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On Hidden Symmetries in $d = 11$ Supergravity and Beyond[1]

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This talk is about hidden symmetries in eleven dimensions, but it is equally a tribute to a scientist and friend, who is eminently visible in four space-time dimensions: François Englert, in whose honor this meeting is being held. Therefore, before entering *dans le vif du sujet* I would like to express my gratitude for having had the opportunity and privilege to learn from him and to work with him, and for all the fun we have had — involving, amongst other things, dinosaurs within dinosaurs [1] (the ancestor of all modern inflationary theories) and their eleven-dimensional avatars [2], as well as higher states of consciousness [3] and monster strings [4].

These days, many of us who have not yet attained the wisdom that comes with an *émeritat*, but who share François' enthusiasm

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for all of physics' mysteries, are participants in the hunt for a still elusive theory, called "M Theory", which is to unify all known consistent string theories and to relate them through a web of non-perturbative dualities [5, 6]. This theory would also accommodate $d = 11$ supergravity [7] as a strong coupling limit via the relations

$$R_{11} = \ell_s g_s \qquad \ell_P^3 = \ell_s^3 g_s$$

where ℓ_P is the $d = 11$ Planck length, ℓ_s the string length, g_s the string coupling constant, and R_{11} the radius of the circle on which $d = 11$ supergravity is compactified to ten dimensions (the limit is taken in such a way that ℓ_P stays finite while $g_s \to \infty$ and $\ell_s \to 0$, hence $R_{11} \to \infty$). It is clear from these relations that present knowledge covers only the "boundary" of M Theory (where either the massive string modes or the $d = 11$ Kaluza Klein modes are sent to infinity), but tells us almost nothing about its "bulk" — the true domain of quantum gravity. Still, we can probably anticipate that it will be a pregeometrical theory in the sense that space time as we know it will emerge as a derived concept and that it should possess a huge symmetry group involving new types of Lie algebras (such as hyperbolic Kac Moody algebras) and perhaps even more general structures such as quantum groups.

According to the currently most popular proposal, M Theory "is" the $N \to \infty$ limit of the maximally supersymmetric quantum mechanical $SU(N)$ matrix model (see e.g. [8] for reviews and many references). This model had already appeared in an earlier study of the $d = 11$ supermembrane in a flat background in the light cone gauge, and for any finite N, it can alternatively be obtained by dimensional reduction of the maximally extended supersymmetric Yang Mills theory in $d = 10$ with gauge group $SU(N)$ to one (time) dimension. However, while matrix theory is pregeometrical in the sense that the target space coordinates are replaced by matrices, thus implying a kind of non-commutative geometry, the symmetries of dimensionally reduced supergravities that we are concerned with here, are hard to come by.

In this contribution, I will briefly describe some recent work done in collaboration with S. Melosch [9], and with K. Koepsell and H. Samtleben [10], which was motivated by recent advances in string theory (as well as the possible existence of an Ashtekar-type canonical formulation of $d = 11$ supergravity). Although, at first sight, this work may seem to have little to do with the issues raised above, it could actually be relevant in the context of M Theory, assuming (as we do) that further progress will crucially depend on the identification of the underlying symmetries, and that the hidden exceptional symmetries of maximal supergravity theories discovered long ago [11, 12] may provide important clues as to where we should be looking. Support for this strategy derives from the fact that some local symmetries of the dimensionally reduced theories can be "lifted" back to eleven dimensions. More precisely, it was shown in [13, 14] that there exist new versions of $d = 11$ supergravity with local $SO(1,3) \times SU(8)$ and $SO(1,2) \times SO(16)$ tangent space symmetry, respectively. In both versions the supersymmetry variations acquire a polynomial form from which the corresponding formulas for the maximal supergravities in four and three dimensions can be read off directly and without the need for complicated duality redefinitions. This reformulation can thus be regarded as a step towards the complete fusion of the bosonic degrees of freedom of $d = 11$ supergravity (i.e. the elfbein and the antisymmetric tensor A_{MNP}) in a way which is in harmony with the hidden symmetries of the dimensionally reduced theories.

The existence of alternative versions of $d = 11$ supergravity, which, though equivalent on-shell to the original version of [7], differ from it off-shell, strongly suggests the existence of a novel kind of "exceptional geometry" for $d = 11$ supergravity, and thus the bigger theory containing it. This new geometry would be intimately tied to the special properties of the exceptional groups, and would be characterized by relations which have no analog in ordinary Riemannian geometry. Much of the ongoing work centers on the role of extended objects (such as 2- and 5-branes in eleven dimensions), which couple to the antisymmetric tensor fields present in $d = 11$ and $d = 10$ supergravities. Since these antisymmetric tensors are here "dualized away", our formulation might open new vistas on a unified description of the basic "objects" of M Theory.

