hatted indices which run over all 249 dimensions, i.e. $0 \leq \hat{a}, \hat{b}, \ldots \leq 248$.

The R-matrix associated with the fundamental representation of $Y(\mathfrak{e}_8)$ is the solution $R(w)$ to the Quantum Yang-Baxter Equation ($\equiv$ QYBE)

$$\overset{12}{R}(u-v)\,\overset{13}{R}(u)\,\overset{23}{R}(v) = \overset{23}{R}(v)\,\overset{13}{R}(u)\,\overset{12}{R}(u-v) \,, \tag{3.3}$$

or, with indices written out,

$$R_{\hat{a}\hat{b}}{}^{\hat{g}\hat{h}}(u-v) R_{\hat{g}\hat{c}}{}^{\hat{p}\hat{i}}(u) R_{\hat{h}\hat{i}}{}^{\hat{q}\hat{r}}(v) = R_{\hat{b}\hat{c}}{}^{\hat{h}\hat{i}}(v) R_{\hat{a}\hat{i}}{}^{\hat{g}\hat{r}}(u) R_{\hat{g}\hat{h}}{}^{\hat{p}\hat{q}}(u-v) \,, \tag{3.4}$$

The classical limit is

$$R(w) = \mathbb{1} - \frac{\mathrm{i}\hbar}{w}\, \Omega_{\mathfrak{e}_8} + \mathcal{O}\left(\frac{\hbar^2}{w^2}\right) \qquad \text{for} \quad w \to \infty \,. \tag{3.5}$$

where the definition of the Casimir element $\Omega_{\mathfrak{e}_8}$ is extended to $\mathbf{1} \oplus \mathbf{248}$ by the trivial (zero) action on the $\mathbf{1}$. We also impose the standard normalization condition

$$R(0) = \Pi \,. \tag{3.6}$$

Within the tensor product $\mathbf{249} \otimes \mathbf{249}$ we introduce in addition to the operators from (2.13) the projector $\mathcal{P}_0$ onto the one-dimensional space $\mathbf{1} \otimes \mathbf{1}$ and the projectors $\mathcal{P}_+$ and $\mathcal{P}_-$ onto the symmetric and antisymmetric part of the space $(\mathbf{248} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{248})$, respectively. Furthermore, there are $\mathfrak{e}_8$ invariant intertwining operators between subspaces of the same dimension, which we denote by $\mathcal{I}_{01}$, $\mathcal{I}_{10}$, $\mathcal{I}_{+248}$, and $\mathcal{I}_{248+}$. They are defined by

$$\begin{aligned} \mathcal{I}_{01}\mathcal{I}_{10} &= \mathcal{P}_0 \qquad & \mathcal{I}_{10}\mathcal{I}_{01} &= \mathcal{P}_1 \,, \\ \mathcal{I}_{+248}\mathcal{I}_{248+} &= \mathcal{P}_+ \qquad & \mathcal{I}_{248+}\mathcal{I}_{+248} &= \mathcal{P}_{248} \,, \end{aligned} \tag{3.7}$$

respectively, up to relative factors between the intertwiners which drop out in the above relations. Explicitly, the new projectors and intertwiners are given by

$$\begin{aligned} (\mathcal{P}_0)_{00}{}^{00} &= 1, \\ (\mathcal{P}_+)_{a0}{}^{b0} &= (\mathcal{P}_+)_{a0}{}^{0b} = (\mathcal{P}_+)_{0a}{}^{b0} = (\mathcal{P}_+)_{0a}{}^{0b} = \frac{1}{2}\delta_b^a, \\ (\mathcal{P}_-)_{a0}{}^{b0} &= -(\mathcal{P}_-)_{a0}{}^{0b} = -(\mathcal{P}_-)_{0a}{}^{b0} = (\mathcal{P}_-)_{0a}{}^{0b} = \frac{1}{2}\delta_b^a, \\ (\mathcal{I}_{01})_{00}{}^{ab} &= \eta^{ab}, \\ (\mathcal{I}_{10})_{ab}{}^{00} &= \frac{1}{248}\eta_{ab}, \\ (\mathcal{I}_{+248})_{0a}{}^{bc} &= (\mathcal{I}_{+248})_{a0}{}^{bc} = \frac{1}{120}f_a{}^{bc}, \\ (\mathcal{I}_{248+})_{ab}{}^{c0} &= (\mathcal{I}_{248+})_{ab}{}^{0c} = -f_{ab}{}^{c}, \end{aligned} \tag{3.8}$$

with all other components vanishing. Again all operators are symmetric w.r.t. interchange of the two subspaces with the exception of the intertwiners $\mathcal{I}_{+248}$ and $\mathcal{I}_{248+}$, which obey

$$\overset{12}{\mathcal{I}}_{+248} = -\overset{21}{\mathcal{I}}_{+248} \,, \qquad \overset{12}{\mathcal{I}}_{248+} = -\overset{21}{\mathcal{I}}_{248+} \;. \tag{3.9}$$

As shown in [15], the R-matrix associated to the fundamental representation of $Y(\mathfrak{e}_8)$ in terms of these projectors and intertwiners is given by

