

SUPERMEMBRANES AND M(ATRIX) THEORY^a

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Abstract

In these lectures, we review the $d = 11$ supermembrane and supersymmetric matrix models at an introductory level. We also discuss some more recent developments in connection with non-perturbative string theory.

1 Introduction

The purpose of these lectures is to give an introduction to supermembranes, with special emphasis on the maximally extended $d = 11$ theory, to supersymmetric matrix models, and to explain the relation between these theories. In doing so, we will not only review “old” results, but also discuss some of the more recent developments. Although we cannot give all the technical details, we will try to be pedagogical and to concentrate on what we consider the salient and most important points.

As is by now well known, the large N limit of the maximally supersymmetric $SU(N)$ matrix model of supersymmetric quantum mechanics is a serious candidate for M-Theory, the still elusive theory unifying $d = 11$ supergravity and superstring theory at the non-perturbative level. The very same model had been encountered more than ten years ago in a study of supermembranes in the light-cone gauge, where the supersymmetric $SU(N)$ matrix model was proposed as a non-perturbative regularization of the supermembrane, from which the full quantum supermembrane can be obtained by taking the limit $N \rightarrow \infty$. Notwithstanding some subtle interpretational distinctions, there is thus no real difference between what is now called M(atr)ix theory and the quantum supermembrane, provided a unique large N limit can be shown to exist. Rather, the

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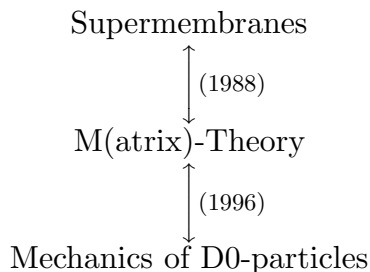


Fig. 1: Two approaches to M(atrrix) Theory

remarkable fact is that the same model can be arrived at in two so different ways. A crucial new insight occasioned by advances in D-branes, which will receive due emphasis in these lectures, is that the quantum supermembrane is a second quantized theory from the very outset.

Despite the recent excitement, however, we do not think that M(atrrix) theory and the $d = 11$ supermembrane in their present incarnation are already the final answer in the search for M-Theory, even though they probably are important pieces of the puzzle. There are still too many ingredients missing that we would expect the final theory to possess. For one thing, we would expect a true theory of quantum gravity to exhibit certain pregeometrical features corresponding to a “dissolution” of space-time and the emergence of some kind of non-commutative geometry at short distances; although the matrix model does achieve that to some extent by replacing commuting coordinates by non-commuting matrices, it seems to us that a still more radical departure from conventional ideas about space and time may be required in order to arrive at a truly background independent formulation (the matrix model “lives” in nine *flat* transverse dimensions only). Furthermore, there should exist some huge and so far completely hidden symmetries generalizing not only the duality symmetries of extended supergravity and string theory, but also the principles underlying general relativity.

2 Basics of supermembranes

Since there exist several reviews of supermembrane theory^{1,2,3,4,5} we here only summarize the basic facts, referring readers there for full details and complementary points of view. By definition, a (super-) p -brane is a p -dimensional extended object moving in a target (super)space whose bosonic d -dimensional

subspace can be curved (subject to certain consistency conditions), but will be taken to be flat \mathbb{R}^d for simplicity here. The dimension of the fermionic subspace is quite generally determined by the number of components of a spinor in d dimensions (possibly with extra factors of $1/2$ or $1/4$ for Majorana and/or Weyl spinors). Unlike the bosonic case, where p and d can be chosen more or less at will, the number of possibilities for supersymmetric extended objects is quite limited (the allowed dimensions are listed in a “brane scan”⁶). This is essentially because the number of spinor components grows exponentially with dimension, whereas the number of components of a vector grows only linearly, and it becomes impossible to match bosonic and fermionic degrees of freedom once d gets too large.

We parametrize the $(p + 1)$ -dimensional world volume by local coordinates

$$\zeta^i = (\zeta^0, \zeta^r) \equiv (\tau, \sigma^r)$$

with indices $r, s, \dots = 1, \dots, p$ labeling the spacelike coordinates on the p -brane. Accordingly, each point in the world volume is mapped to a point in target superspace according to

$$\zeta \longmapsto (X^\mu(\zeta), \theta_\alpha(\zeta))$$

The space time indices μ, ν, \dots run over $0, 1, \dots, d - 1$, and the indices α, β label the components of a spinor in d dimensions. In these lectures we will almost exclusively be concerned with $p = 2$, i.e. supermembranes, which, even as classical theories, can only exist in target spaces of dimension 4,5,7 and 11. Quantum mechanically, there will be further restrictions, just as for the superstring, such that $d = 11$ is presumably the only viable candidate for a consistent quantum supermembrane. Therefore $d = 11$ is the most interesting case, also because it is related to the unique and maximal supergravity theory in eleven dimensions;⁷ in this case, there are 32 real fermionic coordinates θ_α (i.e. $\alpha, \beta, \dots = 1, \dots, 32$) corresponding to the components of a Majorana spinor in eleven dimensions. Note, however, that from the world volume point of view, θ_α transforms as a scalar. This is a general feature of Green Schwarz type actions.