We will here concentrate on the $SO(1,2) \times SO(16)$ invariant version of $d = 11$ supergravity [14]. To derive it from the original formulation of $d = 11$ supergravity, one first breaks the original tangent space symmetry SO(1,10) to its subgroup $SO(1,2) \times SO(8)$ through a partial choice of gauge for the elfbein, and subsequently enlarges it again to $SO(1,2) \times SO(16)$ by introducing new gauge degrees of freedom. The construction thus requires a 3+8 split of the $d = 11$ coordinates and indices, implying a similar split for all tensors of the theory. The symmetry enhancement of the transverse (helicity) group $SO(9) \subset SO(1,10)$ to $SO(16)$ requires suitable redefinitions of the bosonic and fermionic fields, or, more succinctly, their combination into tensors w.r.t. the new tangent space symmetry. It is important, however, that the dependence on all eleven coordinates is retained throughout.

In the bosonic sector, the elfbein and the three-index photon are combined into new objects covariant w.r.t. to $d = 3$ coordinate reparametrizations and the new tangent space symmetry $SO(1,2) \times SO(16)$ (similar redefinitions must be made for the fermionic fields, but we will not give explicit formulas here for lack of space). In a special Lorentz gauge the elfbein takes the form

$$E_M{}^A = \begin{pmatrix} \Delta^{-1} e_\mu{}^a & B_\mu{}^m e_m{}^a \\ 0 & e_m{}^a \end{pmatrix}$$

where curved $d = 11$ indices are decomposed as $M = (\mu, m)$ with $\mu = 0, 1, 2$ and $m = 3, ..., 10$ (with a similar decomposition of the flat indices), and $\Delta := \det e_m{}^a$. It thus contains the (Weyl rescaled) dreibein and the Kaluza Klein vectors $B_\mu{}^m$, all of which are left untouched. By contrast, we will trade the internal achtbein $e_m{}^a$ for a rectangular 248-bein $e^m_{\mathcal{A}} \equiv (e^m_{IJ}, e^m_A)$ containing the remaining "matter-like" degrees of freedom, where the index $\mathcal{A} \equiv ([IJ], A)$ labels the 248-dimensional adjoint representation of $E_{8(8)}$ in the $SO(16)$ decomposition $\mathbf{248} \to \mathbf{120} \oplus \mathbf{128}$. This 248-bein, which in the reduction to three dimensions contains all the propagating bosonic matter degrees of freedom of $d = 3, N = 16$ supergravity, is defined in a special $SO(16)$ gauge by

$$(e^m_{IJ}, e^m_A) := \begin{cases} \Delta^{-1} e_a{}^m \Gamma^a_{\alpha\dot\beta} & \text{if } [IJ] \text{ or } A = (\alpha\dot\beta) \\ 0 & \text{otherwise} \end{cases}$$

where the $SO(16)$ indices IJ or A are decomposed w.r.t. the diagonal subgroup $SO(8) \equiv (SO(8) \times SO(8))_{diag}$ of $SO(16)$ (see [14] for details). Being the inverse densitized internal achtbein contracted with an SO(8) Γ-matrix, this object is similar to the inverse densitized triad in Ashtekar's reformulation of Einstein's theory [15].

In addition we need composite fields $\mathcal{Q}^{\mathcal{A}}_\mu \equiv (Q^{IJ}_\mu, P^A_\mu)$ and $\mathcal{Q}^{\mathcal{A}}_m \equiv (Q^{IJ}_m, P^A_m)$, which together make up an $E_{8(8)}$ connection again *in eleven dimensions*. Their explicit expressions in terms of the $d = 11$ coefficients of anholonomity and the four-index field strength F_{MNPQ} are, however, too lengthy to reproduce here [14].

