

# Gravitational waves from the $r$ -modes of rapidly rotating neutron stars

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**Abstract.** Since the last Amaldi meeting in 1997 we have learned that the  $r$ -modes of rapidly rotating neutron stars are unstable to gravitational radiation reaction in astrophysically realistic conditions. Newborn neutron stars rotating more rapidly than about 100 Hz may spin down to that frequency during up to one year after the supernova that gives them birth, emitting gravitational waves which might be detectable by the enhanced LIGO interferometers at a distance which includes several supernovae per year. A cosmological background of these events may be detectable by advanced LIGO. The spins (about 300 Hz) of neutron stars in low-mass x-ray binaries may also be due to the  $r$ -mode instability (under different conditions), and some of these systems in our galaxy may also produce detectable gravitational waves—see the review by G. Ushomirsky in this volume. Much work is in progress on developing our understanding of  $r$ -mode astrophysics to refine the early, optimistic estimates of the detectability of the gravitational waves.

## I THE $R$ -MODE INSTABILITY

The  $r$ -mode instability has been the subject of about thirty papers over the past two years. I will not be able to do them all justice here.<sup>1</sup> Instead I will summarize the most important (as I see them) results with a direct impact on gravitational-wave detection, beginning with the basic model worked out in 1998 and ending with the latest (end of 1999) developments in this rapidly changing field.

The reason for the excitement is a version of the CFS instability—named for Chandrasekhar, who discovered it in a special case [2], and for Friedman and Schutz, who investigated it in detail and found that it is generic to rotating perfect fluids [3]. The CFS instability allows some oscillation modes of a fluid body to be driven rather than damped by radiation reaction, essentially due to a disagreement between two frames of reference.

The mechanism can be explained heuristically as follows. In a non-rotating star, gravitational waves radiate positive angular momentum from a forward-moving

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<sup>1</sup>) For a recent review with more emphasis on completeness, see Friedman and Lockitch [1].

mode and negative angular momentum from a backward-moving mode, damping both as expected. However, when the star rotates the radiation still lives in a non-rotating frame. If a mode moves backward in the rotating frame but forward in the non-rotating frame, gravitational radiation still removes positive angular momentum—but since the fluid sees the mode as having negative angular momentum, radiation drives the mode rather than damps it. Another example of such an effect due to a disagreement between frames of reference is the well-known Kelvin-Helmholtz instability, which leads to rough airplane rides over the jet stream and pounding surf on the California coast.

Mathematically, the criterion for the CFS instability is

$$\omega(\omega + m\Omega) < 0, \quad (1)$$

with the mode angular frequency  $\omega$  (in an inertial frame) in general a function of the azimuthal quantum number  $m$  and rotation angular frequency  $\Omega$ . For any set of modes of a perfect fluid, there will be some modes unstable above some minimum  $m$  and  $\Omega$ . However, fluid viscosity generally grows with  $m$  and there is a maximum value of  $\Omega$  (known as the Kepler frequency  $\Omega_K$ ) above which a rotating star flies apart. Therefore the instability is only astrophysically relevant if there is some range of frequencies and temperatures (viscosity generally depends strongly on temperature) in which it survives.

The  $r$ -modes are a set of fluid oscillations with dynamics dominated by rotation. They are in some respects similar to the Rossby waves found in the Earth's oceans and have been studied by astrophysicists since the 1970s [4]. The restoring force is the Coriolis inertial “force” which is perpendicular to the velocity. As a consequence, the fluid motion resembles (oscillating) circulation patterns. The (Eulerian) velocity perturbation is

$$\delta\vec{v} = \alpha\Omega R(r/R)^m \vec{r} \times \vec{\nabla} Y_{mm}(\theta, \phi) + O(\Omega^3), \quad (2)$$

where  $\alpha$  is a dimensionless amplitude (roughly  $\delta v/v$ ) and  $R$  is the radius of the star. Since  $\delta\vec{v}$  is an axial vector, mass-current perturbations are large compared to the density perturbations. The Coriolis restoring force guarantees that the  $r$ -mode frequencies are comparable to the rotation frequency,

$$\omega + m\Omega = \frac{2}{m+1}\Omega + O(\Omega^3). \quad (3)$$

It was not until the time of the last Amaldi Conference in mid-1997 that Anderson [5] noticed that the  $r$ -mode frequencies satisfy the mode instability criterion (1) for all  $m$  and  $\Omega$ , and that Friedman and Morsink [6] showed the instability is not an artifact of the assumption of discrete modes but exists for generic initial data. In other words, *all* rotating perfect fluids are subject to the instability.