$$\begin{aligned} f^{-1}(w)R(w) = {} & \frac{w+\mathrm{i}\hbar}{w-\mathrm{i}\hbar}\,\mathcal{P}_{30380} + \mathcal{P}_{27000} + \frac{w^3+15w^2\mathrm{i}\hbar+44w(\mathrm{i}\hbar)^2+60(\mathrm{i}\hbar)^3}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)}\,\mathcal{P}_{248} + \\ & + \frac{(w+\mathrm{i}\hbar)(w+6\mathrm{i}\hbar)}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)}\mathcal{P}_{3875} + \frac{w^3-15w^2\mathrm{i}\hbar+44w(\mathrm{i}\hbar)^2-60(\mathrm{i}\hbar)^3}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)}\mathcal{P}_+ + \\ & + \frac{w+\mathrm{i}\hbar}{w-\mathrm{i}\hbar}\mathcal{P}_- + \frac{w^4+30w^3\mathrm{i}\hbar+269w^2(\mathrm{i}\hbar)^2+660w(\mathrm{i}\hbar)^3+900(\mathrm{i}\hbar)^4}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)(w-15\mathrm{i}\hbar)}\mathcal{P}_1 + \\ & + \frac{w^4-30w^3\mathrm{i}\hbar+269w^2(\mathrm{i}\hbar)^2-660w(\mathrm{i}\hbar)^3+900(\mathrm{i}\hbar)^3}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)(w-15\mathrm{i}\hbar)}\,\mathcal{P}_0 + \\ & + \frac{w(\mathrm{i}\hbar)^3}{\alpha^2(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)(w-15\mathrm{i}\hbar)}\,\mathcal{I}_{01} + \\ & + \frac{248(60\alpha)^2w(\mathrm{i}\hbar)^3}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)(w-15\mathrm{i}\hbar)}\,\mathcal{I}_{10} - \\ & - \frac{60\sqrt{2}w(\mathrm{i}\hbar)^2}{\alpha(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)}\,\mathcal{I}_{+248} + \\ & + \frac{30\sqrt{2}\alpha w(\mathrm{i}\hbar)^2}{(w-\mathrm{i}\hbar)(w-6\mathrm{i}\hbar)(w-10\mathrm{i}\hbar)}\,\mathcal{I}_{248+} \,. \end{aligned}$$

We rewrite this in the form

$$f^{-1}(w)R(w) = \mathbb{1} + \sum_{j=1}^{4} \frac{\mathcal{R}_j}{w - w_j}\,, \tag{3.10}$$

where the poles are located at

$$w_1 = \mathrm{i}\hbar\,, \qquad w_2 = 6\mathrm{i}\hbar\,, \qquad w_3 = 10\mathrm{i}\hbar\,, \qquad w_4 = 15\mathrm{i}\hbar\,, \tag{3.11}$$

and the associated residues are

$$\begin{aligned}
\mathcal{R}_1 &= 2\,\mathcal{P}_{30380} - \frac{14}{5}\,\mathcal{P}_{3875} + \frac{8}{3}\,\mathcal{P}_{248} - \frac{2}{3}\,\mathcal{P}_{+} + 2\,\mathcal{P}_{-} - \frac{62}{21}\,\mathcal{P}_{1} - \\
&\quad - \frac{16}{21}\,\mathcal{P}_0 - \frac{4\sqrt{2}}{3\alpha}\,\mathcal{I}_{+248} + \frac{2\sqrt{2}\alpha}{3}\,\mathcal{I}_{248+} - \frac{9920\alpha^2}{7}\,\mathcal{I}_{10} - \frac{1}{630\alpha^2}\,\mathcal{I}_{01}\,, \\
\mathcal{R}_2 &= \frac{84}{5}\,\mathcal{P}_{3875} - 54\,\mathcal{P}_{248} + 6\,\mathcal{P}_{+} + 124\,\mathcal{P}_1 + 8\,\mathcal{P}_0 + \frac{18\sqrt{2}}{\alpha}\,\mathcal{I}_{+248} - \\
&\quad - 9\sqrt{2}\alpha\,\mathcal{I}_{248+} + 29760\alpha^2\,\mathcal{I}_{10} + \frac{1}{30\alpha^2}\,\mathcal{I}_{01}\,, \\
\mathcal{R}_3 &= \frac{250}{3}\,\mathcal{P}_{248} - \frac{10}{3}\,\mathcal{P}_{+} - \frac{1240}{3}\,\mathcal{P}_1 - \frac{20}{3}\,\mathcal{P}_0 - \frac{50\sqrt{2}}{3\alpha}\,\mathcal{I}_{+248} + \\
&\quad + \frac{25\sqrt{2}\alpha}{3}\,\mathcal{I}_{248+} - 49600\alpha^2\,\mathcal{I}_{10} - \frac{1}{18\alpha^2}\,\mathcal{I}_{01}\,, \\
\mathcal{R}_4 &= \frac{2480}{7}\,\mathcal{P}_1 + \frac{10}{7}\,\mathcal{P}_0 + \frac{148800\alpha^2}{7}\,\mathcal{I}_{10} + \frac{1}{42\alpha^2}\,\mathcal{I}_{01}\,.
\end{aligned} \tag{3.12}$$

The scalar function f is uniquely defined by its functional equation

$$f(w)f(w - 15\mathrm{i}\hbar) = \frac{(w - \mathrm{i}\hbar)(w - 6\mathrm{i}\hbar)(w - 10\mathrm{i}\hbar)(w - 15\mathrm{i}\hbar)}{w(w - 5\mathrm{i}\hbar)(w - 9\mathrm{i}\hbar)(w - 14\mathrm{i}\hbar)}\,, \tag{3.13}$$

and its asymptotic behavior

$$f(w) = 1 - \frac{2\mathrm{i}\hbar}{w} + \mathcal{O}\left(\frac{1}{w^2}\right) \qquad \text{for} \quad w \to \pm\infty\,. \tag{3.14}$$

It allows an explicit expression in terms of Γ-functions which however is not of particular interest for the following. Observe that (3.13) and (3.14) already imply the relations

$$f(w)f(-w) = 1\,, \qquad f(w)^* = f(-w^*)\,.$$

The free parameter α which appears in the solution of the QYBE is basically a consequence of the fact that the singlet in (3.2) may be rescaled with an arbitrary factor; two R matrices (3.10) with different values of α are related by conjugation with $\mathrm{diag}(\alpha_1\alpha_2^{-1}, \mathbb{1}_{120}, \mathbb{1}_{128}) \otimes \mathrm{diag}(\alpha_1\alpha_2^{-1}, \mathbb{1}_{120}, \mathbb{1}_{128})$. Without loss of generality we can thus fix the parameter α to

$$60\alpha^2 := -1\,. \tag{3.15}$$

For this value only, the R-matrix obeys the additional non-covariant relation

$$R_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}}(w) = R_{\hat{c}\hat{d}}{}^{\hat{a}\hat{b}}(w)\,, \tag{3.16}$$

which is proved by inspection and by use of the special (non-covariant) property $f_a{}^{bc} = -f^a{}_{bc}$ of the $\mathrm{E}_{8(8)}$ structure constants (2.4).

