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Finite Yang-Mills integrals

Werner Krauth [a,1], Matthias Staudacher [b,2,3]

[a] *CNRS-Laboratoire de Physique Statistique, Ecole Normale Supérieure, 24, rue Lhomond, F-75231 Paris Cedex 05, France*
[b] *Albert-Einstein-Institut, Max-Planck-Institut für Gravitationsphysik, Schlaatzweg 1, D-14473 Potsdam, Germany*

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Abstract

We use Monte Carlo methods to directly evaluate D-dimensional $SU(N)$ Yang-Mills partition functions reduced to zero Euclidean dimensions, with and without supersymmetry. In the non-supersymmetric case, we find that the integrals exist for $D=3$, $N\geq 4$ and $D=4$, $N\geq 3$ and, lastly, $D\geq 5$, $N\geq 2$. We conclude that the $D=3$ and $D=4$ integrals exist in the large N limit, and therefore lead to a well-defined, new type of Eguchi-Kawai reduced gauge theory. For the supersymmetric case, we check, up to $SU(5)$, recently proposed exact formulas for the $D=4$ and $D=6$ D-instanton integrals, including the explicit form of the normalization factor needed to interpret the integrals as the bulk contribution to the Witten index.

In a recent study [1], we developed Monte Carlo methods in order to deal with dimensionally reduced Yang-Mills theories. We directly work with the gauge potentials, in contradistinction to more conventional numerical studies of lattice gauge theories. Our initial interest was in establishing reliable methods for the numerical calculation of the bulk contribution to the Witten index in supersymmetric field theories. Some further results in this direction are presented below. The methods are actually even more efficient if applied to the non-supersymmetric case, and allow us to settle the question of existence of Yang-Mills theory dimensionally reduced to zero dimensions in a surprising way.

Let us first consider maximally reduced D-dimensional $SU(N)$ Euclidean Yang-Mills theory. The formal functional integral for the partition function then becomes an ordinary integral:

$$\mathcal{Z}_{D,N}=\int\prod_{A=1}^{N^2-1}\prod_{\mu=1}^{D}\frac{dX_\mu^A}{\sqrt{2\pi}}\exp\Big[\tfrac{1}{2}\mathrm{Tr}\,\big[X_\mu,X_\nu\big]\big[X_\mu,X_\nu\big]\Big]. \tag{1}$$

Since there are directions in the integration space which are not suppressed, these integrals were generally

[1] E-mail: krauth@physique.ens.fr.
[2] E-mail: matthias@aei-potsdam.mpg.de.
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believed to be ill-defined. Indeed, it is very easy to show that e.g. for $D = 2$ the integral diverges for all N. Nevertheless, in [1] we obtained an astonishing result: For gauge group $SU(2)$ the exact result reads

$$\mathcal{Z}_{D,2} = \begin{cases} \infty & D \le 4 \\ 2^{-\frac{3}{4}D-1} \dfrac{\Gamma\left(\frac{D}{4}\right)\Gamma\left(\frac{D-2}{4}\right)\Gamma\left(\frac{D-4}{4}\right)}{\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)\Gamma\left(\frac{D-2}{2}\right)} & D \ge 5 \end{cases} \tag{2}$$

Therefore, the reduced bosonic theory is *not necessarily* divergent. It is natural to ask how Eq. (2) generalizes if $N \ge 3$. The methods used to derive Eq. (2) are specific to $SU(2)$, and no analytic result is known for higher gauge groups with $N \ge 3$, except for $D = 2$, as mentioned. However, we can modify the methods of [1] to decide the question of existence by Monte Carlo evaluation. Our results suggest the following intriguing answer:

$$\mathcal{Z}_{D,N} < \infty \quad \text{for} \quad \begin{cases} D = 3 & N \ge 4 \\ D = 4 & N \ge 3 \\ D \ge 5 & N \ge 2 \end{cases}$$

$$Z_{D,N} = \infty \quad \text{otherwise} \tag{3}$$

In particular, this suggests that a well defined large N limit exists in dimensions $D = 3$ and, most interestingly, $D = 4$. It opens the exciting possibility that appropriate large N correlation functions computed for the model (1) can be related to large N Yang-Mills field theory through the Eguchi-Kawai mechanism [2]. For example, one could consider Wilson loop operators such as

$$\mathcal{W}(L,T) = \left\langle \frac{1}{N} \mathrm{Tr}\, e^{iLX_1} e^{iTX_D} e^{-iLX_1} e^{-iTX_D} \right\rangle \tag{4}$$

in the limit $N \to \infty$. In fact, our model Eq. (1) is a new type of continuum (as opposed to lattice) Eguchi-Kawai model. Reduced models are frequently plagued by the need to introduce quenching, but our models are already in the weak coupling phase, so this does not appear to be a problem. However, these important questions are beyond the scope of the present analysis.

