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Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature

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We study compact hypersurfaces M^n, $n \geqq 2$, without boundary, which are smoothly immersed in a Riemannian manifold N^{n+1}. Let $M^n = M_0$ be given locally by some diffeomorphism

$$F_0\colon\ U \subset \mathbb{R}^n \to F_0(U) \subset M_0 \subset N^{n+1}.$$

We want to move M_0 along its mean curvature vector, that is, we want to find a whole family $F(\cdot, t)$ of diffeomorphisms corresponding to surfaces M_t, such that the evolution equation

$$\text{(1)}\qquad \begin{aligned} &\frac{\partial}{\partial t} F(\vec{x}, t) = \vec{H}(\vec{x}, t) \qquad \vec{x} \in U \\ &F(\cdot, 0) = F_0 \end{aligned}$$

is satisfied. Here $\vec{H}(\vec{x}, t)$ is the mean curvature vector of the hypersurface M_t at the point $F(\vec{x}, t)$ and we will see that (1) is a quasilinear parabolic system with a smooth solution at least on some short time interval. If for example M_0 is a sphere of radius $r(0)$ in $\mathbb{R}^{n+1}$, then M_t is a family of concentric spheres of radius

$$r(t) = \sqrt{r^2(0) - 2nt}$$

which shrink towards the center of the initial sphere in finite time. It was shown in [3], that this behaviour is very typical: If the initial hypersurface $M_0 \subset \mathbb{R}^{n+1}$ is uniformly convex, then the surfaces M_t contract smoothly to a single point in finite time and the shape of the surfaces becomes spherical at the end of the contraction.

If the ambient space N is a general Riemannian manifold, the curvature of N will interfere with the motion of the surfaces M_t. We want to show here that the contraction – first to a small sphere and then to a single point – is still

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working in the general case, if we only assume that the initial surface is convex enough to overcome the obstructions imposed by the geometry of N. By 'convex enough' we mean that the principle curvatures of M_0, i.e. the eigenvalues of the second fundamental form on M_0, are bounded from below by a positive constant depending on N. Since we do not have to assume *a priori* that the initial surface M_0 is a sphere, we also obtain results concerning the question when a locally convex hypersurface is the immersion of a sphere and under what conditions a locally convex hypersurface bounds a region diffeomorphic to a ball in N.

1. The result

In the following Latin indices range from 1 to n, Greek indices range from 0 to n and the summation convention is understood. We denote the induced metric and the second fundamental form on M by $g=\{g_{ij}\}$ and $A=\{h_{ij}\}$. The mean curvature of M is the trace of the second fundamental form, $H=g^{ij}h_{ij}$. We write $\bar{R}m=\{\bar{R}_{\alpha\beta\gamma\delta}\}$ and $\bar{\nabla}\bar{R}m=\{\bar{\nabla}_\sigma\bar{R}_{\alpha\beta\gamma\delta}\}$ for the curvature tensor of N and its covariant derivative. Let us denote by $\sigma_x(P)$ the sectional curvature of a 2-plane P at $x\in N$ and let $i_x(N)$ be the injectivity radius of N at x. Let us also agree to write $T_{ij}\geqq 0$ if all eigenvalues of a symmetric tensor $T=\{T_{ij}\}$ are nonnegative.

1.1 Theorem. *Let $n\geqq 2$ and N^{n+1} be a smooth complete Riemannian manifold without boundary which satisfies uniform bounds*

$$-K_1\leqq\sigma_x(P)\leqq K_2,\qquad K_1,K_2\geqq 0$$

$$|\bar{\nabla}\bar{R}m|^2\leqq L^2,\qquad L\geqq 0$$

$$i_x(N)\geqq i(N)>0.$$

Let M_0 be a compact connected hypersurface without boundary which is smoothly immersed in N, and suppose that on M_0 we have

$$Hh_{ij}>nK_1 g_{ij}+\frac{n^2}{H}Lg_{ij} \tag{2}$$

Then (1) *has a smooth solution M_t on a finite time interval $0\leqq t<T$ and the M_t's converge uniformly to a single point $0\in N$ as $t\to T$. If we take for $t\to T$ homothetic expansions of normal coordinates around* 0 *such that the total area of the expanded surfaces $\tilde{M}_t$ is fixed, then the $\tilde{M}_t$ converge to a sphere of that area in the C^∞-topology.*

Remarks. (i) Inequality (2) does not depend on K_2, so positive sectional curvature in the ambient space helps toward mean curvature contraction, whereas negative sectional curvature slows it down. In particular, if N is locally symmetric ($\bar{\nabla}\bar{R}m=0$), we have $L=0$ and condition (2) is satisfied if all eigenvalues of A are bigger than $K_1^{1/2}$. If in addition the sectional curvature in the ambient space is non-negative, Theorem 1.1 takes exactly the same form as in Euclidean space: All locally convex hypersurfaces contract to a single point.

(ii) Condition (2) implies (for a suitable choice of normal to M)

(3) $$H > n K_1^{1/2}.$$

In §4 we show that (2) and (3) remain valid on M_t for all $0 \leqq t < T$. In particular, if N is locally symmetric and the initial surface M_0 is totally umbilic $\left(\text{i.e. } h_{ij} \equiv \frac{1}{n} H g_{ij}\right)$, then this remains so and we have only to assume that (3) holds on M_0.

(iii) We will see in Lemma 7.3 that (2) and (3) are just strong enough to force all eigenvalues of the intrinsic Ricci tensor of M_0 to be positive. Thus in the two dimensional case it follows from the Gauß-Bonnet theorem that M_0 is the immersion of a sphere. In the higher dimensional case this is a consequence of Theorem 1.1: Since M_t is a sphere for t close to T, already M_0 must have been a sphere. We have

1.2 Corollary. *Any isometric immersion $M^n \to N^{n+1}$, with N and M satisfying the conditions in Theorem* 1.1, *is the immersion of a sphere.*

If M_0 is imbedded in N, then it follows from the strong parabolic maximum principle (see Lemma 3.2) that M_t is imbedded for all $0 \leqq t < T$. Thus we have

1.3 Corollary. *If $M^n \to N^{n+1}$ is an isometric imbedding satisfying the assumptions of Theorem* 1.1, *then M bounds a region in N, and the region is diffeomorphic to a ball.*

(iii) Since condition (2) remains valid for all M_t, $0 \leqq t < T$, we obtain from the strong elliptic maximum principle

1.4 Corollary. *If N^{n+1} is a manifold with boundary ∂N and the mean curvature of the boundary $H(\partial N)$ with respect to the inner normal satisfies*

(4) $$\inf H(\partial N) \geqq -n K_1^{\frac{1}{2}},$$

then M_t, $0 \leqq t < T$, cannot touch ∂N and all results stated above remain true.