The following further properties of the R-matrix are easily verified:

$$\overset{12}{R}(w)\,\overset{21}{R}(-w) = \mathbb{1}, \tag{3.17}$$

$$\overset{12}{R}(w)^* = \overset{21}{R}(-w^*)\,, \tag{3.18}$$

where the second equation is only valid for imaginary α, which is compatible with our choice (3.15) above. In the context of two-dimensional scattering theory, these relations express the requirements of unitarity and hermiticity of the S-matrix, respectively. With indices written out they acquire the following explicit form

$$R_{\hat{a}\hat{b}}{}^{\hat{g}\hat{h}}(w)R_{\hat{h}\hat{g}}{}^{\hat{c}\hat{d}}(-w) = \delta_{\hat{a}}^{\hat{d}}\delta_{\hat{b}}^{\hat{c}}\,, \tag{3.19}$$

$$\left(R_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}}(w)\right)^* = R_{\hat{b}\hat{a}}{}^{\hat{d}\hat{c}}(-w^*)\,. \tag{3.20}$$

The occurrence of poles at $w = w_j$ and relation (3.17) together imply that $R(w)$ is non-invertible at the points $w = -w_j$. More specifically, (3.17) yields the relations

$$\overset{12}{\mathcal{R}_j}\,\overset{21}{R}(-w_j) = 0\,. \tag{3.21}$$

From the formulae given above it is straightforward to check that the residue $\mathcal{R}_4$ at $w_4 = 15\mathrm{i}\hbar$ is singled out by its property of being proportional to a one-dimensional projector:

$$(\mathcal{R}_4)_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}} = \frac{10}{7}\,\eta_{\hat{a}\hat{b}}\eta^{\hat{c}\hat{d}}\,, \tag{3.22}$$

where $\eta_{\hat{a}\hat{b}}$ denotes the natural extension of the Cartan-Killing form into $\mathbf{249}\otimes\mathbf{249}$ given by the additional entry $\eta_{00} = 60\alpha^2 = -1$. Evaluating the QYBE (3.4) at $u-v = 15\mathrm{i}\hbar$ then gives rise to the following relation

$$(\mathcal{R}_4)_{\hat{a}\hat{b}}{}^{\hat{g}\hat{h}}R_{\hat{g}\hat{c}}{}^{\hat{p}\hat{i}}(u)R_{\hat{h}\hat{i}}{}^{\hat{q}\hat{r}}(u-15\mathrm{i}\hbar) = \delta_{\hat{c}}^{\hat{r}}\;(\mathcal{R}_4)_{\hat{a}\hat{b}}{}^{\hat{p}\hat{q}}\,. \tag{3.23}$$

From these observations, we can deduce the crossing invariance property of the R-matrix:

$$R_{\hat{b}}{}^{\hat{a}\hat{d}}{}_{\hat{c}}(w) \equiv \eta^{\hat{a}\hat{g}}R_{\hat{b}\hat{g}}{}^{\hat{d}\hat{h}}(w)\eta_{\hat{h}\hat{c}} = R_{\hat{c}\hat{b}}{}^{\hat{a}\hat{d}}(15\mathrm{i}\hbar - w)\,. \tag{3.24}$$

The knowledge of the R-matrix associated with an irreducible representation of (3.1) now gives rise to another equivalent presentation of the Yangian algebra itself [16]. Consider the associative algebra with generators $(T_{(n)})_{\hat{a}}{}^{\hat{b}}$, $(0 \leq \hat{a},\hat{b} \leq 248)$, $n \in \mathbb{N}$ and defining relations

$$R_{\hat{a}\hat{b}}{}^{\hat{e}\hat{f}}(u-v)T_{\hat{e}}{}^{\hat{c}}(u)T_{\hat{f}}{}^{\hat{d}}(v) = T_{\hat{b}}{}^{\hat{f}}(v)T_{\hat{a}}{}^{\hat{e}}(u)R_{\hat{e}\hat{f}}{}^{\hat{c}\hat{d}}(u-v)\,, \tag{3.25}$$

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where $T_{\hat{a}}{}^{\hat{b}}(u)$ denotes the formal series

$$T_{\hat{a}}{}^{\hat{b}}(u) = \delta_{\hat{a}}^{\hat{b}} + \sum_{n=1}^{\infty} (T_{(n)})_{\hat{a}}{}^{\hat{b}}\, u^{-n}\,. \tag{3.26}$$

The QYBE (3.4) ensures compatibility of the exchange relations (3.25) with associativity of the multiplication. Their evaluation at $u-v = 15\mathrm{i}\hbar$ shows that there exists an invariant scalar quantity $q(T(u))$, the "quantum determinant", which is bilinear in the matrix entries of T:

$$q(T(u))\mathcal{R}_4 := \mathcal{R}_4 \overset{1}{T}(u{+}15\mathrm{i}\hbar) \overset{2}{T}(u) = \overset{2}{T}(u) \overset{1}{T}(u{+}15\mathrm{i}\hbar)\, \mathcal{R}_4\,. \tag{3.27}$$

Using (3.23) one checks that $q(T(u))$ lies in the center of the algebra (3.25). Thus, we may pass to the quotient of this algebra over the two-sided ideal generated by the central element by setting $q(T) = 1$ or equivalently

$$T_{\hat{a}}{}^{\hat{c}}(u-15\mathrm{i}\hbar) T_{\hat{b}}{}^{\hat{d}}(u) \eta_{\hat{c}\hat{d}} = \eta_{\hat{a}\hat{b}}\,. \tag{3.28}$$

It has been stated by Drinfeld [16] that this quotient is isomorphic to the Yangian $Y(\mathfrak{e}_8)$ as defined at the beginning of this section (3.1).[2] The precise isomorphism requires knowledge of the universal R-matrix of $Y(\mathfrak{e}_8)$ which is certainly beyond our scope here; one may however easily identify the generating elements $\mathcal{X}^a$ and $\mathcal{Y}^a$

$$\mathcal{X}^a = \mathrm{tr}\,\left[X^a T_{(1)}\right]\,, \qquad \mathcal{Y}^a = \mathrm{tr}\,\left[X^a \left(T_{(2)} - \frac{1}{2} T_{(1)} T_{(1)}\right)\right]\,. \tag{3.29}$$