Let us explain how to obtain convincing evidence for the result Eq. (3). In [1] we computed *ratios* of integrals for different dimensions D. Such a strategy is rendered necessary by the strong fluctuations of the integrand in Eq. (1). One key point in our procedure [1] consists in *compactifying* the integrals: after introducing polar coordinates $(x_1, \ldots, x_d) = (\Omega_d, R)$ ($d = D(N^2-1)$ being the total dimension of the integral), we exactly perform the R-integration. In the present (bosonic) case of eq(1), $\mathcal{Z}_{D,N}$ can be written as

$$\mathcal{Z}_{D,N} = \frac{\int \mathcal{D}\Omega_d \, z_{D,N}(\Omega_d)}{\int \mathcal{D}\Omega_d} \tag{5}$$

with

$$z_{D,N}(\Omega_d) = 2^{-(N^2-1)\frac{D}{2}-1} \frac{\Gamma\left((N^2-1)\frac{D}{4}\right)}{\Gamma\left((N^2-1)\frac{D}{2}\right)} \times \frac{1}{\left[\mathcal{S}(\Omega_d,1)\right]^{\frac{D}{4}(N^2-1)}} \tag{6}$$

where $\mathcal{S}(\Omega_d,1)$ is the action of the partition sum Eq. (1), restricted to the unit sphere S^{d-1}. To obtain the absolute value of $\mathcal{Z}_{D,N}$, we now consider a series of interpolating functions $z^{\alpha_i}_{D,N}(\Omega_d)$ with $1 = \alpha_0 < \alpha_1 <$

$\ldots < \alpha_{l-1} < \alpha_l = 0$ (notice that $z_{D,N} \geq 0$). These interpolating functions allow us to connect $z_{D,N}(\Omega_d)$ to a constant in much the same way as we can (sometimes) compute the free energy of a statistical physics system by simulating at various temperatures from ∞ down to the temperature of interest. The term with $\alpha_l = 0$ plays the role of the exactly solvable high-temperature limit, since we can integrate the constant function $z_{D,N}^{\alpha_l} = 1$ analytically on the d-sphere. We now write

$$\mathcal{Z}_{D,N} = \left[\frac{\int \mathcal{D}\Omega_d \, z^{\alpha_1} \times z^{\alpha_0 - \alpha_1}}{\int \mathcal{D}\Omega_d \, z^{\alpha_1}}\right]\left[\frac{\int \mathcal{D}\Omega_d \, z^{\alpha_2} \times z^{\alpha_1 - \alpha_2}}{\int \mathcal{D}\Omega_d \, z^{\alpha_2}}\right] \ldots \left[\frac{\int \mathcal{D}\Omega_d \, 1 \times z^{\alpha_{l-1}}}{\int \mathcal{D}\Omega_d \, 1}\right] \tag{7}$$

Each of the terms [ ] in eq(7) is computed in a separate Monte Carlo run, in which points are sampled according to $\pi(\Omega) \sim z^{\alpha_i}$, using the Metropolis algorithm. The fluctuations of the operator to be evaluated, $z^{\alpha_{i-1} - \alpha_i}$, are much damped if $\alpha_{i-1} \sim \alpha_i$. In practice, we have used $l = 10$, corresponding to the number of work stations available to us. Judiciously chosen values of α_i, closely spaced as we approach 1, allow us to compute any finite bosonic integral with ease. As a simple check, we have computed $\mathcal{Z}_{N=2,D=5}$ (cf. Eq. (2)) to within 1% precision in about 30 minutes of individual CPU time on 10 machines.

The finiteness of the integrals was also checked with the extremely powerful *qualitative* Monte Carlo method described in [1]. In that method, we solely monitor the autocorrelation function of the Metropolis random walk. The cases analytically known to diverge were easily identified ($D = 2$ for all N, $D = 2,3,4$ for $N = 2$). Of these cases, the $N = 2, D = 4$ random walk appears less divergent, which agrees with the analytic result that the divergence is marginal (the $N = 2, D = 4 + \epsilon$ integral exists). A similar behavior is observed for $N = 3, D = 3$, which we believe to be marginally divergent as well.

Using the techniques just described, we are able to present Table 1.

The computation becomes simpler both with D (it is completely trivial for $D > 5$) and N. We are thus confident about the statements expressed for large N and arrive at the prediction (3). This is the central observation of the present paper.