The new geometry is encoded into constraints between the vielbein components, which rely in an essential way on special properties of the exceptional group $E_{8(8)}$. With the $E_{8(8)}$ indices $\mathcal{A}, \mathcal{B}, \ldots (= 1, \ldots, 248)$, we have

$$(\mathcal{P}_j)_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}\mathcal{D}} e^m_{\mathcal{C}} e^n_{\mathcal{D}} = 0$$

where $\mathcal{P}_j$ are the projectors onto the $j = \mathbf{1}$, $\mathbf{248}$ and $\mathbf{3875}$ representations of $E_{8(8)}$. (Note that the projectors onto the $j = \mathbf{27000}$ and $\mathbf{30380}$ representations do *not* vanish.) In addition, the 248-bein and the new connection fields are subject to a "vielbein postulate" similar to the usual vielbein postulate, which states the covariant constancy of the 248-bein w.r.t. to an $E_{8(8)}$ covariant derivative involving the $E_{8(8)}$ connection $\mathcal{Q}^{\mathcal{A}}_M$. For instance, for $M = m$ we have

$$\partial_m e^n_{\mathcal{A}} + f_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}} \mathcal{Q}^{\mathcal{B}}_m e^n_{\mathcal{C}} = 0$$

where $f^{\mathcal{A}\mathcal{B}\mathcal{C}}$ are the $E_{8(8)}$ structure constants. (The relations with $M = \mu$ involve the Kaluza Klein vectors $B_\mu{}^m$ and are slightly more complicated). The supersymmetry variations of $d = 11$ supergravity can now be re-expressed entirely in terms of these new variables and their fermionic partners [14, 9].

Despite the "$E_{8(8)}$ covariance" of these relations, it must be stressed, however, that the full theory does not respect $E_{8(8)}$ invariance, as is already obvious from the fact that the fermions do not fit into representations of $E_{8(8)}$. However, the algebraic relations given above can be exploited to show [10] that there exists an $E_{8(8)}$ matrix $\mathcal{V}$ *in eleven dimensions* such that

$$e^m_{\mathcal{A}} = \frac{1}{60} \mathrm{Tr}\Big(Z^m \mathcal{V} X_{\mathcal{A}} \mathcal{V}^{-1} \Big)$$

where the $X_{\mathcal{A}}$ are the generators of $E_{8(8)}$, and the Z^m span an eight-dimensional nilpotent subalgebra of $E_{8(8)}$ (there are altogether

$36 = 8 + 28$ such nilpotent generators, whose role in relating the various dualized forms of dimensionally reduced supergravity has been explained in [16]). Because the fundamental and the adjoint representations of $E_{8(8)}$ are the same, we have $\mathcal{V} X_{\mathcal{A}} \mathcal{V}^{-1} = X_{\mathcal{B}} \mathcal{V}^{\mathcal{B}}{}_{\mathcal{A}}$ and can thus rewrite this relation in the form

$$e^m_{\mathcal{A}} = \mathcal{V}^m{}_{\mathcal{A}}$$

This means that the (inverse densitized) achtbein, which itself is part of the elfbein of $d = 11$ supergravity, has become part of an $E_{8(8)}$ matrix $\mathcal{V}$ in eleven dimensions! Furthermore, it then follows from the generalized vielbein postulate stated above that the $M = m$ part of the $E_{8(8)}$ connection $\mathcal{Q}^{\mathcal{A}}_M$ can be simply expressed in terms of this matrix via

$$\mathcal{Q}_m = \mathcal{V}^{-1} \partial_m \mathcal{V}$$

This simple formula, however, does not work for the low dimensional components $\mathcal{Q}^{\mathcal{A}}_\mu$.

The results obtained so far suggest further extensions incorporating infinite dimensional symmetries. More specifically, the fact that the construction outlined above works with a 4+7 and 3+8 split of the indices suggests that we should be able to construct versions of $d = 11$ supergravity with infinite dimensional tangent space symmetries, which would be based on a 2+9 or even a 1+10 split of the indices. This would also be desirable in view of the fact that the new versions are "simple" only in their internal sectors, as put in evidence by the above formula for $\mathcal{Q}^{\mathcal{A}}_m$. The general strategy would thus be to further enlarge the internal sector by absorbing more and more degrees of freedom into it, such that in the final step, only an einbein would be left in the low dimensional sector. However, it is also clear that the elaboration of these ideas will not be an easy task. After all, it took a considerable effort extending over many years to show that the general pattern continues when one descends to $d = 2$ and that the hidden symmetries become infinite dimensional, generalizing the Geroch group of general relativity [17].