We can further expand (3.25) around $u{=}\infty$ and use (3.5) to obtain the commutator

$$\left[\overset{1}{T}_{(1)}\,, \overset{2}{T}(w)\right] = \mathrm{i}\hbar\left[\Omega_{\mathfrak{e}_8}\,, \overset{2}{T}(w)\right]\,, \tag{3.30}$$

which in particular reproduces the first two commutation relations of (3.1). We close the general discussion here with two well-known properties of the presentation (3.25) of the Yangian

- Any representation ρ of $Y(\mathfrak{e}_8)$ defines a one-parameter family of representations ρ_a labeled by a complex number a:

  $$\rho_a(T(w)) := \rho(T(w{-}a))\,. \tag{3.31}$$

  The fundamental representation **249** in particular gives rise to the family $\mathbf{249}_a$:

  $$\rho_a^{\mathbf{249}}(T(w)) := R(w{-}a)\,, \tag{3.32}$$

  with the R-matrix from (3.10). Note that the relation (3.28) in these representations corresponds to the normalization (3.23) of the R-matrix.

- The coproduct of $Y(\mathfrak{e}_8)$ takes the simple form:

  $$\Delta\left(T_{\hat{a}}{}^{\hat{b}}(w)\right) = T_{\hat{a}}{}^{\hat{c}}(w) \otimes T_{\hat{c}}{}^{\hat{b}}(w)\,. \tag{3.33}$$

[2]However, we presently cannot exclude the possibility that the center of (3.25) contains elements of higher degree in the T's which are not generated by $q(T)$.

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4. Classical Yangian symmetries in $N=16$ supergravity

This section is a brief review of the classical symmetries and the algebra of nonlocal charges in two-dimensional $N=16$ supergravity [13, 14]. The scalar sector of this model is described by an $E_{8(8)}$-valued matrix $\mathcal{V}$ which transforms under a global E_8 symmetry and a local SO(16) gauge symmetry in the usual way

$$\mathcal{V}(x) \mapsto g\mathcal{V}(x)h(x)\,, \qquad g \in E_{8(8)}\,, \quad h(x) \in SO(16)\,. \tag{4.1}$$

Thus, its bosonic configuration space is given by the coset space $E_{8(8)}/SO(16)$. It may be parametrized by the symmetric E_8-valued matrix

$$\mathcal{M} \equiv \mathcal{V}\mathcal{V}^T\,, \qquad \text{i.e.} \qquad \mathcal{M}_{ab} \equiv \mathcal{V}_a{}^c\mathcal{V}_b{}^c = \mathcal{M}_{ba}\,, \tag{4.2}$$

which is evidently gauge (= SO(16)) invariant. The symmetry of $\mathcal{M}$ may be characterized algebraically by the fact that it is annihilated by the antisymmetric projectors

$$(\mathcal{P}_{248})_{ab}{}^{cd}\mathcal{M}_{cd} = 0 = (\mathcal{P}_{30380})_{ab}{}^{cd}\mathcal{M}_{cd}\,, \tag{4.3}$$

whereas for the symmetric projectors from (2.15) one finds the following identities

$$\begin{aligned}
(\mathcal{P}_1)_{ab}{}^{cd}\mathcal{M}_{cd} &= \frac{1}{31}\eta_{ab} \quad \Longleftrightarrow \quad \eta^{ab}\mathcal{M}_{ab} = 8\,,\\
(\mathcal{P}_{3875})_{ab}{}^{cd}\mathcal{M}_{cd} &= 0\,,\\
(\mathcal{P}_{27000})_{ab}{}^{cd}\mathcal{M}_{cd} &= \mathcal{M}_{ab} - \frac{1}{31}\eta_{ab}\,.
\end{aligned} \tag{4.4}$$

To verify these relations one needs the E_8-invariance of the projectors and the Cartan-Killing form:

$$(\mathcal{P}_j)_{ab}{}^{cd}\mathcal{M}_{cd} = \mathcal{V}_a{}^c\mathcal{V}_b{}^d(\mathcal{P}_j)_{cd}{}^{ee}\,, \qquad \mathcal{V}_a{}^c\mathcal{V}_b{}^d\,\eta_{cd} = \eta_{ab}\,.$$

The second relation in (4.4) requires the additional formula

$$f^d{}_{ac}f_{dbc} = \begin{cases} 8\delta^{IJ}_{KL} & \text{if } (ab) = (IJ, KL) \\ 0 & \text{otherwise}\,, \end{cases}$$

where the summation over the E_8 index c is with the "wrong metric" (i.e. with the SO(16)-covariant δ_{ab} rather than η_{ab}).

The scalar fields represented by the matrix $\mathcal{M}$ satisfy equations of motion which allow a Lax pair formulation [7, 6, 4, 8, 9] similar to the principal chiral model. In particular this allows the construction of an infinite family of nonlocal integrals of motion which are obtained from the transition matrices associated to the Lax pair [13, 14]. These integrals of motion are encoded in a symmetric E_8-valued matrix $\mathcal{M}(w)$ obtained by integrating the Lax connection over certain space intervals and depending on a complex spectral parameter w. This matrix parametrizes the full scalar sector of the phase space in the sense that for real values of w the matrix $\mathcal{M}(w)$ coincides with the physical scalar fields $\mathcal{M}(x)$ evaluated on the particular axis in space-time where the dilaton field ρ vanishes

$$\mathcal{M}(w) = \mathcal{M}(x)\Big|_{\rho(x)=0,\, \tilde{\rho}(x)=w} \;. \tag{4.5}$$

This relation has been formulated in the coordinate system where the two-dimensional world-sheet is parametrized by the dilaton field ρ and its dual axion $\tilde{\rho}$. For instance, for cylindrically symmetry spacetimes the matrix $\mathcal{M}(w)$ carries the values of physical scalar fields along the symmetry axis.