Corollary 1.4 can be used to obtain results in manifolds N without a lower bound on the injectivity radius. If for example N admits an exhaustion $B_1 \subset\subset B_2 \subset\subset B_3 \ldots$ by compact regions B_l, $l \in \mathbb{N}$, such that each boundary ∂B_l satisfies (4) with respect to the inner normal, then these boundaries act as obstacles for the evolution of M_0. Thus we have an automatic lower bound on the injectivity radius since the surfaces M_t remain in one of the compact regions B_l and Theorem 1.1 applies. We illustrate this with an example which also shows that inequalities (2) and (4) are optimal.

Example. Let $N = N^3$ be as in ([7], §5) a non-compact hyperbolic three-manifold with a finite number of ends $E_1, \ldots, E_k$ and assume that each end is homeomorphic to $T^2 \times [0, \infty)$, where T^2 is the 2-torus. Suppose that each end is isometric to the quotient of a region in $\mathbb{H}^3$ (hyperbolic three-space in the upper half-space representation) above an interior horizontal euclidean plane by a group which is generated by two parabolic transformations which leave the point at infinity fixed. Then $L=0$, $K_1=1$ and the injectivity radius tends to zero in each end. All tori $T^2\times\{s\}$ are flat and all principal curvatures with respect to the inner normal are equal to -1 such that relation (4) is satisfied with equality. Thus, choosing a sequence $s_l\to\infty$ in each end, we can construct an exhaustion of N as mentioned above and all results quoted before are true in this manifold.

The proof of Theorem 1.1 follows the proof in the euclidean case [3]. After proving in §4 that the assumptions (2), (3) are preserved as the evolution goes on, we show in §5 that the eigenvalues of the second fundamental form approach each other, an idea which was originally used by Hamilton, [1], for a different problem. Using this we can show that the diameter of the surfaces M_t tends to zero at some stage and the result then follows from the assumption that the injectivity radius of N is bounded from below.

2. Preliminaries

Let ν be the outer unit normal to M_t, i.e., we choose ν such that inequalities (2) and (3) hold with respect to $-\nu$ and the surfaces are moving in direction $-\nu$. Then for a fixed time t we choose a local field of frames $e_0, e_1, \ldots, e_n$ in N such that restricted to M_t, we have $e_0=\nu$, $e_i=\frac{\partial F}{\partial x_i}$. We use the same notation as in [3] and write in particular

$$\begin{aligned} H&=g^{ij}h_{ij}=h_i^i\\ |A|^2&=g^{ij}g^{kl}h_{ik}h_{jl}=h_{ik}h^{ik}\\ C&=g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}=h_{ik}h^k{}_l h^{li}\\ Z&=H\cdot C-|A|^4. \end{aligned}$$

If we mean the metric or the connection on N, this will be indicated by a bar, for example $\bar g_{\alpha\beta}$, $\bar\Gamma^\alpha_{\beta\gamma}$ and $\bar\nabla$. The Riemann curvature tensors of M and N will be denoted by $Rm=\{R_{ijkl}\}$ and $\bar Rm=\{\bar R_{\alpha\beta\gamma\delta}\}$. The relation between A, Rm and $\bar Rm$ is then given by the equations of Gauß and Codazzi:

$$\begin{aligned} R_{ijkl}&=\bar R_{ijkl}+h_{ik}h_{jl}-h_{il}h_{jk}\\ \nabla_k h_{ij}-\nabla_j h_{ik}&=\bar R_{0ijk}. \end{aligned}$$

These relations now imply Simons' identity, [6], for the Laplacian of the second fundamental form on M. See also [5] for a simple derivation.

2.1 Lemma. *We have the identities*

(i) $$\Delta h_{ij}=\nabla_i\nabla_j H+Hh_{il}h^l{}_j-|A|^2h_{ij}+H\bar R_{0i0j} -h_{ij}\bar R_{0l0}{}^l+h_{jl}\bar R^l{}_{mi}{}^m+h_{il}\bar R^l{}_{mj}{}^m-2h_{lm}\bar R^l{}_i{}^m{}_j+\bar\nabla_j\bar R_{0li}{}^l+\bar\nabla_l\bar R_{0ij}{}^l.$$

(ii) $\frac{1}{2}\Delta|A|^2=\langle h_{ij}, \nabla_i\nabla_j H\rangle+|\nabla A|^2+Z$
$$+Hh^{ij}\bar{R}_{0i0j}-|A|^2\bar{R}_{0l0}{}^{l}+2h^{ij}h_{jl}\bar{R}^{l}{}_{mi}{}^{m}-2h^{ij}h^{lm}\bar{R}_{limj}$$
$$+h^{ij}(\bar{\nabla}_j\bar{R}_{0li}{}^{l}+\bar{\nabla}_l\bar{R}_{0ij}{}^{l}).$$

We also need an extension of ([3], Lemma 2.2) to hypersurfaces in general Riemannian manifolds. For that purpose we denote by $w=\{w_i\}$ the vector with components $w_i=\bar{R}_{0li}{}^{l}$, i.e., w is the projection of $\bar{\mathrm{R}}\mathrm{ic}(v,\cdot)$ on M.

2.2 Lemma. *For any $\eta>0$ we have the inequality*

$$\text{(i)}\quad |\nabla A|^2\geqq\left(\frac{3}{n+2}-\eta\right)|\nabla H|^2-\frac{2}{n+2}\left(\frac{2}{n+2}\eta^{-1}-\frac{n}{n-1}\right)|w|^2$$

and in particular

$$\text{(ii)}\quad |\nabla A|^2-\frac{1}{n}|\nabla H|^2\geqq\frac{n-1}{2n+1}|\nabla A|^2-\frac{2n}{(n-1)(2n+1)}|w|^2$$
$$\geqq\frac{n-1}{2n+1}|\nabla A|^2-C(n,K_1,K_2).$$

Proof. First note that the second inequality follows from the first one with $\eta=\frac{2(n-1)}{n(n+2)}$. To prove (i), we decompose the tensor $\nabla A=\{\nabla_i h_{jk}\}$ as follows:

$$\nabla_i h_{jk}=E_{ijk}+F_{ijk}$$

where

$$E_{ijk}=\frac{1}{n+2}(\nabla_i H g_{jk}+\nabla_j H g_{ik}+\nabla_k H g_{ij})$$
$$-\frac{2}{(n+2)(n-1)}w_i g_{jk}+\frac{n}{(n+2)(n-1)}(w_j g_{ik}+w_k g_{ij}).$$

Then E_{ijk} has the same traces as $\nabla_i h_{jk}$ in view of the Codazzi equations and

$$\langle E_{ijk}, F_{ijk}\rangle=0.$$

Furthermore

$$|E|^2=\frac{3}{n+2}|\nabla H|^2+\frac{2n}{(n+2)(n-1)}|w|^2-\frac{4}{n+2}\langle w_i,\nabla_i H\rangle$$
$$\geqq\left(\frac{3}{n+2}-\eta\right)|\nabla H|^2+\frac{2}{n+2}\left(\frac{n}{n-1}-\frac{2}{n+2}\eta^{-1}\right)|w|^2$$

which proves the Lemma. It is worth noting that in case of an Einstein manifold N the vector w vanishes identically and therefore η can be chosen equal to zero.