Turning now to the supersymmetric case, the integrals reduced to zero dimensions read

$$\mathcal{Z}_{D,N}^{\mathcal{N}} := \int \prod_{A=1}^{N^2-1} \left(\prod_{\mu=1}^{D} \frac{dX_\mu^A}{\sqrt{2\pi}} \right) \left(\prod_{\alpha=1}^{\mathcal{N}} d\Psi_\alpha^A \right) \exp\left[\tfrac{1}{2} \mathrm{Tr}\left[X_\mu, X_\nu\right]\left[X_\mu, X_\nu\right] + \mathrm{Tr}\Psi_\alpha \left[\Gamma_{\alpha\beta}^\mu X_\mu, \Psi_\beta \right] \right] \tag{8}$$

The only possible dimensions are $D = 3,4,6,10$ corresponding to $\mathcal{N} = 2,4,8,16$ real supersymmetries, respectively. Integrating out the fermions, we find an integral that differs from Eq. (1) only by a modified measure:

$$\mathcal{Z}_{D,N}^{\mathcal{N}} = \int \prod_{A=1}^{N^2-1} \prod_{\mu=1}^{D} \frac{dX_\mu^A}{\sqrt{2\pi}} \exp\left[\tfrac{1}{2} \mathrm{Tr}\left[X_\mu, X_\nu\right]\left[X_\mu, X_\nu\right] \right] \mathcal{P}_{D,N}(X) \tag{9}$$

The integrand is weighted by a very special homogeneous polynomial $\mathcal{P}_{D,N}$ of degree $k = (D-2)(N^2-1) = \frac{1}{2}\mathcal{N}(N^2-1)$ in the variables X_μ^A. See e.g. [1] for further details. In a very recent calculation, Moore et.al.

Table 1
Direct evaluation of bosonic Yang-Mills integrals

N	$D = 2$	$D = 3$	$D = 4$
2	∞	∞	∞
3	∞	∞	$1.9 \cdot 10^{-3}$
4	∞	$6.9 \cdot 10^{-3}$	$2.9 \cdot 10^{-8}$
5	∞	$9.9 \cdot 10^{-7}$	$2.1 \cdot 10^{-15}$
6	∞	$4.8 \cdot 10^{-12}$	$4.1 \cdot 10^{-25}$

[3], heavily using cohomological field theory techniques (and therefore supersymmetry), evaluated, for $D = 4,6,10$ analytically a class of related integrals which are initially defined in Minkowski space and are Wick rotated in the course of the evaluation. These cohomologically deformed integrals are argued, after an appropriate prescription for the Wick rotation is found, to take the same values as the integrals (9). They find the integrals to be of the form

$$\mathcal{Z}_{D,N}^{\mathcal{N}} = \mathcal{F}_N \times \begin{cases} \dfrac{1}{N^2} & D = 4, \quad \mathcal{N} = 4 \\ \dfrac{1}{N^2} & D = 6, \quad \mathcal{N} = 8 \\ \displaystyle\sum_{m|N} \frac{1}{m^2} & D = 10, \quad \mathcal{N} = 16 \end{cases} \tag{10}$$

Let us point out that the $D = 10$ result had previously been conjectured by Green and Gutperle [4]. For $SU(3)$, we had also performed a careful numerical check of the form of Eq. (10) in [1]. $\mathcal{F}_N$ is a group-theoretic factor not worked out explicitly in [3]. In fact, it is unclear to us how to derive the explicit form of $\mathcal{F}_N$ from their approach. In [1], it was found to be

$$\mathcal{F}_N = \frac{2^{\frac{N(N+1)}{2}} \pi^{\frac{N-1}{2}}}{2\sqrt{N} \prod_{i=1}^{N-1} i!} \tag{11}$$

It is important to stress again our finding that the Euclidean supersymmetric partition sums (8), (9) are not some formal mathematical entities, but perfectly converging ordinary multiple integrals, just as their bosonic counterparts. It is very tempting to speculate here once more that appropriate correlation functions computed for $N \to \infty$ could be related to correlators of the full susy gauge field theories.