We may further introduce its Riemann-Hilbert decomposition

$$\mathcal{M}_{ab}(w) \equiv U_+(w)_a{}^c\, U_-(w)_b{}^c\,, \tag{4.6}$$

into E_8-valued functions $U_\pm(w)$ which are holomorphic in the upper and the lower half of the complex w-plane, respectively. They are related by complex conjugation

$$(U_+(w))^* = U_-(w^*)\,. \tag{4.7}$$

In [13, 14] it was shown that these phase space quantities are subject to the following symplectic structure:

$$\begin{aligned}\{\mathcal{M}_{ab}(v), \mathcal{M}_{cd}(w)\} = \frac{1}{v-w}\Big(&(\Omega_{\mathfrak{e}_8})_{ac}{}^{mn}\mathcal{M}_{mb}(v)\mathcal{M}_{nd}(w) + \\ &+ \mathcal{M}_{am}(v)\mathcal{M}_{cn}(w)(\Omega_{\mathfrak{e}_8})_{bd}{}^{mn} - \mathcal{M}_{am}(v)(\Omega^\tau_{\mathfrak{e}_8})_{mc}{}^{bn}\mathcal{M}_{nd}(w) - \\ &- \mathcal{M}_{cm}(w)(\Omega^\tau_{\mathfrak{e}_8})_{an}{}^{md}\mathcal{M}_{nb}(v)\Big)\,,\end{aligned} \tag{4.8}$$

with $\Omega_{\mathfrak{e}_8}$ and $\Omega^\tau_{\mathfrak{e}_8}$ from (2.10) and (2.12), respectively. One may check that these Poisson brackets are covariant under E_8 and compatible with the symmetry of $\mathcal{M}$ (4.3), as required for consistency. For the purpose of quantization to be addressed in the next section it is further convenient to decompose this structure according to (4.6) into the following brackets

$$\begin{aligned}\Big\{\overset{1}{U}_\pm(v)\,,\, \overset{2}{U}_\pm(w)\Big\} &= \left[\frac{2\Omega_{\mathfrak{e}_8}}{v-w}\,,\, \overset{1}{U}_\pm(v)\, \overset{2}{U}_\pm(w)\right]\,, \\ \Big\{\overset{1}{U}_\pm(v)\,,\, \overset{2}{U}_\mp(w)\Big\} &= \frac{2\Omega_{\mathfrak{e}_8}}{v-w}\, \overset{1}{U}_\pm(v)\, \overset{2}{U}_\mp(w) - \overset{1}{U}_\pm(v)\, \overset{2}{U}_\mp(w)\frac{2\Omega^\tau_{\mathfrak{e}_8}}{v-w}\,.\end{aligned} \tag{4.9}$$

In a theory with local symmetries, observables such as the conserved non-local charges contained in $U_\pm(w)$ must weakly commute with the associated canonical constraints. For the above charges this was shown to be the case in [14]. Namely, for the traceless components $T'_{\mu\nu} := T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\rho{}_\rho$ of the energy momentum tensor (generating local translations along the lightcone), we simply have

$$\Big\{T'_{\mu\nu}(z)\,,\, U_\pm(w)\Big\} = 0\,. \tag{4.10}$$

This relation expresses the invariance of the charges $U_\pm(w)$ under general coordinate transformations, which thus indeed constitute "observables" in the sense of Dirac.

In supergravity, we have in addition the constraints $S^I_\alpha(z)$ generating $N=16$ local supersymmetry transformations (α is a spinor index in two dimensions). As shown in [14], the relations expressing the invariance of the charges $U_\pm(w)$ under local supersymmetry are considerably more complicated than (4.10). Recalling that the integrals of motion $U_\pm(w)$ are obtained from certain transition matrices $U(x,y;w)$ associated to the Lax pair of the model, we found that they obey Poisson bracket of the following type (for $x<z<y$):

$$\Big\{ U(x,y;w), S^I_\alpha(z) \Big\} \sim U(x,z;w)\, X^{IJ} S^J_\alpha(z)\, U(z,y;w)\,, \tag{4.11}$$

which vanish indeed on the constraint surface $S^I_\alpha(x)=0$. Apart from questions of operator ordering, it is clear from the form of (4.11) that the combined algebra of nonlocal charges and supersymmetry constraints does not close. It remains an open problem at this point whether one can arrive at a closed structure upon sufficient enlargement of the algebra. Its quantization would entail the existence of a novel type of exchange relations between the conserved charges and the local supersymmetry constraints. The full algebra should then contain the Yangian charge algebra to be presented in the next section as well as a quantized version of the $N=16$ superconformal algebra, into which the supersymmetry constraints close.[3] Note however, that a consistent quantum formulation of the latter is a highly nontrivial task due to the nonlinear nature of the $N=16$ superconformal algebra. For instance, — and in contrast to the standard extended superconformal algebras — free field realizations are not even known at the classical level.

5. Quantization

We now wish to quantize the symplectic structure of the classical charge algebra by means of the R-matrix described above. This amounts to replacing the Poisson brackets (4.9) by quantum exchange relations, leading to a "twisted" Yangian double with central extension c. More precisely, we employ the construction (3.25) to replace the classically conserved non-local charges $U_\pm(w)$ (which by their definition are 248×248 matrices) by a corresponding set of 249×249 matrices $T_\pm(w)$ with operator-valued entries subject to the exchange relations

$$\overset{12}{R}(v-w)\,\overset{1}{T}_\pm(v)\,\overset{2}{T}_\pm(w) = \overset{2}{T}_\pm(w)\,\overset{1}{T}_\pm(v)\,\overset{12}{R}(v-w)\,, \tag{5.1}$$
$$\overset{12}{R}(v-w-\mathrm{i}\hbar c)\,\overset{1}{T}_-(v)\,\overset{2}{T}_+(w) = \overset{2}{T}_+(w)\,\overset{1}{T}_-(v)\,\overset{12}{Q}(v-w)\,, \tag{5.2}$$

[3] There exist alternative extended superconformal algebras with $N>8$ in the literature, see [22, 23, 24]. However, the $N=16$ algebra considered here and in [14] does not fit into any of these schemes.