3. The evolution equations

In a general Riemannian manifold N^{n+1} the Gauß-Weingarten relations take the form

$$\frac{\partial^2 F^\alpha}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial F^\alpha}{\partial x_k} + \bar{\Gamma}_{\rho\sigma}^\alpha \frac{\partial F^\rho}{\partial x_i} \frac{\partial F^\sigma}{\partial x_j} = -h_{ij} \nu^\alpha \tag{5}$$

$$\frac{\partial \nu^\alpha}{\partial x_j} + \bar{\Gamma}_{\rho\sigma}^\alpha \frac{\partial F^\rho}{\partial x_j} \nu^\sigma = h_{jl} g^{lm} \frac{\partial F^\alpha}{\partial x_m}$$

and evolution Eq. (1) becomes

$$\begin{aligned} \frac{\partial}{\partial t} F^\alpha(\vec{x}, t) &= \vec{H}^\alpha(\vec{x}, t) = -H(\vec{x}, t)\, \nu(\vec{x}, t) \\ &= \Delta_t F^\alpha(\vec{x}, t) + \left\{ \bar{\Gamma}_{\rho\sigma}^\alpha \frac{\partial F^\rho}{\partial x_i} \frac{\partial F^\sigma}{\partial x_j} g^{ij} \right\} (\vec{x}, t) \end{aligned} \tag{1}$$

where Δ_t is the Laplace-Beltrami operator on M_t and the indices α, ρ, σ refer to a local coordinate system y^α in N^{n+1}. This is a quasi linear parabolic system and we obtain a smooth solution at least on some short time interval, cf. [1].

3.1 Lemma. *If the initial surface M_0 is smooth, then* (1) *has a smooth solution on some maximal open time interval $0 \leqq t < T \leqq \infty$.*

Since (1) is parabolic, we can also show that two surfaces moving by their mean curvature cannot overtake each other:

3.2 Lemma. (i) *Let $M_{1,t}$ and $M_{2,t}$ be two smooth closed surfaces moving by their mean curvature for $0 \leqq t \leqq t_1$. If M_1 and M_2 are disjoint for $t=0$, they stay disjoint on the whole interval $0 \leqq t \leqq t_1$.*

(ii) *If $M_{1,t}$ is imbedded for $t=0$, then this remains so for $0 \leqq t \leqq t_1$.*

Proof. If the surfaces were intersecting at one stage, there was a first time $0 < t_0$ such that M_{1,t_0} touches M_{2,t_0} at some point $p \in N$. Let S be some fixed reference surface which is tangential to the surfaces M_{1,t_0} and M_{2,t_0} at p and assume that we have Gaussian coordinates in a neighbourhood of S, i.e., $y^0(q)$ is the length of the geodesic arc perpendicular to S through q, and $y^i(q) = x_i(q)$ are the coordinates of the basepoint of the geodesic in S. Then locally around p we can write $M_{1,t}$ and $M_{2,t}$ for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ as graphs of functions $u_1(t)$ and $u_2(t)$ on S. The unit normal to M_i, $i = 1, 2$, is then given by

$$\nu_i = (1 + |\nabla u_i|^2)^{-\frac{1}{2}} \left(1, -\frac{\partial}{\partial x_1} u_i, \ldots, -\frac{\partial}{\partial x_n} u_i\right)$$

and u_i, $i = 1, 2$, satisfies the evolution equation

$$\frac{\partial}{\partial t} u_i = -(1 + |\nabla u_i|^2)^{-\frac{1}{2}} \cdot H_i \tag{6}$$

where H_i is the mean curvature of M_i. We have $\nabla u_1 = \nabla u_2 = 0$ at (p, t_0) and (6) becomes a uniformly parabolic equation in a small neighbourhood of (p, t_0). By assumption we have $u_1(t) > u_2(t)$ (say) for $t < t_0$ and the contradiction follows from the strong parabolic maximum principle, see for example ([4], §§3.3, 3.7). The same argument applies for the second part of the Lemma.

Now we want to establish evolution equations for the induced metric and the second fundamental form on M_t. It will be convenient to assume that at a fixed point $\vec{x}_0$ and a fixed time t_0 we have $g_{ij}(\vec{x}_0, t_0) = \delta_{ij}$ and that the coordinates y^α, $0 \leqq \alpha \leqq n$ for N are normal coordinates at $F(\vec{x}_0, t_0)$. We can also arrange that in these coordinates $v^\alpha = -\delta_0^\alpha$ and $\frac{\partial F^\alpha}{\partial x_i} = \delta_i^\alpha$ at $F(\vec{x}_0, t_0)$. Then all Christoffel symbols of the connection $\bar{\Gamma}$ vanish at $F(\vec{x}_0, t_0)$ and we have only to take derivatives of the Christoffel symbols into account, which will lead to curvature terms eventually. Using the Gauß-Weingarten relations (5) and the fact that $\frac{\partial}{\partial y^\delta} \bar{g}_{\alpha\beta}$ vanishes at $F(\vec{x}_0, t_0)$ for $0 \leqq \delta \leqq n$ in our coordinates, we derive exactly as in ([3], Lemma 3.2 and 3.3):

3.3 Lemma. *The metric and the normal of M_t satisfy the evolution equations*

(i) $\frac{\partial}{\partial t} g_{ij} = -2H h_{ij}$

(ii) $\frac{\partial}{\partial t} v = \nabla H.$

Furthermore we have

3.4 Theorem. *The second fundamental form of M_t satisfies the evolution equation*

$$\begin{aligned} \frac{\partial}{\partial t} h_{ij} = &\Delta h_{ij} - 2H h_{il} h^l{}_j + |A|^2 h_{ij} + h_{ij} \bar{R}_{0l0}{}^l \\ &- h_{jl} \bar{R}^l{}_{mi}{}^m - h_{il} \bar{R}^l{}_{mj}{}^m + 2 h_{lm} \bar{R}^l{}_i{}^m{}_j \\ &- \bar{\nabla}_j \bar{R}_{0li}{}^l - \bar{\nabla}_l \bar{R}_{0ij}{}^l. \end{aligned}$$