## II DRIVING VS. DAMPING

The universe is inhabited not by balls of perfect fluid, but by stars subject to internal viscous processes which tend to damp out oscillation modes. To evaluate the stability of modes in realistic neutron stars, we must compare driving and damping timescales.

In the small-amplitude limit, a mode is a driven, damped harmonic oscillator with an exponential damping timescale [7]

$$\frac{1}{\tau} = -\frac{1}{2E} \frac{dE}{dt} = -\frac{1}{2E} \left[ \left( \frac{dE}{dt} \right)_G + \sum_V \left( \frac{dE}{dt} \right)_V \right] = \frac{1}{\tau_G} + \sum_V \frac{1}{\tau_V}. \quad (4)$$

Here  $E$  is the energy of the mode in the rotating frame and  $dE/dt$  is the sum of contributions from gravitational radiation (subscript  $G$ ) and all viscous processes (subscript  $V$ ). The mode is stable if the damping timescale  $\tau$  is positive, unstable if  $\tau$  is negative. The gravitational radiation timescale  $\tau_G$  depends on the rotation frequency  $\Omega$ , and the viscous timescales generally depend also on the temperature  $T$ . Therefore we define a critical frequency  $\Omega_c$  such that

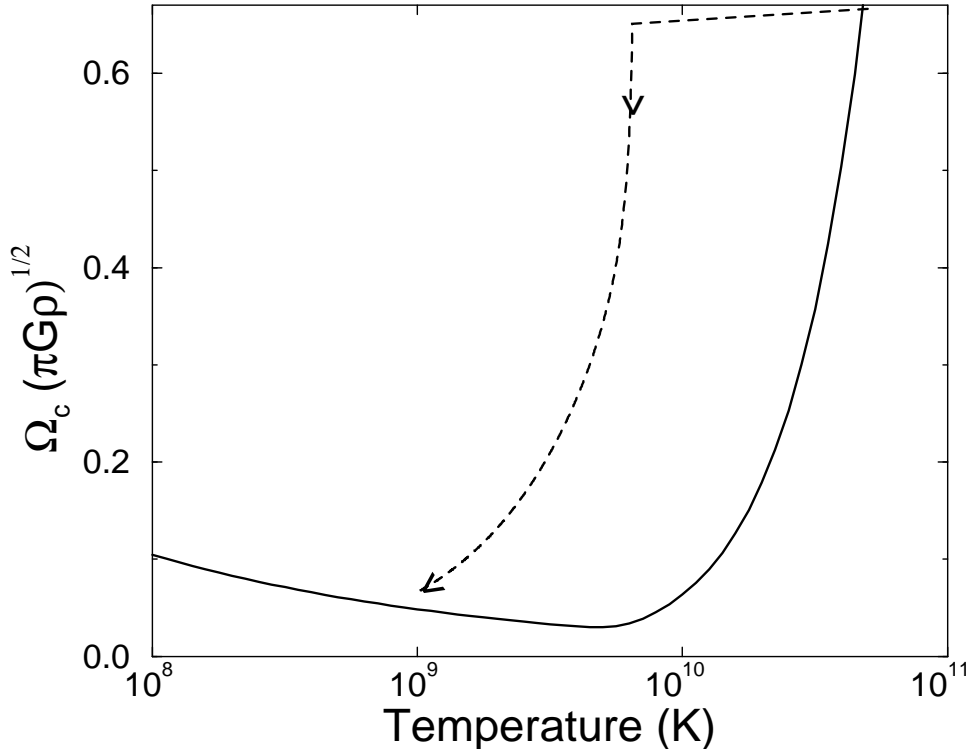
$$\frac{1}{\tau(\Omega_c, T)} = 0 \quad (5)$$

and decide if a given mode is astrophysically interesting by examining the curve  $\Omega_c(T)$ .