For completeness, let us sketch the derivation of (11), in particular since its precise relationship with the non-explicit normalization of [3] remains obscure. In [5] is was argued how the D-dimensional *Euclidean* matrix model emerges from the functional integral for the supersymmetric gauge quantum mechanics of $D-1$ matrices when computing the bulk part of the Witten index

$$\lim_{\beta \to 0} \mathrm{Tr}\,(-1)^F e^{-\beta H} \tag{12}$$

One needs to integrate over the group $SU(N)$ in order to project onto gauge invariant states. This integration becomes, in the limit $\beta \to 0$, an integration over the hermitean generators of the group

$$\mathcal{D}U \to \frac{1}{\mathcal{F}_N} \prod_{A=1}^{N^2-1} \frac{dX_D^A}{\sqrt{2\pi}} \tag{13}$$

where $\mathcal{F}_N$ is precisely the constant relating the bulk part of the index to the Euclidean matrix model. It is easily found if we investigate how the normalized Haar measure $\mathcal{D}U$ and the flat measure $\prod_{A=1}^{N^2-1} dX_D^A$ act on class functions. The latter depend only on the eigenvalues $z_i = e^{i\lambda_i}$ of the unimodular unitary matrices U. One easily checks (e.g. by verifying orthonormality of group characters) that group integration on class functions $f(U)$ is performed as

$$\int \mathcal{D}U f(U) = \frac{1}{N!} \oint \prod_{k=1}^{N} \frac{dz_k}{2\pi i z_k} \Delta(z_i)\, \Delta(z_i^*)\, 2\pi \delta_p(\lambda_1 + \ldots \lambda_N) f(z_1, \ldots, z_N) \tag{14}$$

where δ_p is the 2π-periodic δ-function and $\Delta(z_i) = \prod_{i<j}(z_i - z_j)$. On the other hand, the flat measure for the hermitean matrices X_D becomes

$$\prod_{A=1}^{N^2-1} \frac{dX_D^A}{\sqrt{2\pi}} = \frac{2^{\frac{N(N-1)}{2}}}{\prod_{i=1}^{N} i!} \sqrt{N} \prod_{k=1}^{N} \frac{d\lambda_k}{\sqrt{\pi}} \Delta^2(\lambda_i) \sqrt{\pi}\, \delta(\lambda_1 + \ldots + \lambda_N) \tag{15}$$

where the factors are easily checked by computing a Gaussian integral normalized to one (the λ_i are the eigenvalues of X_D). Then, replacing (since $\beta \to 0$, see [5])

$$z_i = 1 + i\lambda_i + \ldots \tag{16}$$

comparing (14) and (15) and multiplying by an extra factor of N, due to the fact that the unitary matrices localize on N values because of invariance under the center of the group, we can read off from Eq. (13) the result Eq. (11) for $\mathcal{I}_N$.

In order to test the group factor (11) as well as the form (10) of the $D = 4$ bulk part of the Witten index, we have applied the methods of direct Monte Carlo evaluation explained above to the $D = 4$ and $N \leq 5$ integrals. As conjectured in [1], we find that the integrand is always positive if $D = 4$. This means that we can use the method of direct evaluation described earlier. The results are presented in Table 2.

We have also checked that the $D = 6$ integrals equal the $D = 4$ integrals to good precision by the ratio method of [1].

Finally, let us mention the open problem of the evaluation of the supersymmetric $D = 3$ integral. In [1], we conjectured

$$\mathcal{I}_{D=3,N}^{\mathcal{N}=2} = 0 \tag{17}$$

This result is trivial for N even, but non-trivial for odd N. We suspect that the supersymmetric $D = 3$ integrals are absolutely convergent (except for $SU(2)$, and possibly $SU(3)$), just like their non-supersymmetric counterparts, cf. Eq. (3), and that their well-defined value is given by Eq. (17) [6]. Another interesting conjecture was presented in [3], where it was suggested that a modified $D = 3$ integral $\tilde{\mathcal{I}}_{D=3,N}^{\mathcal{N}=2}$, where the action is complemented by a Chern-Simons term, leads to

$$\tilde{\mathcal{I}}_{D=3,N}^{\mathcal{N}=2} = \mathcal{I}_N \frac{1}{N^2} \tag{18}$$

Actually, for the solvable case of $SU(2)$, it can be shown that Eq. (18) is *not* true, since one easily finds $\tilde{\mathcal{I}}_{D=3,N=2}^{\mathcal{N}=2} = \infty$. However, our bosonic result (3) suggests that the susy integrals modified by Chern-Simons exist for at least $N \geq 4$, and therefore it is feasible to test the conjecture Eq. (18) for generic gauge groups by our approach [6].

Table 2
Direct evaluation of the $D = 4$ D-instanton integral

N	Eqs. (10), (11)	Monte Carlo	Error
2	1.25	1.25	< 0.01
3	3.22	3.22	< 0.01
4	7.42	7.6	± 0.2
5	10.04	10.2	± 0.2

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