Proof. From (1) and (5) we derive

$$\begin{aligned} \frac{\partial}{\partial t} h_{ij} &= -\frac{\partial}{\partial t}\left(\frac{\partial^2 F}{\partial x_i \partial x_j}, v\right) + H \bar{g}_{\alpha\beta} \frac{\partial}{\partial y^\delta} \bar{\Gamma}^\alpha_{i\rho} v^\delta \frac{\partial F^\rho}{\partial x_j} v^\beta \\ &= \left(\frac{\partial^2}{\partial x_i \partial x_j}(H v), v\right) - \left(\frac{\partial^2 F}{\partial x_i \partial x_j}, \frac{\partial}{\partial x_l} H g^{lm} \frac{\partial F}{\partial x_m}\right) \\ &\quad + H \bar{g}_{\alpha\beta} \frac{\partial}{\partial y^\delta} \bar{\Gamma}_{i\rho}{}^\alpha v^\delta \frac{\partial F^\rho}{\partial x_j} v^\beta \end{aligned}$$

where we used the notation (,) for the inner product in N^{n+1}. Using again (5) this is equal to

$$\begin{aligned} &\frac{\partial^2}{\partial x_i \partial x_j} H - \Gamma^k_{ij} \frac{\partial}{\partial x_k} H + H \bar{g}_{\alpha\beta} \frac{\partial}{\partial y^\delta} \bar{\Gamma}^\alpha_{i\delta} v^\delta \frac{\partial F}{\partial x_j} v^\beta \\ &\quad + H \left(h_{jl} g^{lm} \frac{\partial^2 F}{\partial x_i \partial x_m}, v\right) - H \bar{g}_{\alpha\beta} \frac{\partial}{\partial x_i} \bar{\Gamma}^\alpha_{\rho\sigma} v^\delta \frac{\partial F^\rho}{\partial x_j} v^\beta \\ &= \nabla_i \nabla_j H - H h_{il} h^l{}_j + H \bar{R}_{0i0j} \end{aligned}$$

and the conclusion follows from Lemma 2.1.

From this we derive as in [3]

3.5 Corollary. *We have the evolution equations*

(i) $\frac{\partial}{\partial t} H = \Delta H + H(|A|^2 + \bar{\mathrm{R}}\mathrm{ic}(v,v)),$

(ii) $\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^2(|A|^2 + \bar{\mathrm{R}}\mathrm{ic}(v,v))$

$$-4(h^{ij}h_j^m \bar{R}_{mli}{}^l - h^{ij}h^{lm}\bar{R}_{milj})$$
$$-2h^{ij}(\bar{\nabla}_j \bar{R}_{0li}{}^l + \bar{\nabla}_l \bar{R}_{0ij}{}^l)$$

(iii) $\frac{\partial}{\partial t}\left(|A|^2 - \frac{1}{n}H^2\right) = \Delta\left(|A|^2 - \frac{1}{n}H^2\right) - 2\left(|\nabla A|^2 - \frac{1}{n}|\nabla H|^2\right)$

$$+2\left(|A|^2 - \frac{1}{n}H^2\right)(|A|^2 + \bar{\mathrm{R}}\mathrm{ic}(v,v)) - 2h^{ij}(\bar{\nabla}_j \bar{R}_{0li}{}^l + \bar{\nabla}_l \bar{R}_{0ij}{}^l)$$
$$-4(h^{ij}h_j^m \bar{R}_{mli}{}^l - h^{ij}h^{lm}\bar{R}_{milj}).$$

where $\bar{\mathrm{R}}\mathrm{ic}(v,v) = \bar{R}_{0l0}{}^l$.

Let us also note that in view of Lemma 3.3(i) the time derivative of the measure $d\mu_t = \mu_t\, dx$ on M_t is the same as in the euclidean case:

$$\frac{\partial}{\partial t}\mu_t = -H^2 \mu_t$$

and the area of the surfaces M_t is decreasing very rapidly.

4. A lower bound for the eigenvalues of A

In this section we want to show that our convexity assumptions, i.e., inequalities (2) and (3) are preserved during the evolution of M_t. In view of the strict inequality in (2) there are some $\varepsilon_1, \varepsilon_2 > 0$ such that

(7a) $$H^2 \geqq n^2 K_1 + n\varepsilon_2 H^2,$$

(7b) $$H h_{ij} \geqq n K_1 g_{ij} + \frac{n^2}{H} L g_{ij} + \varepsilon_1 (H^2 - n^2 K_1) g_{ij}$$

holds on M_0. Since $|A|^2 \geqq \frac{1}{n} H^2$ and $\bar{\mathrm{R}}\mathrm{ic}(v,v) = \bar{R}_{0l0}{}^l \geqq -nK_1$, it follows from Corollary 3.5(i) and the maximum principle, that (7a) is preserved with the same $\varepsilon_2 > 0$ for all $0 \leqq t < T$. Then we have

$$\frac{\partial}{\partial t} H \geqq \Delta H + \varepsilon_2 H^3$$

and as in ([3], Lemma 5.8) we conclude that this inequality can have a bounded solution only on a finite time interval since $\min_{M_0} H = H_{\min}(0) > 0$. We have

4.1 Lemma. *If* (7a) *holds on* M_0, *then it remains true on* M_t *for* $0 \leqq t < T$ *and we have* $T \leqq \frac{1}{2}\varepsilon_2^{-1} H_{\min}^{-2}(0)$.

Now we derive a lower bound for the eigenvalues of A.

4.2 Theorem. *If for some* $0 < \varepsilon_1 < \frac{1}{n}$ *the inequality*

$$H h_{ij} \geqq n K_1 g_{ij} + \frac{n^2}{H} L g_{ij} + \varepsilon_1 (H^2 - n^2 K_1) g_{ij}$$

is valid on M_0, *then it remains true on* M_t, $0 \leqq t < T$.

Proof. We are going to show that all eigenvalues of

$$M_{ij} = \frac{h_{ij}}{H} - \varepsilon_1 g_{ij} - \frac{n(1 - n\varepsilon_1)}{H^2} K_1 g_{ij} - \frac{n^2}{H^3} L g_{ij}$$

remain non-negative. First of all we need an evolution equation for M_{ij}. Using the evolution equation for h_{ij} in Theorem 3.4 and the fact that by Corollary 3.5(i)