There is some reason to believe that a generalization along these lines will take us beyond $d = 11$ supergravity. The fundamental object of the theory could then turn out to be an infinite generalization of the vierbein of general relativity, which would be acted upon from one side by a vast extension of the Lorentz group, containing not only space-time, but also internal symmetries, and perhaps even local supersymmetries. For the left action, one would have to appeal to some kind of generalized covariance principle, which would involve the E_{11-d} symmetries.

To put these ideas into perspective, let us recall some facts about dimensionally reduced maximal supergravity to two dimensions. Following the empirical rules of dimensional reduction one is led to predict $E_9 = E_8^{(1)}$ as a symmetry for the dimensional reduction of $d = 11$ supergravity to two dimensions [12, 18]. This expectation is borne out by the existence of a linear system for maximal $N = 16$ supergravity in two dimensions [19] (see [20] for the bosonic theory, and [21] for a more recent summary). As is usually the case for integrable systems, the linear system requires the introduction of an extra spectral parameter t, and the extension of the σ-model matrix $\mathcal{V}(x)$ to a matrix $\widehat{\mathcal{V}}(x;t)$ depending on this extra parameter t. An unusual feature is that, due to the presence of gravitational degrees of freedom, this parameter becomes coordinate dependent, i.e. we have $t = t(x;w)$, where w is an integration constant, sometimes referred to as the "constant spectral parameter" whereas t itself is called the "variable spectral parameter".

The (finite dimensional) coset structure of the higher dimensional theories has a natural continuation in two dimensions, with the only difference that the symmetry groups are infinite dimensional. This property is manifest from the transformation properties of the linear system matrix $\widehat{\mathcal{V}}$, with a global affine symmetry acting from the left, and a local symmetry corresponding to some "maximal compact" subgroup acting from the right:

$$\widehat{\mathcal{V}}(x;t) \longrightarrow g(w)\widehat{\mathcal{V}}(x;t)h(x;t)$$

Here $g(w) \in E_{9(9)}$ with affine parameter w, and the subgroup to which $h(x;t)$ belongs is defined as follows [18, 20, 21]. Let τ be the involution characterizing the coset space $E_{8(8)}/SO(16)$: then $h(t) \in SO(16)^\infty$ is defined to consist of all τ^∞ invariant elements of $E_{9(9)}$, where the extended involution τ^∞ is defined by $\tau^\infty(h(t)) := \tau h(\varepsilon t^{-1})$, with $\varepsilon = +1$ (or -1) for a Lorentzian (Euclidean) worldsheet. Observe that $SO(16)^\infty$ is different from the affine extension of $SO(16)$ for either choice of sign.

Introducing a suitable triangular gauge and taking into account the compensating $SO(16)^\infty$ transformations to re-establish the chosen gauge where necessary, one finds that these symmetries are realized in a non-linear and non-local fashion on the basic physical fields. Moreover, they act as duality transformations in the sense that they mix scalar fields with their duals. At the linear level, a scalar field φ and its dual $\tilde{\varphi}$ in two dimensions are related by

$$\partial_\mu \tilde{\varphi} = \epsilon_{\mu\nu} \partial^\nu \varphi$$

If we were just dealing with free fields (as in conformal field theory), there would not be much more to duality than this simple equation, since a second dualization obviously brings us back to the original field (up to an integration constant). The crucial difference here is that, as a consequence of the non-linearity of the field equations, there are *infinitely many* dual potentials because each dualization now produces a new (i.e. higher order) dual potential. It is basically this non-linearity inherited from Einstein's equations which explains why the group of duality transformations becomes infinite dimensional in two dimensions. Remarkably, however, already the free field relation above (with φ replaced by any target space coordinate) is central to modern string duality — for instance implying the emergence of D(irichlet) branes through the interchange of Neumann and Dirichlet boundary conditions for open strings [22]. It is furthermore well known that the integration constant arising in the dualization of a compactified string target space coordinate is associated with string winding modes, and that duality interchanges Kaluza Klein and winding modes. Since we here get infinitely many such integration constants (i.e. one for every dualization), we are led to predict the existence of an infinite tower of novel "winding modes" over and above the ones seen so far seen in string theory. These could be related to the mysterious states found in [23] that cannot be accounted for by the standard counting arguments.