To construct the action of the supermembrane, one starts from the well-known Nambu-Goto action principle:

$$\text{action} = \text{world volume}$$

We thus first of all need an expression for the induced metric on the world volume allowing us to define a volume element. This is simply obtained by pulling back the target space metric using the coordinate functions:

$$g_{ij}(X, \theta) = E_i^\mu E_j^\nu \eta_{\mu\nu}$$

where the $\{E_i^\mu | i = 0, 1, 2\}$ form a dreibein tangent to the world volume. For a target *superspace*, this dreibein becomes a supervielbein with extra fermionic components $\partial_i\theta$. The bosonic part of the vielbein for the supermembrane reads

$$E_i^\mu := \partial_i X^\mu + \bar{\theta}\Gamma^\mu\partial_i\theta.$$

The 32×32 matrices Γ^μ generate the target space Clifford algebra, i.e.

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$$

These ingredients are all that is needed to write down the supermembrane action in a flat eleven-dimensional target space^{8,1}

$$\begin{aligned} \mathcal{L} = & -\sqrt{-g(X, \theta)} - \\ & -\epsilon^{ijk} \left[\frac{1}{2} \partial_i X^\mu (\partial_j X^\nu + \bar{\theta}\Gamma^\nu\partial_j\theta) + \frac{1}{6} \bar{\theta}\Gamma^\mu\partial_i\theta\bar{\theta}\Gamma^\nu\partial_j\theta \right] \bar{\theta}\Gamma_{\mu\nu}\partial_k\theta \end{aligned} \quad (2.1)$$

which represents a generalization of the Green Schwarz action for the superstring (which exists for target space dimensions $d = 3, 4, 6$ and 10). This action is not so easy to guess, and is most conveniently derived in the general superspace formulation of⁸ which also exists for non-trivial backgrounds. The second term can be interpreted as a WZW term in the target superspace.⁹

Of course, this action will have extra terms once a non-trivial supergravity background is switched on; consistency then requires that the background fields satisfy the equations of motion of $d = 11$ supergravity^{8,1} (see¹⁰ for a more recent analysis of the supermembrane in a curved background). Note also that for simplicity we have set the inverse membrane tension T_p , which multiplies the action to unity. This parameter can be easily put back into all formulas by simple dimensional arguments. Let us mention that we can alternatively treat the world volume metric g_{ij} as an independent variable on the $(p + 1)$ dimensional world volume. Solving the equations of motion that follow from

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}g^{ij}E_i^\mu E_{j\mu} + \frac{1}{2}(p-1)\sqrt{-g}$$

for the world volume metric, and substituting the on-shell metric, brings us back to the previous Nambu-Goto-like action. Unfortunately, this is not enough to set up a formulation of the supermembrane which would be the world volume analog of the Ramond and Neveu-Schwarz formulations of the superstring.

We also record the Euler-Lagrange equations of motion following from the above Lagrangian

$$\begin{aligned}\partial_i(\sqrt{-g}g^{ij}E_j^\mu) &= \epsilon^{ijk}E_j^\nu\partial_j\bar{\theta}\Gamma^\mu{}_\nu\partial_k\theta, \\ (\mathbb{1} + \Gamma)g^{ij}\mathbb{F}_i\partial_j\theta &= 0\end{aligned}$$

where the field dependent matrix

$$\Gamma := \frac{\epsilon^{ijk}}{6\sqrt{-g}}E_i^\mu E_j^\nu E_k^\rho \Gamma_{\mu\nu\rho}$$

obeys $\Gamma^2 = \mathbb{1}$, whence $\mathbb{1} \pm \Gamma$ are projection operators.