Neutron stars are complicated objects, but a simple model suffices to estimate the most important driving and damping timescales in the very young ones. When hotter than  $10^9\text{K}$  (younger than about a year), most of the star is a ball of ordinary, barotropic (equation of state independent of temperature) fluid. Given a putative equation of state, the gravitational radiation timescale  $\tau_G$  can be calculated by standard multipole integrals [8], although the  $r$ -modes are nonstandard in that the leading-order (in  $\Omega$ ) contribution is not from the mass multipoles but from the mass-current multipoles [9]. Viscous damping is due both to shearing of the fluid and to compression and rarefaction of individual fluid elements (bulk viscosity). The shear viscosity is stronger (timescale is shorter) at lower temperatures (like everyday experience with motor oil) and can be calculated from neutron-neutron scattering cross-sections [10]. The bulk viscosity is a weak nuclear interaction effect and thus is much stronger at higher temperatures. Compression and rarefaction of the fluid by the mode disturbs the density-dependent equilibrium  $p + e \leftrightarrow n$ , generating neutrinos which efficiently carry energy away [11]. As the star cools the viscous mechanisms change (see Sec. V), but this model is good enough for a first look.

The net damping timescale of the most unstable ( $m = 2$ )  $r$ -mode can be written in terms of fiducial timescales (written with tildes)

$$\frac{1}{\tau(\Omega, T)} = \frac{1}{\tilde{\tau}_G} \frac{\Omega^6}{(\pi G \bar{\rho})^3} + \frac{1}{\tilde{\tau}_S} \left( \frac{10^9\text{K}}{T} \right)^2 + \frac{1}{\tilde{\tau}_B} \left( \frac{T}{10^9\text{K}} \right)^6 \frac{\Omega^2}{\pi G \bar{\rho}}, \quad (6)$$



**FIGURE 1.** The solid curve is critical angular velocity  $\Omega_c$  as a function of temperature  $T$ , assuming ordinary fluid viscosity given by Lindblom, Mendell, and Owen [13]. Above this curve, the fastest-growing ( $m = 2$ )  $r$ -mode is unstable. The dashed curve is the evolutionary track of a neutron star in the first year of its life if the  $r$ -modes begin at amplitude  $\alpha = 10^{-5}$  and saturate at  $\alpha = 1$ , although it is not very sensitive to these values.

where  $\bar{\rho}$  is the mean density of the equilibrium star. The numerical values of the fiducial timescales (for a simplistic equation of state) have been evaluated as [9,12,13]

$$\tilde{\tau}_G = -3.3 \text{ s}, \quad \tilde{\tau}_S = 2.5 \times 10^8 \text{ s}, \quad \tilde{\tau}_B = 2.0 \times 10^{11} \text{ s}. \quad (7)$$

The numbers change by factors of two or so for different neutron-star models, but the curve plotted in Fig. 1 and its conclusion are very robust. The  $r$ -modes are unstable in realistic neutron stars over an interesting range of frequencies and temperatures. Neutron stars born rotating at or near the Kepler frequency will spin down and emit gravitational radiation in the process; the question now is how much.

### III SPINDOWN-COOLING MODEL

To detect gravitational waves from the  $r$ -modes, we need to know how the modes grow beyond the limits of perturbation theory and spin down the neutron star as

it cools during the first year of its life. Even in the simple approximation of an ordinary fluid ball, this involves nonlinear hydrodynamics and radiation reaction, which are both tricky subjects and could take years to explore properly. In the meantime we make do with a simple model [15] developed to make the first, rough estimates of detectability.

In this model, we consider three coupled systems—a uniformly rotating fluid background (with angular velocity  $\Omega$ ), the most unstable  $r$ -mode (with dimensionless amplitude  $\alpha$ ), and the rest of the universe. The two systems in the star (mode and background) couple to each other by viscosity and nonlinear fluid effects. The mode couples to the universe by gravitational radiation; the background does not. The mode’s energy evolves by gravitational radiation and viscous damping; the behavior of the background is determined by conservation of energy and angular momentum. Although some of the mode’s energy goes into heating the star, the standard neutrino cooling law  $(T/10^9\text{K}) \sim (1 \text{ yr}/t)^{1/6}$  is all but unaffected.