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{H^\alpha} &= \Delta\left(\frac{1}{H^\alpha}\right) - \alpha(\alpha+1)\frac{1}{H^{\alpha+2}}|\nabla H|^2 \\
&\quad - \frac{\alpha}{H^\alpha}(|A|^2 + \bar{\mathrm{R}}\mathrm{ic}(v, v)) \\
&= \Delta\left(\frac{1}{H^\alpha}\right) + \frac{2}{H}\left\langle \nabla_l H, \nabla_l\left(\frac{1}{H^\alpha}\right)\right\rangle \\
&\quad - \alpha(\alpha-1)\frac{1}{H^{\alpha+2}}|\nabla H|^2 - \frac{\alpha}{H^\alpha}(|A|^2 + \bar{\mathrm{R}}\mathrm{ic}(v, v)),
\end{aligned}$$

we derive as in ([3], §4) that

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + \frac{2}{H}\langle \nabla_l H, \nabla_l M_{ij}\rangle + N_{ij}$$

where

$$\begin{aligned}
N_{ij} = &-2h_{il}h^l_j + 2\varepsilon_1 H h_{ij} + \frac{2n(1-n\varepsilon_1)}{H} K_1 h_{ij} + \frac{2n^2}{H^2} L h_{ij} \\
&+ \frac{2n(1-n\varepsilon_1)}{H^4} K_1 |\nabla H|^2 g_{ij} + \frac{6n^2}{H^5} L |\nabla H|^2 g_{ij} \\
&+ \frac{1}{H}(2h_{lm}\bar{R}^{l\ \,m}_{\ i\ \ j} - h_{jl}\bar{R}^l_{\ mi}{}^m - h_{il}\bar{R}^l_{\ mj}{}^m) - \frac{1}{H}(\bar{\nabla}_j \bar{R}_{0li}{}^l + \bar{\nabla}_l \bar{R}_{0ij}{}^l) \\
&+ \left(\frac{2n(1-n\varepsilon_1)}{H^2} K_1 + \frac{3n^2}{H^3} L\right)(|A|^2 + \bar{\mathrm{R}}\mathrm{ic}(v, v)) g_{ij}.
\end{aligned}$$

In ([1], Theorem 9.1) a maximum principle for such an evolution equation was proved under the assumption that the absolute term N_{ij} is a polynomial of M_{ij} and g_{ij}. Since $\bar{R}m$ is smooth, it is easy to see that the argument is valid in our case as well. We have then only to consider the first time t_0, where at some point $p \in M_{t_0}$ a zero eigenvector $v=\{v^i\}$ of M_{ij} occurs, and Theorem 4.2 is proved if we can show that $N_{ij} v^i v^j$ is non-negative. For that purpose we choose an orthonormal basis $(e_1, \ldots, e_n)$ for $T_p M_{t_0}$ such that h_{ij} (and thus M_{ij}) becomes diagonal. Let us assume that $v=e_1$ and that $\kappa_1, \ldots, \kappa_n$ are the eigenvalues of h_{ij} at p. Then from $M_{11}=0$ it follows that at p

$$\kappa_1 = \varepsilon H + \frac{n(1-n\varepsilon)}{H} K_1 + \frac{n^2}{H^2} L$$

and we obtain

$$\begin{aligned} N_{ij} v^i v^j = N_{11} \geqq & \frac{2}{H} \sum_{l=2}^{n} \bar{R}_{1l1l}(\kappa_l - \kappa_1) - \frac{2n}{H} L + 2(1-n\varepsilon) K_1 \\ & + \frac{3n}{H} L - \frac{2n^2(1-n\varepsilon)}{H^2} K_1^2 - \frac{3n^3}{H^3} L K_1. \end{aligned}$$

Here we used $|A|^2 \geqq \frac{1}{n} H^2$, $\bar{\mathrm{R}}\mathrm{ic}(v,v) \geqq -nK_1$, and $|\bar{\nabla}_\alpha \bar{R}_{\beta\gamma\delta\sigma}| \leqq L$. Since κ_1 is the smallest eigenvalue of h_{ij} it follows that

$$\begin{aligned} \frac{2}{H} \sum_{l=2}^{n} \bar{R}_{1l1l}(\kappa_l - \kappa_1) & \geqq -\frac{2}{H} K_1 \sum_{l=2}^{n} (\kappa_l - \kappa_1) \\ & = -\frac{2}{H} K_1 (H - n\kappa_1) = -2K_1 + \frac{2n}{H} K_1 \left(\varepsilon_1 H + \frac{n(1-n\varepsilon_1)}{H} K_1 + \frac{n^2}{H^2} L \right). \end{aligned}$$

Thus we obtain

$$N_{ij} v^i v^j \geqq \frac{n}{H} L - \frac{n^3}{H^3} L K_1 \geqq 0$$

by Lemma 4.1 and the Theorem follows.

5. The pinching estimate

We will show that the eigenvalues of the second fundamental form come close together if the mean curvature becomes very large.

5.1 Theorem. *There are constants $\delta > 0$ and $C_0 < \infty$ depending only on M_0 and the curvature bounds K_1, K_2, L and $i(N)$ such that*

$$|A|^2 - \frac{1}{n} H^2 \leqq C_0 \cdot H^{2-\delta}$$

holds on $0 \leqq t < T$.

Proof. We want to bound the function

$$f_\sigma = \frac{|A|^2 - \frac{1}{n} H^2}{H^{2-\sigma}}$$

for some small $\sigma > 0$. Using the evolution equations in §3 we derive similar as in [3]

5.2 Lemma. *Let $\alpha = 2 - \sigma$. Then for any σ*

$$\begin{aligned}\frac{\partial}{\partial t} f_\sigma = \Delta f_\sigma &+ \frac{2(\alpha-1)}{H} \langle \nabla_l H, \nabla_l f_\sigma \rangle \\ &- \frac{2}{H^{\alpha+2}} |\nabla_i H \cdot h_{kl} - \nabla_i h_{kl} \cdot H|^2 - \frac{(2-\alpha)(\alpha-1)}{H^{\alpha+2}} \left(|A|^2 - \frac{1}{n} H^2\right) |\nabla H|^2 \\ &+ (2-\alpha)(|A|^2 + \bar{\mathrm{R}}\mathrm{ic}(\nu,\nu)) f_\sigma \\ &- \frac{1}{H^\alpha} [4(h^{ij} h_{jl} \bar{R}^l{}_{mi}{}^m - h^{ij} h^{lm} \bar{R}_{iljm}) + h^{ij}(\bar{\nabla}_j \bar{R}_{0li}{}^l + \bar{\nabla}_l \bar{R}_{0ij}{}^l)].\end{aligned}$$

We now need the following consequences of inequality (7b) and Theorem 4.2.

5.3 Lemma. *If $H > 0$ and (7b) is valid with some $\varepsilon_1 > 0$, then*

(i) $Z \geqq n \varepsilon_1^2 H^2 \left(|A|^2 - \frac{1}{n} H^2\right)$

(ii) $|\nabla_i h_{kl} \cdot H - \nabla_i H h_{kl}|^2 \geqq \frac{1}{4} \varepsilon_1^2 H^2 |\nabla H|^2 - \varepsilon_1^{-2} c_n \max(K_1^2, K_2^2) H^2$

where c_n here and in the following denotes a constant only depending on n.