where for the "twisted" R-matrix Q we require the classical expansion (cf. (3.5))

$$Q(w) = \mathbb{1} - \frac{\mathrm{i}\hbar}{w}\Omega^{\tau}_{\mathfrak{e}_8} + \mathcal{O}\left(\frac{\hbar^2}{w^2}\right) , \tag{5.3}$$

and the compatibility relations

$$\begin{aligned} \overset{12}{Q}(u-v)\,\overset{13}{Q}(u)\,\overset{23}{R}(v) &= \overset{23}{R}(v)\,\overset{13}{Q}(u)\,\overset{12}{Q}(u-v)\,, \\ \overset{12}{R}(u-v)\,\overset{13}{Q}(u)\,\overset{23}{Q}(v) &= \overset{23}{Q}(v)\,\overset{13}{Q}(u)\,\overset{12}{R}(u-v)\,, \end{aligned} \tag{5.4}$$

whose derivation is completely analogous to (3.4). For $60\alpha^2 = -1$, the solution is given by

$$Q_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}}(w) \;:=\; R_{\hat{d}\hat{a}}{}^{\hat{b}\hat{c}}(-w) = R_{\hat{b}\hat{c}}{}^{\hat{d}\hat{a}}(-w)\,, \tag{5.5}$$

i.e. by interchanging the two subspaces and taking the transpose of the original R-matrix in one of them. The interchange of subspaces here is necessary because $\overset{12}{R}\neq\overset{21}{R}$. It is easy to check that the above definition yields the correct first order term displayed in (5.3). Furthermore, although transposing the indices is a non-covariant operation, it turns out that all summations in (5.4) are again covariant, such that with a little algebra these relations can be reduced to the original QYBE (3.4).

We emphasize that the shift c (alias the central charge) in (5.2) is compatible with all of our requirements so far and therefore still arbitrary at this point. It is important here that the algebras for different c are not isomorphic; in particular, they may have different ideals. The central charge c will be fixed later by requiring symmetry of the quantum monodromy matrix. Note that a possible additional shift in the argument of Q in (5.5) has been absorbed into a redefinition of T_-.

As for the singular points, there is an important difference between eqs. (5.1) and (5.2): whereas the poles on the l.h.s. and r.h.s. of (5.1) always match, this is not so for (5.2) due to the shift. Thus, either some of the mixed operator products are singular, or the regularity on one side imposes the vanishing of certain residues on the other side. These questions as well as the proper quantum analogue of the classical holomorphy properties of the $T_\pm$'s may however only be addressed after specializing to a particular representation of (5.1), (5.2). To be on the safe side here, we will use the exchange relations only at the generic points where the R-matrices are nonsingular.

In addition to these exchange relations we demand that the quantum determinant for both T_+ and T_- be equal to unity, viz. (3.28)

$$T_\pm(w-15\mathrm{i}\hbar)_{\hat{a}}{}^{\hat{c}}\,T_\pm(w)_{\hat{b}}{}^{\hat{d}}\,\eta_{\hat{c}\hat{d}} = \eta_{\hat{a}\hat{b}}\,. \tag{5.6}$$

Due to (3.13), Q in (5.5) is normalized such that the l.h.s. of this equation indeed lies in the center of the full algebra (5.1), (5.2). The hermiticity (3.18) of the R-matrix shows that the full quantum algebra is compatible with the following $*$-structure — suggested by the classical relation (4.7) —

$$\left(T_{\pm}(w)_{\hat{a}}{}^{\hat{b}}\right)^{*} = T_{\mp}(w^{*})_{\hat{a}}{}^{\hat{b}}\,, \tag{5.7}$$

for the purely imaginary choice of the parameter α we have made.[4]

Let us show that the algebra (5.1)–(5.7) has the correct classical limit for $\hbar \to 0$. If we embed the original non-local charges $U_{\pm}(w)$ by identifying them with the upper left 248×248 block of $T_{\pm}(w)$, the exchange relations (5.1), (5.2) reduce to the Poisson brackets (4.9) in the limit $\hbar \to 0$. The $U_{\pm}(w)$ being E_8-valued matrices, the condition (5.6) can then be viewed as the quantum analog of the statement that any element of $\mathrm{E}_{8(8)}$ also belongs to $\mathrm{SO}(128,120)$. While this submatrix of $T_{\pm}$ is evidently the quantum analog of the classical charges, one may wonder about the significance of the extra components $T_0{}^a(w), T_a{}^0(w)$ and the singlet $T_0{}^0(w)$. The exchange relations may be read in such a way, that the off-diagonal components can be solved to become functions of the 248 degrees of freedom originally present. In order to make the dependence explicit, we evaluate the defining relation (3.25) at the remaining poles $u - v = w_j$ $(j = 1, \ldots, 3)$ (the residue at w_4 has already been exploited to derive (3.28)):

$$\overset{12}{\mathcal{R}}_j \overset{1}{T}(u) \overset{2}{T}(u-w_j) = \overset{2}{T}(u-w_j) \overset{1}{T}(u) \overset{12}{\mathcal{R}}_j \;.$$

After expansion around $u = \infty$ this equation can be solved order by order to get expressions for the off-diagonal components $(T_{(n)})_0{}^a$ and $(T_{(n)})_a{}^0$, respectively. In first order this yields

$$\overset{12}{\mathcal{P}}_j \left(\overset{1}{T}_{(1)} + \overset{2}{T}_{(1)}\right) = \left(\overset{2}{T}_{(1)} + \overset{1}{T}_{(1)}\right) \overset{12}{\mathcal{P}}_j \,,$$

for all projectors from (2.15) and (3.8). Hence,

$$(T_{(1)})_a{}^b \in \mathfrak{e}_8\,, \qquad \text{and} \qquad (T_{(1)})_0{}^a = (T_{(1)})_a{}^0 = (T_{(1)})_0{}^0 = 0\,. \tag{5.8}$$

In second order we get the equations

$$\begin{aligned} \overset{12}{\mathcal{R}}_j \left(\overset{1}{T}_{(2)} - \tfrac{1}{2}(\overset{1}{T}_{(1)})^2 + \overset{2}{T}_{(2)} - \tfrac{1}{2}(\overset{2}{T}_{(1)})^2 + w_j \overset{2}{T}_{(1)} + \tfrac{1}{2}\left[\overset{1}{T}_{(1)}, \overset{2}{T}_{(1)}\right]\right) = \\ = \left(\overset{1}{T}_{(2)} - \tfrac{1}{2}(\overset{1}{T}_{(1)})^2 + \overset{2}{T}_{(2)} - \tfrac{1}{2}(\overset{2}{T}_{(1)})^2 + w_j \overset{2}{T}_{(1)} - \tfrac{1}{2}\left[\overset{1}{T}_{(1)}, \overset{2}{T}_{(1)}\right]\right) \overset{12}{\mathcal{R}}_j \;. \end{aligned}$$