The most important feature of the above action are its symmetries. The global ones are determined by the (bosonic and fermionic) Killing symmetries of the $d = 11$ background geometry in which the supermembrane moves. For a *flat* $d = 11$ target space, they correspond to the target space super-Poincaré transformations

$$\begin{aligned}\delta X^\mu &= a^\mu + \omega^{\mu\nu}X_\nu - \bar{\epsilon}\Gamma^\mu\theta \\ \delta\theta &= \frac{1}{4}\omega_{\mu\nu}\Gamma^{\mu\nu}\theta + \epsilon\end{aligned}\tag{2.2}$$

As is well known, the global invariances give rise to conserved charges. For instance, the current associated with $d = 11$ target space supersymmetry reads

$$\begin{aligned}J^i &= -2\sqrt{-g}g^{ij}\mathbb{F}_j\theta - \epsilon^{ijk}\left\{E_j^\mu E_k^\nu\Gamma_{\mu\nu}\theta + \right. \\ &\quad \left. + \frac{4}{3}\left[\Gamma^\nu\theta(\bar{\theta}\Gamma_{\mu\nu}\partial_j\theta) + \Gamma_{\mu\nu}\theta(\bar{\theta}\Gamma^\nu\partial_j\theta)\right]\left(E_k^\mu - \frac{2}{5}\bar{\theta}\Gamma^\mu\partial_k\theta\right)\right\}\end{aligned}\tag{2.3}$$

such that the global supersymmetry variations are generated by the Noether supercharges

$$Q = \int d^2\sigma J^0$$

The other currents are given by standard expressions and are not presented here.

The local (gauge) symmetries are associated with the invariance under reparametrizations of the world volume coordinates along a vector field ξ and a fermionic κ -symmetry^{8,11,12}

$$\begin{aligned}\delta X^\mu &= \xi^i\partial_i X^\mu + \bar{\kappa}(\mathbb{1} - \Gamma)\Gamma^\mu\theta \\ \delta\theta &= \xi^i\partial_i\theta + (\mathbb{1} - \Gamma)\kappa\end{aligned}$$

Here κ is again a 32 component Majorana spinor. The WZW term in the Lagrangian is essential for κ -symmetry to hold. As already pointed out above, the combination $(\mathbb{1} - \Gamma)$ is a projector: It eliminates half of the spinor components of κ . This means that κ -symmetry halves the number of physical spinor components. The κ invariance requires the Fierz identity

$$\bar{\psi}_{[1}\Gamma^\mu\psi_2\bar{\psi}_3\Gamma_{\mu\nu}\psi_4] = 0 \quad (2.4)$$

which holds only in $d = 4, 5, 7, 11$ dimensions (this identity is analogous to the one in supersymmetric gauge theories¹³ and the standard Green Schwarz action, which leads to the values $d = 3, 4, 6, 10$). This is the technical reason why supermembranes exist only in dimensions $d = 4, 5, 7$, and 11. As we already said, in these lectures we shall restrict ourselves mostly to the case $d = 11$, since this is the one relevant to M-theory.

The local invariances allow us to eliminate unphysical (i.e. off-shell) degrees of freedom such that on-shell, we are left with an equal number of bosonic and fermionic degrees of freedom, as required by supersymmetry. Namely, the reparametrization invariance can be used to eliminate (on the mass-shell) three polarizations of X^μ tangential to the world volume. This gauge is not the same as the light-cone gauge that we will discuss later, but gives the right count more quickly. For the fermions, we invoke κ symmetry to get rid of one half of them as explained above. Altogether, we then end up with the following count of degrees of freedom in the allowed dimensions for the supermembranes.

	$d = 4$	$d = 5$	$d = 7$	$d = 11$
X^μ	$4 \rightarrow 1$	$5 \rightarrow 2$	$7 \rightarrow 4$	$11 \rightarrow 8$
θ	$4 \rightarrow 2$	$8 \rightarrow 4$	$16 \rightarrow 8$	$32 \rightarrow 16$

This achieves the matching on the mass shell, because, canonically, half of the (real) spinor components must be treated as coordinates and the other half as momenta. Thus we get another factor of 1/2.

We had to fix all gauge degrees of freedom to make this counting work and to exhibit the supersymmetry of the spectrum explicitly. It is one of the outstanding problems to find auxiliary fields which would also match the degrees of freedom off-shell. Solving it is probably as hard as finding an off-shell formulation for $N = 8$ supergravity (still unknown after 20 years). In fact, it is likely that no off-shell formulation in the conventional sense (with finitely many auxiliary fields) exists at all. Nevertheless, some equivalent of an off-shell version of the supermembrane or an equally powerful tool is needed if we want to settle the problem of renormalizability vs. non-renormalizability once and for all, for instance by simply excluding the existence of supersymmetric counterterms depending on the world volume curvature.