Proof. This is a generalization of the result in ([3], Lemma 2.3). The proof of the first inequality carries over unchanged and to obtain the second inequality we estimate

$$\begin{aligned}|\nabla_i h_{kl} H - \nabla_i H h_{kl}|^2 &\geqq \tfrac{1}{4} |(\nabla_i h_{kl} - \nabla_k h_{il}) H - (\nabla_i H h_{kl} - \nabla_k H h_{il})|^2 \\ &= \tfrac{1}{4} |\bar{R}_{0lki} H - (\nabla_i H h_{kl} - \nabla_k H h_{il})|^2.\end{aligned}$$

Rotating now the coordinates as in [3] such that $\nabla H = e_1 |\nabla H|$, we see that this is larger than

$$\begin{aligned}&\tfrac{1}{4} |\bar{R}_{0221} H - |\nabla H| h_{22}|^2 + \tfrac{1}{4} |\bar{R}_{0212} H + |\nabla H| h_{22}|^2 \\ &\quad \geqq \tfrac{1}{2} \varepsilon_1^2 H^2 |\nabla H|^2 + \tfrac{1}{2} H^2 \bar{R}_{0212}^2 + H h_{22} |\nabla H| \bar{R}_{0212} \\ &\quad \geqq \tfrac{1}{4} \varepsilon_1^2 H^2 |\nabla H|^2 - \varepsilon_1^{-2} H^2 \bar{R}_{0212}^2 \\ &\quad \geqq \tfrac{1}{4} \varepsilon_1^2 H^2 |\nabla H|^2 - c_n \varepsilon_1^{-2} \max(K_1^2, K_2^2) H^2\end{aligned}$$

since $h_{22} \geqq \varepsilon_1 H$ by assumption.

Choosing now again coordinates such that at a fixed point we have $h_{ij} = \kappa_i \delta_{ij}$ we get (see also [5], 1.24)

$$h^{ij} h_{jl} \bar{R}^{l}{}_{mi}{}^{m} - h^{ij} h^{lm} \bar{R}_{iljm} = \sum_{l<m} (\kappa_l - \kappa_m)^2 \bar{R}_{lmlm} \tag{8}$$

$$\geqq -K_1 \sum_{l<m} (\kappa_l - \kappa_m)^2 = -n K_1 \left(|A|^2 - \frac{1}{n} H^2 \right).$$

Furthermore we have

$$h^{ij} (\bar{\nabla}_j \bar{R}_{0li}{}^{l} + \bar{\nabla}_l \bar{R}_{0ij}{}^{l}) = \mathring{h}^{ij} (\bar{\nabla}_j \bar{R}_{0li}{}^{l} + \bar{\nabla}_l \bar{R}_{0ij}{}^{l}) \tag{9}$$

where $\mathring{h}_{ij} = h_{ij} - \frac{1}{n} H g_{ij}$ is the traceless second fundamental form. We have $|\mathring{h}_{ij}|^2 = |A|^2 - \frac{1}{n} H^2$ and combining (8), (9) with Lemma 5.3, we derive from Lemma 5.2

5.4 Corollary. *We have the inequality*

$$\frac{\partial}{\partial t} f_\sigma \leqq \Delta f_\sigma + \frac{2(\alpha-1)}{H} \langle \nabla_l H, \nabla_l f_\sigma \rangle - \frac{1}{2} \varepsilon_1^2 \frac{1}{H^\alpha} |\nabla H|^2$$

$$+ \sigma |A|^2 f_\sigma + C \frac{1}{H^\alpha} + C f_\sigma$$

where C only depends on n, ε_1, K_1, K_2 and L.

We want to exploit the negative term on the right hand side involving $|\nabla H|^2$. First we conclude from Lemma 2.1 (ii) that

$$\tfrac{1}{2} \Delta |A|^2 \geqq \langle h_{ij}, \nabla_i \nabla_j H \rangle + Z + |\nabla A|^2$$
$$- C H^2 - C$$

where $C = C(n, K_1, K_2, L)$. Then it follows that

$$\Delta f_\sigma \geqq \frac{2}{H^\alpha} \langle \mathring{h}_{ij}, \nabla_i \nabla_j H \rangle + \frac{2}{H^\alpha} Z$$

$$- \frac{2(\alpha-1)}{H} \langle \nabla_l H, \nabla_l f_\sigma \rangle - \frac{\alpha}{H} f_\sigma \Delta H$$

$$- C H^{2-\alpha} - C H^{-\alpha}$$

and we derive as in ([3], Lemma 5.4) for any $p \geqq 2$, $\eta > 0$

$$n \varepsilon_1^2 \int f_\sigma^p H^2 \, d\mu \leqq (2\eta p + 5) \int \frac{1}{H^\alpha} f_\sigma^{p-1} |\nabla H|^2 \, d\mu$$

$$+ \eta^{-1}(p-1) \int f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu + c \int |\nabla H| \frac{1}{H^\alpha} f_\sigma^{p-1} \, d\mu$$

$$+ C \int H^2 f_\sigma^{p-1} \, d\mu$$

where C depends on n, K_1, K_2, L and $H_{\min}^{-1}(0)$. Using now Young's inequality

$$xy \leqq \varepsilon x^p + \varepsilon^{-q/p} y^q, \qquad \varepsilon > 0,\ \frac{1}{p} + \frac{1}{q} = 1$$

we obtain

5.5 Lemma. *Let $p \geqq 2$. Then for any $\eta > 0$ and any $0 \leqq \sigma \leqq \frac{1}{2}$ we have the estimate*

$$\tfrac{1}{2} n \varepsilon_1^2 \int f_\sigma^p H^2 \, d\mu \leqq (2\eta p + 5) \int \frac{1}{H^\alpha} f_\sigma^{p-1} |\nabla H|^2 \, d\mu$$
$$+ \eta^{-1}(p-1) \int f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu + C^p$$

where C depends on ε_1, M_0, K_1, K_2 and L.

Now we can bound L^p-norms of f_σ.

5.6 Lemma. *There is a constant $C_1 < \infty$ depending only on M_0, K_1, K_2 and L such that for all*

$$\begin{aligned} &p \geqq 200 \varepsilon_1^{-2} \\ &\sigma \leqq n 2^{-5} \varepsilon_1^3 p^{-\frac{1}{2}} \end{aligned} \tag{10}$$

we have the estimate

$$\Big(\int_{M_t} f_\sigma^p \, d\mu \Big)^{1/p} \leqq C_1, \qquad 0 \leqq t < T.$$

Proof. Using the same calculations as in ([3], Lemma 5.5) we obtain from Corollary 5.4 and Lemma 5.5 for σ and p as in (10)

$$\frac{\partial}{\partial t} \int_{M_t} f_\sigma^p \, d\mu \leqq p \cdot C \int_{M_t} f_\sigma^p \, d\mu + p\, C^p$$

where C depends on M_0, K_1, K_2 and L. Thus

$$\sup_{[0,T)} \int_{M_t} f_\sigma^p \, d\mu \leqq \int f_\sigma^p \, d\mu \big|_{t=0} + p\, C^p T e^{CT}$$

and the conclusion follows from Lemma 4.1.