[4]It is helpful to note that like for $\mathfrak{g} = \mathfrak{sl}_2$ [13] the algebra (5.1), (5.2) may in fact be mapped to the usual (untwisted) centrally extended Yangian double $\mathcal{D}Y(\mathfrak{e}_8)_c$ by the (noncovariant) map

$$T_{+}(w)_{\hat{a}}{}^{\hat{b}} \mapsto T_{+}(w)_{\hat{a}}{}^{\hat{c}}\eta_{\hat{c}\hat{b}}\,, \qquad T_{-}(w)_{\hat{a}}{}^{\hat{b}} \mapsto T_{-}(w)_{\hat{a}}{}^{\hat{b}}\,.$$

The additional relation (3.16) is required to show that this is indeed an automorphism of (5.1). With respect to (5.7) this map is, however, no $*$-isomorphism; the representation theory of (5.1), (5.2) will thus differ considerably from the one of $\mathcal{D}Y(\mathfrak{e}_8)_c$. (Needless to say that even the latter is far from being developed.)

Together with (3.30), (3.28) and the explicit form of the residues of the R-matrix in (3.12) these relations can be used to deduce

$$
\begin{aligned}
&\Big(T_{(2)} - \frac{1}{2} T_{(1)} T_{(1)}\Big)_a{}^b \;\in\; \mathfrak{e}_8\,, \\
&(T_{(2)})_0{}^a = \frac{\sqrt{2}}{\alpha}\, \mathrm{i}\hbar f^{ab}{}_c\, (T_{(1)})_c{}^b\,, \qquad (T_{(2)})_0{}^0 = 0\,, \qquad \text{etc.}
\end{aligned}
\tag{5.9}
$$

In this fashion one may in principle determine the components $T(w)_0{}^a, T(w)_a{}^0$ in all orders as functions of the $T(w)_a{}^b$ which vanish in the classical limit $\hbar \to 0$. Thus, we have consistently

$$
T_{\pm}(w)_{\hat{a}}{}^{\hat{b}} \;\xrightarrow[\hbar\to 0]{}\; \left(\begin{array}{c|c} U_{\pm}(w)_a{}^b & 0 \\ \hline 0 & 1 \end{array}\right). \tag{5.10}
$$

Recall now that the classical phase space was parametrized by the symmetric E_8-valued matrix $\mathcal{M}_{ab}(w)$. On the quantum side we define this object in analogy to (4.6) as

$$
\mathcal{M}_{\hat{a}\hat{b}}(w) \equiv T_+(w)_{\hat{a}}{}^{\hat{c}}\, T_-(w)_{\hat{b}}{}^{\hat{c}}\,, \tag{5.11}
$$

where the operator ordering on the r.h.s. is fixed by this relation. The matrix entries of $\mathcal{M}$ are distinguished elements in the algebra (5.1), (5.2) in that they satify the exchange relations

$$
\begin{aligned}
T_+(v)_{\hat{c}}{}^{\hat{d}}\, \mathcal{M}_{\hat{a}\hat{b}}(w) &= R_{\hat{a}\hat{c}}{}^{\hat{p}\hat{k}}(w-v) \mathcal{M}_{\hat{p}\hat{q}}(w) R_{\hat{b}\hat{k}}{}^{\hat{q}\hat{l}}(w-v-\mathrm{i}\hbar c)\, T_+(v)_{\hat{l}}{}^{\hat{d}}\,, \\
T_-(v)_{\hat{c}}{}^{\hat{d}}\, \mathcal{M}_{\hat{a}\hat{b}}(w) &= R_{\hat{a}\hat{c}}{}^{\hat{p}\hat{k}}(w-v+\mathrm{i}\hbar c) \mathcal{M}_{\hat{p}\hat{q}}(w) R_{\hat{b}\hat{k}}{}^{\hat{q}\hat{l}}(w-v)\, T_-(v)_{\hat{l}}{}^{\hat{d}}\,,
\end{aligned}
\tag{5.12}
$$

as well as the closed algebra

$$
\begin{aligned}
&R_{\hat{a}\hat{b}}{}^{\hat{m}\hat{n}}(v-w)\; \mathcal{M}_{\hat{m}\hat{k}}(v)\; R_{\hat{c}\hat{n}}{}^{\hat{k}\hat{l}}(v-w-\mathrm{i}\hbar c)\; \mathcal{M}_{\hat{l}\hat{d}}(w) = \\
&\qquad\qquad = \mathcal{M}_{\hat{b}\hat{m}}(w)\; R_{\hat{k}\hat{a}}{}^{\hat{m}\hat{n}}(w-v-\mathrm{i}\hbar c)\; \mathcal{M}_{\hat{n}\hat{l}}(v)\; R_{\hat{d}\hat{c}}{}^{\hat{k}\hat{l}}(w-v)\,,
\end{aligned}
\tag{5.13}
$$

which we hence view as the quantized version of (4.8).[5]

While the classical matrix $\mathcal{M}$ was manifestly symmetric (cf. (4.3), (4.4)) this is not necessarily true for its quantum analog. Rather, we must now impose some quantum version of this condition in order to ensure that the number of degrees of freedom in the quantum structure matches the classical phase space. In other words, we still have to implement the quantum analogue of the classical coset structure. In algebraic language this amounts to dividing out another ideal from (5.1)–(5.2). Unlike the quantum determinant condition (5.6) (which we still assume to hold), this new condition will involve T_+ and T_- simultaneously.

[5]Like its classical counterpart (4.8) the algebra (5.13) belongs to the general class of quadratic algebras which has been considered in [21].