3 Light-cone gauge

The gauge which is traditionally chosen to analyze the physical content of a gauge theory is the so-called lightcone gauge. The Hamiltonian formulation of the bosonic membrane in this gauge was given in^{14,15}. For the supermembrane this gauge was first studied in^{16,17}. To implement it, we make partial use of the world volume reparametrization invariance to impose the lightcone gauge condition

$$X^+(\zeta) = X_0^+ + \tau \iff \partial_i X^+ = \delta_{i0},$$

Here we have introduced the standard lightcone coordinates

$$X^\pm = \frac{1}{\sqrt{2}}(X^{10} \pm X^0).$$

We will denote transverse coordinates by $\vec{X}(\zeta) = X^a(\zeta)$, with $a = 1, \dots, 9$. This has reduced the number of bosonic coordinates from 11 to 9. Similarly, we define

$$\Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^{10} \pm \Gamma^0)$$

Next, we use κ -symmetry to eliminate 16 of the 32 fermionic coordinates by imposing

$$\Gamma^+ \theta = 0.$$

With these substitutions, we find

$$\begin{aligned} g_{rs} &\equiv \bar{g}_{rs} = \partial_r X^a \partial_s X^a \equiv \partial_r \vec{X} \cdot \partial_s \vec{X} \\ g_{0r} &\equiv u_r = \partial_r X^- + \partial_0 \vec{X} \partial_r \vec{X} + \bar{\theta} \Gamma^- \partial_r \theta \\ g_{00} &= 2\partial_0 X^- + (\partial_0 \vec{X})^2 + 2\bar{\theta} \Gamma^- \partial_0 \theta. \end{aligned}$$

Now, the Lagrangian simplifies significantly:

$$\mathcal{L} = -\sqrt{\bar{g}\Delta} + \epsilon^{rs} \partial_r X^a \bar{\theta} \Gamma^- \Gamma_a \partial_s \theta$$

Here, we defined $\bar{g} \equiv \det \bar{g}_{rs}$ and $\Delta \equiv -g_{00} + u_r \bar{g}^{rs} u_s$.

Since here our main interest is in studying the relation between the supermembrane and the matrix model, we pass on to the Hamiltonian formulation of the supermembrane without further ado. As a first step, let us work out the

canonical momenta:

$$\begin{aligned}\vec{P} &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \vec{X})} = \sqrt{\frac{\bar{g}}{\Delta}} (\partial_0 \vec{X} - u_r \bar{g}^{rs} \partial_s \vec{X}) \\ P^+ &= \frac{\partial \mathcal{L}}{\partial(\partial_0 X^-)} = \sqrt{\frac{\bar{g}}{\Delta}} \\ S &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \theta)} = -\sqrt{\frac{\bar{g}}{\Delta}} \Gamma^- \theta\end{aligned}$$

After some algebra, the Hamiltonian density is found to be

$$\begin{aligned}\mathcal{H} &\equiv \vec{P} \cdot \partial_0 \vec{X} + P^+ \partial_0 X^- + \bar{S} \partial_0 \theta - \mathcal{L} \\ &= \frac{\vec{P}^2 + \bar{g}}{2P^+} - \epsilon^{rs} \partial_r X^a \bar{\theta} \Gamma^- \Gamma_a \partial_s \theta\end{aligned}\tag{3.1}$$

The Hamiltonian is then the integral of this density over the membrane, viz.

$$H \equiv -P_0^- = \int_{\mathcal{M}} d^2 \sigma \mathcal{H}(\sigma)\tag{3.2}$$

We can further simplify the above expressions for the metric components by making use of the following residual invariance of the gauge conditions under spatial diffeomorphisms. Putting $\xi^0 = 0$, we transform σ^r according to

$$\sigma^r \mapsto \sigma^r + \xi^r(\tau, \sigma)$$

in order to achieve

$$u^r = 0.\tag{3.3}$$

A little further thought (see e.g. ¹⁷) then shows that the Hamiltonian equations in this gauge imply that

$$\partial_0 P^+ = 0.$$

Keeping in mind that $P^+(\sigma)$ transforms as a density under diffeomorphisms, we can easily solve this equation

$$P^+(\sigma) = P_0^+ \sqrt{w(\sigma)}$$

where the function $\sqrt{w(\sigma)}$ is normalized as

$$\int d^2 \sigma \sqrt{w(\sigma)} = 1.$$

$w(\sigma)$ should be viewed as the metric determinant on the membrane associated with a fiducial background metric $w_{rs}(\sigma)$ (this is a 2-by-2 spatial metric on the membrane itself, and should not be confused with the Lorentzian metric on the membrane world volume). This metric is assumed to be non-singular, but otherwise arbitrary. Of course, we will have to make sure eventually that no physical quantity depends on this choice. Quite generally, this independence will follow from the invariance of the lightcone theory under area preserving diffeomorphisms (which by definition leave the metric density $\sqrt{w(\sigma)}$ invariant), and the fact that all the relevant quantities — with the exception of the Lorentz boost generators — involve the metric w_{rs} only through its determinant. The check will be rather more subtle for the Lorentz generators which depend explicitly on the associated Laplace-Beltrami operator.