To proceed further, we need a Sobolev inequality for submanifolds of Riemannian manifolds, which was derived in [2]. In our case it takes the form

5.7 Lemma. *Let v be a Lipschitz function on M. Then*

$$\Big(\int_M |v|^{\frac{n}{n-1}} \, d\mu \Big)^{\frac{n-1}{n}} \leqq c_n \Big\{ \int_M |\nabla v| \, d\mu + \int_M H |v| \, d\mu \Big\}$$

provided

$$K_2^2 (1-\alpha)^{-\frac{2}{n}} (\omega_n^{-1} |\operatorname{supp} v|)^{\frac{2}{n}} \leqq 1$$

and

$$2\rho_0 \leqq i(N)$$

where ω_n is the volume of the unit ball and

$$\rho_0 = K_2^{-1} \arcsin \{ K_2 (1-\alpha)^{-\frac{1}{n}} (\omega_n^{-1} |\operatorname{supp} v|)^{\frac{1}{n}} \}.$$

Here α is a free parameter, $0<\alpha<1$, *and*

$$c_n=\pi 2^{n-1}\alpha^{-1}(1-\alpha)^{-\frac{1}{n}}\frac{n}{n-1}\omega_n^{-\frac{1}{n}}.$$

Now let $f_{\sigma,k}=\max(f_\sigma-k,0)$ for all $k\geqq k_0=\sup_{M_0} f_\sigma$ and denote by $A(k)$ the set where $f_\sigma>k$. If we set $v=f_{\sigma,k}^{p/2}$ for $p\geqq 200\varepsilon_1^{-2}$ then we derive as in [3] from Corollary 5.4

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{A(k)} v^2\,d\mu+\int_{A(k)}|\nabla v|^2\,d\mu\\ &\leqq \sigma p\int_{A(k)} H^2 f_\sigma^p\,d\mu+C_p\int_{A(k)}\frac{1}{H^\alpha}f_{\sigma,k}^{p-1}\,d\mu+C_p\int_{A(k)} f_\sigma^p\,d\mu\\ &\leqq C_p\int_{A(k)} H^2 f_\sigma^p\,d\mu\end{aligned}$$

where C depends on M_0, K_1, K_2 and L. We have from Lemma 5.6

$$|A(k)|=\int_{A(k)} d\mu\leqq\frac{1}{k}\int_M f_\sigma\,d\mu\leqq\frac{1}{k}C$$

where C depends on C_1 and $|M_0|$. Thus we can choose $k_1\geqq k_0$ so large that the conditions in Lemma 5.7 for $|A(k)|=|\operatorname{supp} v|$ are satisfied. Then k_1 depends on k_0, $i(N)$, M_0, K_1, K_2, L and we can now apply the Sobolev inequality as in [3] to derive a bound for f_σ, if σ is small.

6. The gradient bound

The gradient estimate for the mean curvature in [3] is also valid in the context of Riemannian manifolds.

6.1 Theorem. *For any $\eta>0$ there is a constant $C_\eta<\infty$ depending on η, C_0, δ, M_0, n, K_1, K_2 and L such that*

$$|\nabla H|^2\leqq\eta H^4+C_\eta.$$

Proof. Proceeding as in ([3], Lemma 6.1) and observing that

$$\Delta(\nabla_k H)=\nabla_k(\Delta H)+g^{ij}\nabla_i H(H h_{kj}-h_{km}g^{mn}h_{nj}+\bar R_{kj}),$$

$$\nabla_i(\overline{\mathrm{Ric}}(v,v))=\bar\nabla_i\bar R_{0l0}{}^l+2\bar R_{ml0}{}^l h^m{}_i$$

we obtain

6.2 Lemma. *We have the evolution equation*

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla H|^2&=\Delta|\nabla H|^2-2|\nabla^2 H|^2+2|A|^2|\nabla H|^2\\ &\quad+2\langle\nabla_i H h_{mj},\nabla_j H h_{im}\rangle+2H\langle\nabla_i H,\nabla_i|A|^2\rangle\\ &\quad+2\overline{\mathrm{Ric}}(v,v)|\nabla H|^2-2\bar R_{ij}\nabla^i H\nabla^j H\\ &\quad+2H\langle\bar\nabla_i R_{0l0}{}^l,\nabla_i H\rangle+4H\langle\bar R_{ml0}{}^l h^m{}_i,\nabla_i H\rangle.\end{aligned}$$

6.3 Corollary. *We have the estimate*

$$\frac{\partial}{\partial t}|\nabla H|^2 \leqq \Delta |\nabla H|^2 - 2|\nabla^2 H|^2 + 6|A|^2|\nabla H|^2 + 2H\langle \nabla_i H, \nabla_i |A|^2\rangle + C|\nabla H|^2 + CH^2$$

where C depends on K_1, K_2 and L.

6.4 Lemma. *We have*

(i) $\frac{\partial}{\partial t}H^3 \geqq \Delta H^3 - 6H|\nabla H|^2 + 3\varepsilon_2 H^5$

(ii) $\frac{\partial}{\partial t}\left(H\left(|A|^2 - \frac{1}{n}H^2\right)\right) \leqq \Delta\left(H\left(|A|^2 - \frac{1}{n}H^2\right)\right) - \frac{n-1}{2n+1}H|\nabla A|^2 + C_2|\nabla A|^2 + C_3H^3 + 3|A|^2\cdot H\left(|A|^2 - \frac{1}{n}H^2\right)$

where C_2 and C_3 depend on M_0, C_0, K_1, K_2, δ and L.

Proof. (i) We have

$$\frac{\partial}{\partial t}H^3 = \Delta H^3 - 6H|\nabla H|^2 + 3H^3(|A|^2 + \bar{\mathrm{R}}\mathrm{ic}(\nu, \nu))$$

and in view of $|A|^2 \geqq \frac{1}{n}H^2$ the first inequality follows from Lemma 4.1.

(ii) From Lemma (iii) we derive

$$\frac{\partial}{\partial t}\left(H\left(|A|^2 - \frac{1}{n}H^2\right)\right) \leqq \Delta\left(H\left(|A|^2 - \frac{1}{n}H^2\right)\right) - 2\left(|\nabla A|^2 - \frac{1}{n}|\nabla H|^2\right)H - 2\left\langle \nabla_i H, \nabla_i\left(|A|^2 - \frac{1}{n}H^2\right)\right\rangle + 3|A|^2\cdot H\left(|A|^2 - \frac{1}{n}H^2\right) + CH^3$$

where C depends on K_1, K_2, L and $H_{\min}^{-1}(0)$. Using Theorem 5.1 one estimates

$$\begin{aligned}\left|\left\langle \nabla_i H, \nabla_i\left(A^2 - \frac{1}{n}H^2\right)\right\rangle\right| &= 2|\langle \nabla_i H \mathring{h}_{kl}, \nabla_i \mathring{h}_{kl}\rangle| \\ &\leqq 2|\nabla H|\,|\mathring{h}_{kl}|\,|\nabla A| \\ &\leqq 2nC_0^{\frac{1}{2}}H^{1-\delta/2}|\nabla A|^2 \\ &\leqq \frac{n-1}{2n+1}H|\nabla A|^2 + C(n, C_0, \delta)|\nabla A|^2\end{aligned}$$

and the second inequality follows then from Lemma 2.2(ii).