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To this end we return to the exchange algebra (5.1)–(5.2) with arbitrary, but fixed central charge c, and consider the set of elements

$$\phi_{\hat a\hat b}(w) \;\equiv\; \begin{cases} \mathrm{Res}\Big|_{v=\mathrm{i}\hbar c} R_{\hat a\hat b}{}^{\hat c\hat d}(v)\;\mathcal{M}_{\hat c\hat d}(w) & \text{if } R(\mathrm{i}\hbar c) \text{ is singular} \\ R_{\hat a\hat b}{}^{\hat c\hat d}(\mathrm{i}\hbar c)\mathcal{M}_{\hat c\hat d}(w) & \text{else}\,. \end{cases} \tag{5.14}$$

Use of the exchange relations (5.12) then yields in a first step

$$T_+(u)_{\hat c}{}^{\hat d}\;\phi_{\hat a\hat b}(w) = R_{\hat a\hat b}{}^{\hat p\hat q}(\mathrm{i}\hbar c)R_{\hat p\hat c}{}^{\hat m\hat k}(w-u)R_{\hat q\hat k}{}^{\hat n\hat l}(w-u-\mathrm{i}\hbar c)\;\mathcal{M}_{\hat m\hat n}(w)\,T_+(u)_{\hat l}{}^{\hat d}\,.$$

The necessity of the choice $v = \mathrm{i}\hbar c$ in (5.14) becomes evident at this point: it is the only value of the argument of the R-matrix in (5.14) for which we can exploit the QYBE to re-arrange the indices (this would not be possible if the first factor on the r.h.s. were $R_{\hat a\hat b}{}^{\hat p\hat q}(v)$ with arbitrary argument v). Thus, by use of (3.4) we finally obtain

$$\begin{aligned} &= R_{\hat b\hat c}{}^{\hat q\hat k}(w-u-\mathrm{i}\hbar c)R_{\hat a\hat k}{}^{\hat p\hat l}(w-u)R_{\hat p\hat q}{}^{\hat m\hat n}(\mathrm{i}\hbar c)\;\mathcal{M}_{\hat m\hat n}(w)\,T_+(u)_{\hat l}{}^{\hat d} \\ &= R_{\hat b\hat c}{}^{\hat q\hat k}(w-u-\mathrm{i}\hbar c)R_{\hat a\hat k}{}^{\hat p\hat l}(w-u)\;\phi_{\hat p\hat q}(w)\,T_+(u)_{\hat l}{}^{\hat d}\,. \end{aligned} \tag{5.15}$$

A similar calculation for $T_-(u)$ gives the same result. We conclude that the elements $\phi_{\hat a\hat b}(w)$ constitute the basis of a two-sided ideal of (5.1)–(5.2). Obviously, these ideals are nontrivial only if (5.14) does not contain $\mathcal{M}$ entirely, i.e. only if $R(\mathrm{i}\hbar c)$ or the relevant residue is singular or non-invertible. This happens only at the special values $c = \pm w_j$.[6]

There is (of course) a more group-theoretical interpretation of this construction: in view of (3.32) and (3.33), the exchange relations (5.12) express the fact, that under the adjoint action of $T_+(v)$ and $T_-(v)$, respectively, the matrix $\mathcal{M}(w)$ transforms in the tensor product $\mathbf{249}_w \otimes \mathbf{249}_{w-\mathrm{i}\hbar c}$. I.e. the existence of nontrivial ideals in $\mathcal{M}$ amounts to the reducibility of the tensor product $\mathbf{249}_{\mathrm{i}\hbar c} \otimes \mathbf{249}$ which is in correspondence with the singular points of the associated R-matrix [15] as we have explicitly seen here.

Returning to the problem of identifying the proper quantum analogue of the symmetry of $\mathcal{M}$, let us now examine (5.14) for all critical choices of the central extension c with the desired conditions (4.3), (4.4). With the explicit form of (3.12) one confirms that there is a unique value of c such that the algebra (5.1)–(5.2) has an ideal which in the classical limit indeed reduces to (4.3) and (4.4). The correct choice is

$$c = 1\,. \tag{5.16}$$

[6]Together with the fact (3.22) that $\mathcal{R}_4$ is proportional to a one-dimensional projector, (5.15) in particular shows the well-known infinite-dimensional enlargement of the center of the algebra (5.1)–(5.2) at the critical level $c = 15$ [18]. However, to achieve consistency with the classical coset structure $\mathrm{E}_8/\mathrm{SO}(16)$ we need another value of the central extension here.

Dividing out the ideal corresponding to (5.14) now amounts to imposing the additional set of relations $\phi_{\hat{a}\hat{b}} = 0$, or

$$(\mathcal{R}_1)_{\hat{a}\hat{b}}{}^{\hat{c}\hat{d}} \mathcal{M}_{\hat{c}\hat{d}}(w) = 0\,. \tag{5.17}$$

The ensuing relations can be written more succinctly by splitting $\mathcal{R}_1$ into the E_8-invariant projectors:

$$\begin{aligned}
(\mathcal{P}_{248})_{ab}{}^{cd}\mathcal{M}_{cd} &= -\frac{\alpha}{8}\, f_{ab}{}^{c}(\mathcal{M}_{0c} + \mathcal{M}_{c0})\,, \\
(\mathcal{P}_{3875})_{ab}{}^{cd}\mathcal{M}_{cd} &= 0\,, \\
(\mathcal{P}_{30380})_{ab}{}^{cd}\mathcal{M}_{cd} &= 0\,, \\
\mathcal{M}_{0c} &= \mathcal{M}_{c0}\,, \\
\mathcal{M}_{00} &= -\frac{1}{480\alpha^2}\,\eta^{ab}\mathcal{M}_{ab}\,.
\end{aligned} \tag{5.18}$$

Since the off-diagonal components $\mathcal{M}_{0a}$ are of order $\mathcal{O}(\hbar)$ by (5.10), it now follows with our choice $60\alpha^2 = -1$ that the relations (5.18) indeed encompass the classical coset relations (4.3) and (4.4) in the limit $\hbar \to 0$.

In conclusion, the quantum algebra which replaces the classical Poisson algebra (4.9) is given by (5.1)–(5.7) with central extension $c = 1$ divided by the ideal which is generated by (5.18). The operator (5.11) consistently represents the quantum analogue of the classical matrix $\mathcal{M}(w)$ related to the physical scalar fields on a certain axis in space-time. Matrix elements of (5.11) in particular representations should thus carry the information about quantum spectra and fluctuations of the original fields. These issues as well as the general representation theory of the twisted Yangian doubles remain to be investigated.

Acknowledgments

We thank D. Bernard, B. Julia, J.M. Maillet and M. Niedermaier for helpful comments and conversations. This work was supported by EU contract ERBFMRX-CT96-0012.

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