If a star is born spinning at  $\Omega_K$  at temperature  $10^{11}\text{K}$  (and recent observations [14] suggest that some stars are), the evolution falls into three distinct phases. The *growth phase* begins when the  $r$ -modes go unstable of order one second after the supernova. During this phase a small initial perturbation  $\alpha$  grows exponentially on a timescale of order one minute while  $\Omega$  remains almost constant (the mode is too small to emit much angular momentum). In this regime linearized hydrodynamics (which is all we know at the moment) is a good approximation. Within at most a few minutes after the supernova,  $\alpha$  becomes so large that nonlinear hydrodynamic effects can no longer be neglected. Previous studies of other modes [16] indicate that the main effect might be a saturation of the mode amplitude at some constant value, which we can treat as a phenomenological parameter. In this *saturation phase*, the star spins down very rapidly ( $d\Omega/dt \sim \Omega^7$ ) and emits gravitational radiation of strain amplitude

$$h(t) = 4 \times 10^{-24} \left( \frac{\Omega}{\pi G \bar{\rho}} \right)^3 \left( \frac{20 \text{ Mpc}}{D} \right) \alpha_{\text{max}}, \quad (8)$$

at a detector at distance  $D$  (normalized here to the distance at which we expect several events per year). As the star spins down, the gravitational radiation gets much weaker (recall  $1/\tau_G \sim \Omega^6$ ). Also, viscous damping becomes stronger, especially since other mechanisms come into play—for instance, when the neutrons become superfluid after cooling to about  $10^9\text{K}$ . Thus, within a year the star has moved along a track such as that in Fig. 1 and entered the *decay phase*, where the  $r$ -modes are stabilized by viscosity and  $\alpha$  slowly dies away without changing  $\Omega$  much. The final spin frequency  $\Omega_{\text{end}}$  is in practice another phenomenological parameter, since it depends on the more complicated viscous processes of cooler neutron stars as well as on  $\alpha_{\text{max}}$ .

## IV DETECTABILITY OF GRAVITATIONAL WAVES

Even at its strongest, an  $r$ -mode signal is below the strain noise of a gravitational-wave detector. But electromagnetic astronomers have been pulling faint pulsar signals out of noisy data for decades, and their data analysis techniques can be adapted for the  $r$ -modes.

Surprisingly, even the crude model of the source given in Sec. III is good enough to estimate the detectability of the gravitational waves. The quantity of interest is not the raw strain  $h(t)$  but rather a *characteristic strain*

$$h_c(f) = h[t(f)]\sqrt{f^2/|df/dt|}, \quad (9)$$

where  $df/dt$  is the time derivative of the gravitational wave frequency. The optimal (filtered) signal-to-noise ratio is

$$(S/N)^2 = 2 \int (d \ln f) (h_c/h_{\text{rms}})^2, \quad (10)$$

where the rms strain noise is related to the detector's power spectral noise density by

$$h_{\text{rms}} = \sqrt{f S_h(f)}. \quad (11)$$

Thus  $(S/N)^2$  can be estimated by looking at a plot such as Fig. 2. For the  $r$ -modes, we find that [15]

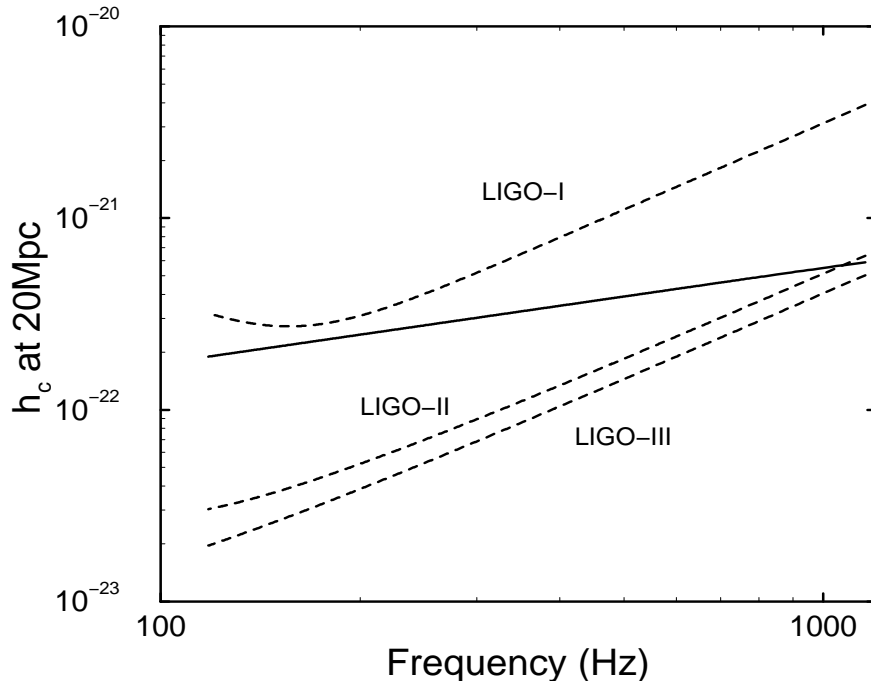
$$h_c = 6 \times 10^{-22} \left( \frac{f}{1 \text{ kHz}} \right)^{1/2} \left( \frac{20 \text{ Mpc}}{D} \right) \quad (12)$$