By representing the "moduli space of solutions" $\mathcal{M}$ of the bosonic equations of motion of $d = 11$ supergravity with nine commuting space-like Killing vectors as

$$\mathcal{M} = \frac{\text{solutions of field equations}}{\text{diffeomorphisms}} = \frac{E_{9(9)}}{SO(16)^\infty}$$

one has managed to endow this space, which a priori is very complicated, with a group theoretic structure that makes it much easier to handle. In particular, the integrability of the system is directly linked to the fact that $\mathcal{M}$ possesses an infinite dimensional "isometry group" $E_{9(9)}$. The introduction of infinitely many gauge degrees of freedom embodied in the subgroup $SO(16)^\infty$ linearizes and localizes the action of this isometry group on the space of solutions. Of course, in making such statements, one should keep in mind that a mathematically rigorous construction of such spaces is a thorny problem. We can ignore these subleties here, not least because these spaces ultimately will have to be "quantized" anyway.

Elevating the local symmetries of maximal supergravity in two dimensions to eleven dimensions would thus require the existence of yet another extension of the theory, for which the Lorentz group $SO(1,10)$ is replaced by $SO(1,1) \times SO(16)^{\infty}$ (the subgroup $SO(16)^{\infty}$ can be interpreted as an extension of the transverse group $SO(9)$ in eleven dimensions). Accordingly, we would now decompose the elfbein into a zweibein and nine Kaluza Klein vectors $B_{\mu}{}^{m}$ (with $m = 2, ..., 10$). The remaining internal neunbein would have to be replaced by an "Unendlichbein" (or "∞-bein", for short) $e^{m}_{\mathcal{A}}(x;t)$. The parameter t is necessary in order to parametrize the infinite dimensional extension of the symmetry group; whether it would still be a "spectral parameter" in the conventional sense of the word for the "lifted" theory, remains to be seen. One important difference with the dimensionally reduced theory is, however, clear: in eleven dimensions, there is no anolog of the dualization mechanism, which would ensure that despite the existence of infinitely many dual potentials, there are only finitely many physical degrees of freedom. This means that the construction will almost certainly take us beyond $d = 11$ supergravity.

Some information can be deduced from the requirement that in the dimensional reduction to $d = 2$, there should exist a formula relating $e^{m}_{\mathcal{A}}(x;t)$ to the linear system matrix $\widehat{\mathcal{V}}(x;t)$, analogous to the one relating $e^{m}_{\mathcal{A}}(x)$ to the $E_{8(8)}$ matrix $\mathcal{V}(x)$ before. For this purpose, we would need a ninth nilpotent generator to complement the Z^{m}'s; an obvious candidate is the central charge generator c, since it obeys $\langle c|c\rangle = \langle c|Z^{m}\rangle = 0$ for all $m = 3, ..., 10$. The parameter t, introduced somewhat ad hoc for the parametrization of the ∞-bein, must coincide in the dimensional reduction with the spectral parameter of the $d = 2$ theory. Furthermore, the generalized "∞-bein postulate" should reduce to the linear system of $d = 2$ supergravity in this reduction.

One difference with the previous situation, where the tangent space symmetry was still finite, is that the Lie algebra of $SO(16)^\infty$ also involves the non-compact $E_{8(8)}$ generators, but in such a way that the generalized Cartan Killing form on $E_{9(9)}$ is still positive on all these generators. This follows from consideration of the t dependence of the linear system of the dimensionally reduced theory and shows that the new connections would constitute an $SO(16)^\infty$ rather than an $E_{9(9)}$ gauge connection. This means that the covariantizations in the generalized vielbein postulate would be in precise correpondence with the local symmetries, in contrast with the previous relations which looked $E_{8(8)}$ covariant, whereas the full theory was actually invariant only under $SO(16)$. Another curious feature is the following: in two dimensions, the linear system matrix contains all degrees of freedom, including the fermionic ones, and the local $N = 16$ supersymmetry can be bosonized into a local $SO(16)^\infty$ gauge transformation [24]. This could mean that there is a bosonization of fermions in the sense that $e^m_{\mathcal{A}}(x;t)$ would describe bosonic and fermionic degress of freedom.

What has been said here could be summarized as follows: in searching for a possible candidate M Theory, one should not only concentrate on dimensionally reduced maximally extended *rigidly* supersymmetric theories (= supersymmetric Yang Mills theories), but also consider the dimensionally reduced maximally extended *locally* supersymmetric theory. The idea (already proposed in [19]) is that a third quantized version of maximal supergravity in two dimensions would give rise via a kind of bootstrap to a theory beyond $d = 11$ supergravity that would contain the latter in the same way as superstring theories contain $d = 10$ supergravity and $d = 10$ super-Yang-Mills theories as special limits. However, it is not clear how (and if) this idea fits with presently accepted points of view.

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