Now proceeding exactly as in [3], we study the function

$$f = \frac{|\nabla H|^2}{H} + P\left(|A|^2 - \frac{1}{n}H^2\right)H + PC_4|A|^2 - \eta H^3$$

where P depending only on N is large and $C_4>0$ depends on K_1, K_2, L and C_2. Using Corollary 6.3, Lemma 6.4 and Corollary 3.5(ii), we obtain as in [3]

$$\frac{\partial}{\partial t} f \leqq \Delta f + C$$

since all terms which do not already occur in the case $N=\mathbb{R}^{n+1}$ are of lower order. Here C depends on η, M_0, C_0, δ, K_1, K_2, L and ε_2. This implies the estimate in Theorem 6.1.

7. Contraction to a point

Let again $0 \leqq t < T < \infty$ be the maximal time interval where the smooth solution of (1) exists.

7.1 Theorem. *The quantity* $\max_{M_t} |A|^2$ *becomes unbounded as* $t \to T$.

Proof. If the Lemma is false, there is some $C_5 < \infty$ such that

$$\max_{M_t} |A|^2 \leqq C_5 \tag{11}$$

on $0 \leqq t < T$. It follows that for $\hat{x} \in U$, $0 < \sigma < \rho < T$

$$\operatorname{dist}(F(\hat{x}, \rho), F(\hat{x}, \sigma)) \leqq \int_\sigma^\rho H(\hat{x}, \tau)\, d\tau \leqq C(\rho - \sigma) \tag{12}$$

and $F(\cdot, t)$ converges uniformly to some continuous limit function $F(\cdot, T)$. We want to show that $F(\cdot, T)$ actually represents a smooth limit surface M_T. This is then a contradiction to the maximality of T in view of the local existence result in Lemma 3.1. In order to show that $F(\cdot, T)$ represents a smooth surface M_T, we have only to establish uniform bounds for all derivatives of the second fundamental form on M_t, $0 \leqq t < T$, (see [3], section 8).

7.2 Lemma. *If* (11) *holds, then for each* $m \geqq 0$ *there is* $C_m < \infty$ *depending on* m, C_5, M_0 *and* N *such that* $\max_{M_t} |\nabla^m A|^2 \leqq C_m$ *for all* $0 \leqq t < T$.

Proof. Since M_t stays in a compact region of N in view of (12), we have $\max_{0 \leqq l \leqq m} |\bar{\nabla}^l \bar{R}m| \leqq \tilde{C}_m$ for fixed constants $\tilde{C}_m$. Now, starting from the evolution equation for A in Theorem 3.4, one derives as in [3] and ([1], §13) evolution equations for all iterated derivatives $\nabla^m A$ and obtains

$$\begin{aligned}\frac{\partial}{\partial t} |\nabla^m A|^2 \leqq{}& \Delta |\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 \\ &+ C(n,m)\Big\{ \sum_{i+j+k=m} |\nabla^i A|\,|\nabla^j A|\,|\nabla^k A|\,|\nabla^m A| \\ &+ \tilde{C}_m \sum_{i \leqq m} |\nabla^i A|\,|\nabla^m A| + \tilde{C}_{m+1} |\nabla^m A| \Big\}.\end{aligned}$$

The generalized Hölder inequality and interpolation yields

$$\frac{d}{dt}\int |\nabla^m A|^2 d\mu + 2\int |\nabla^{m+1} A|^2 d\mu$$
$$\leqq C(\max_{M_t} |A|^2 + 1)\{\int |\nabla^m A|^2 d\mu + (\int |\nabla^m A|^2 d\mu)^{\frac{1}{2}}\}$$

where C depends on n, m and $\tilde{C}_{m+1}$. Then the assertion follows as in [3] from the Sobolev inequality, proving Theorem 7.1.

To proceed further, we need a lower bound for the intrinsic Ricci curvature R_{ij} of the surfaces M_t.

7.3 Lemma. *The intrinsic Ricci curvature* R_{ij} *of* M_t *satisfies*

$$R_{ij} \geqq (n-1)\varepsilon_1 \varepsilon_2 H^2 g_{ij}.$$

Proof. The Ricci curvature on M is given by Gauß' equation

$$R_{ij} = \bar{R}_{ilj}{}^l + H h_{ij} - h_{il} h^l{}_j.$$

Let us suppose that R_{ij} is diagonal at the point of consideration, then $\bar{R}_{ili}{}^l$ is the sum of $(n-1)$ sectional curvatures and therefore larger than $-(n-1)K_1$.

Any eigenvalue of $H h_{ij} - h_{il} h^l{}_j$ is larger than $\frac{n-1}{n} H \kappa_1$, where κ_1 is the smallest eigenvalue of h_{ij}. But from (2) and (7) we obtain

$$H\kappa_1 \geqq \varepsilon_1 (n^2 K_1 + n\varepsilon_2 H^2) + n K_1 - n^2 \varepsilon_1 K_1$$

and the conclusion follows.

Combining now Theorem 6.1, Theorem 7.1 and Lemma 7.3 exactly as in [3], we derive

7.4 Theorem. *We have* $H_{\max}/H_{\min} \to 1$ *as* $t \to T$.

Once this is established it follows from Theorem 7.1 that both $H_{\max}$ and $H_{\min}$ tend to infinity as $t \to T$ and therefore the diameter of M_t tends to zero. Since the injectivity radius of N is bounded from below, there is $\theta < T$ such that M_θ is contained in a ball $B_\rho(p) = \{q \in N \mid \mathrm{dist}_N(p,q) < \rho\}$ where ρ is small compared to $i(N)$ and $(K_1 + K_2)^{-1}$. It is well known that then $B_\rho(p)$ is a convex region. In view of the elliptic maximum principle the M_t's will then stay in $B_\rho(p)$ for all $\theta \leqq t < T$. As $H_{\min} \to \infty$ for $t \to T$, we see from Theorem 5.1 that all ratios of principal curvatures tend to one as $t \to T$. Thus for t close to T, M_t is an imbedded sphere bounding a convex region. The region enclosed by M_{t_2} is contained in the region enclosed by M_{t_1} for $t_2 > t_1 \geqq \theta$ since the surfaces are shrinking and so the M_t's converge to a single point as $t \to T$. The last statement of Theorem 1.1 is proved in exactly the same way as in the euclidean case ([3], §10), since for t close to T all quantities arising from the metric of N are negligible compared to the mean curvature H of the hypersurface.

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