with  $(S/N) = 8$  for the projected LIGO-II (enhanced) noise curve as of 1998. This result is independent of much of the detailed physics of the source, including  $\alpha_{\text{max}}$ .<sup>2</sup> However, it does depend on the detailed astrophysics through the final low-frequency cutoff, which does not change much even if the viscous damping changes by orders of magnitude.

The optimal signal-to-noise ratio is only an upper limit—it assumes matched filtering, which requires precise tracking of the signal phase. While our knowledge of astrophysics will never be good enough to track an  $r$ -mode signal to within one cycle out of  $10^9$ , there are alternatives. The lower limit has been set by Brady and Creighton [17] using the simplest possible search algorithm, patterned on the techniques used to find pulsar signals in electromagnetic data. Assuming the supernova has been observed optically and a sky position is available, the Doppler shifts due to the Earth's motion can be removed to obtain a signal which is sinusoidal but for

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<sup>2</sup> To my knowledge the argument was first made by R. D. Blandford in 1984 (but never published) that such a robust result holds for any system evolving mainly via gravitational radiation—like the  $r$ -modes in the saturation phase.



**FIGURE 2.** Characteristic signal amplitude  $h_c$  and noise  $h_{\text{rms}}$ . The signal is from a neutron star with mass  $1.4M_{\odot}$  and radius 12.5 km at a distance of 20 Mpc (several events per year). The noise is for the three versions of LIGO used in Ref. [15]. Integrated (optimal) signal-to-noise ratio is 8; realistic signal-to-noise is somewhat less.

the (slow) intrinsic frequency evolution of the source. Even without any modeling of this evolution, i.e. by expanding

$$f(t) = f_0 \left( 1 + \sum_k f_k t^k \right) \quad (13)$$

for short Fourier transforms (integrating for year is computationally too expensive) and combining the transforms in some way for different trial values of the spindown parameters  $f_k$ , it is possible to obtain one fifth of the optimal signal-to-noise. This is in spite of the fact that the search is computationally limited by the requirement that data analysis keep pace with data acquisition and by the fact that the  $r$ -modes evolve so quickly that many terms  $f_k$  are needed. With constraints—even rough ones—from a physical model, the  $f_k$  are no longer all independent and the efficiency of data analysis could be increased. Since event rate goes roughly as  $(S/N)^3$ , it is important to beat the lower limit.

A stochastic background from the superposition of many faint  $r$ -mode signals out to cosmological distances will also exist. However, it is much fainter than a single signal and thus detectable only by (advanced) LIGO-III [15,18].

## V OPEN QUESTIONS

We are now (at the end of 1999) in the midst of a renewed flurry of activity on  $r$ -mode astrophysics. Several effects neglected in the first simple scenario are being worked out. Some of them could damp the  $r$ -modes much more effectively than previously thought, pushing the detectability of the gravitational waves from LIGO-II to LIGO-III. However, this is far from certain and the astrophysicists are having an exciting time working it out. Here is a list of the effects that (I think) have the most direct impact on detection prospects.

*Superfluid viscosity.* One of the most eagerly awaited papers has been the calculation of the damping effects of “mutual friction”, a process which paradoxically increases the viscous damping when a neutron star cools to a superfluid. At temperatures below about  $10^9$ K this viscous mechanism was expected to dominate, and the big question was whether the damping was sufficient to stabilize the  $r$ -modes in stars older than about a year—especially the low-mass x-ray binaries (see the review by G. Ushomirsky in this volume). The answer [19] is a definite maybe. The damping timescale varies by several orders of magnitude, depending on a parameter of superfluid physics (the neutron-proton entrainment coefficient) which is yet poorly known. More work is in progress, but recently mutual friction has been upstaged by other issues.

*Relativistic effects.* Most  $r$ -mode calculations to date have assumed Newtonian gravity. Relativity was thought to simply multiply various numbers by redshift factors of order unity, but there are two important qualitative differences with Newtonian gravity. First there is the claim by Kojima [20] that the  $r$ -mode frequency becomes smeared over a finite bandwidth. This claim is contradicted, however, by Lockitch [21]. With Andersson and Friedman, he [22] finds that relativistic  $r$ -modes do however have an increased coupling to bulk viscosity similar to that of the “generalized  $r$ -modes” [23,24] of Newtonian stars, which are still unstable but less so.

*Nonlinear fluid dynamics.* At least two groups [25,26] are working on codes to numerically solve the fully nonlinear fluid equations for the  $r$ -modes and determine the saturation amplitude. However, the problem is complicated and the investment of coding and formalism is large, so expect results in a year or two at best. Order of magnitude arguments [27] have been made to claim that the  $r$ -modes saturate due to mode-mode coupling at a very small amplitude  $\alpha \sim 10^{-5}$ , which would render the signal undetectable. However, these arguments neglect the unique symmetries of the  $r$ -modes; and based on work on the  $g$ -modes of white dwarfs [28] it seems that the  $r$ -modes could indeed grow much larger. Semi-analytical analyses [29] of mode-mode coupling may give some indications about mode saturation while we wait for numerical results.

*Magnetic fields.* If the growth of the  $r$ -modes leads to substantial differential rotation, it could wind up magnetic field lines frozen into the fluid of a young neutron star, amplifying any seed field and saturating the  $r$ -modes at a small amplitude. Two recent papers [30,31] claim that the  $r$ -modes produce differential rotation,



but this is a nonlinear effect which the authors have tried to treat with linear perturbation theory. Strictly speaking the gravitational radiation is also nonlinear (quadratic in  $\alpha$ ), but the canonical energy and angular momentum are global quantities whose perturbation can be derived self-consistently from a Lagrangian principle. It is not clear how to do this for local, dynamical quantities such as vorticity.

*Crust formation.* Perhaps the most important new result is that the formation of a solid crust (below about  $10^{10}\text{K}$ ) can act to strongly stabilize the mode. Bildsten and Ushomirsky [32] find that shear viscosity in the fluid boundary layer just below the crust decreases the damping timescale by  $10^5$ – $10^7$ . They conclude that the  $r$ -modes are completely suppressed in low-mass x-ray binaries and that the signal-to-noise ratio is reduced by three for newborn neutron stars. But it is not clear to me that this result is correct for newborn neutron stars. If an  $r$ -mode is already excited when the crust starts to form (of order a minute after the supernova), the intense and localized shear heating in the boundary layer can re-melt the crust if the pre-existing  $r$ -mode is strong enough. In this case, the outer layers stay in a self-regulating equilibrium at the melting temperature and the old model of the evolution is largely unaffected. I estimate that, in this case, “strong enough” means an  $r$ -mode amplitude of  $\alpha = 10^{-3}$ . This points out some interesting questions for future research: First, what is the initial value of  $\alpha$  when the  $r$ -modes first go unstable? The first model [15] used gratuitously small values to make a point, but no one knows yet what are reasonable values. Also, what exactly is the melting temperature of a new crust? If it is  $8 \times 10^9\text{K}$  rather than  $10^{10}\text{K}$  then the  $r$ -mode could have plenty of time to grow, and in astrophysics a 20% error is considered high precision.

Although I have skipped over many astrophysics issues, I realize even this short list may be bewildering to the experimenters and data analysts who are the main audience at the Amaldi Conference. If I had to distill my presentation into one sentence, I would say: Let the theorists argue for another two years; the  $r$ -modes are not as good a bet as binaries, but they may not be far behind.

## VI ACKNOWLEDGMENTS

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