As in string theory, we can utilize the constraints to eliminate X^- . Namely, from (3.3) we read off

$$\partial_r X^- = -\partial_0 \vec{X} \cdot \partial_r \vec{X} - \bar{\theta} \Gamma^- \partial_r \theta. \quad (3.4)$$

In contrast to string theory, we here get two equations (for higher dimensional p -branes we would get p equations) whose compatibility is not automatic: the equation for X^- can only be solved on a subspace of the full supermembrane phase space. To ensure the compatibility, we must demand that the vector field has vanishing curl, which immediately yields the constraint

$$\phi = \epsilon^{rs} (\partial_r \vec{P} \cdot \partial_s \vec{X} + \partial_r \bar{\theta} \Gamma^- \partial_s \theta) \approx 0 \quad (3.5)$$

Here, we adopt the standard notion of “weakly zero” (≈ 0).¹⁸ This means that we restrict the phase space to the submanifold where the constraints vanish but nevertheless, as functions on phase space, the constraints may have non-vanishing Poisson (or Dirac) brackets with other phase space variables. With the help of the canonical brackets

$$\begin{aligned} \{P^a(\sigma), X^b(\sigma')\} &= \delta^{ab} \delta^{(2)}(\sigma - \sigma') \\ \{\theta_\alpha(\sigma), \bar{\theta}_\beta(\sigma')\} &= \frac{1}{4\sqrt{w(\sigma)}} \Gamma_{\alpha\beta}^+ \delta^{(2)}(\sigma - \sigma'), \end{aligned} \quad (3.6)$$

it is now straightforward to verify that the constraint has vanishing canonical brackets with the Hamiltonian, and can thus be used to reduce the number of X^a fields from nine to eight. Let us also mention that for topologically non-trivial membranes, there are extra consistency conditions corresponding to the non-contractible cycles on the membrane.

The fact that the constraint commutes with the Hamiltonian implies the existence of a residual gauge symmetry of the lightcone Hamiltonian. This is the invariance under area preserving diffeomorphisms, which we will discuss in detail in section 5.

4 Some Properties of the Lightcone Hamiltonian

A lot about the qualitative features of supermembrane theory can be learnt by studying some general properties of the Hamiltonian (3.1). A first, and rather obvious, observation is that the zero modes

$$\begin{aligned}\vec{X}_0 &\equiv \int d^2\sigma \sqrt{w(\sigma)} \vec{X}(\sigma) \\ \theta_0 &\equiv \int d^2\sigma \sqrt{w(\sigma)} \theta(\sigma)\end{aligned}$$

do not appear in this Hamiltonian. Likewise, the center of mass momenta

$$\begin{aligned}\vec{P}_0 &\equiv \int d^2\sigma \vec{P}(\sigma) \\ P_0^- &\equiv - \int d^2\sigma \mathcal{H}\end{aligned}$$

decouple from the non-zero mode degrees of freedom of the membrane. By subtracting this contribution from the Hamiltonian, we arrive at the mass formula

$$\mathcal{M}^2 = -2P_0^+ P_0^- - \vec{P}_0^2 = \int d^2\sigma \left\{ \frac{[\vec{P}^2]' + \bar{g}}{\sqrt{w(\sigma)}} - 2P_0^+ \epsilon^{rs} \partial_r X^a \bar{\theta} \Gamma^- \Gamma_a \partial_s \theta \right\} \quad (4.1)$$

by substituting the expression (3.2) for P_0^- in terms of the transverse degrees of freedom. The prime indicates that the zero modes are to be omitted. This formula contains all the non-trivial dynamics of the membrane, whereas the center of mass motion is governed by the kinematics of a free relativistic particle. Similarly, the fact that the fermionic zero modes decouple will be used later to show that, if there exists a massless state, it will give rise to precisely one massless supermultiplet of $d = 11$ supergravity.

The bosonic part of the Hamiltonian (4.1) is of the standard form

$$H = \mathcal{M}^2 = T + V$$

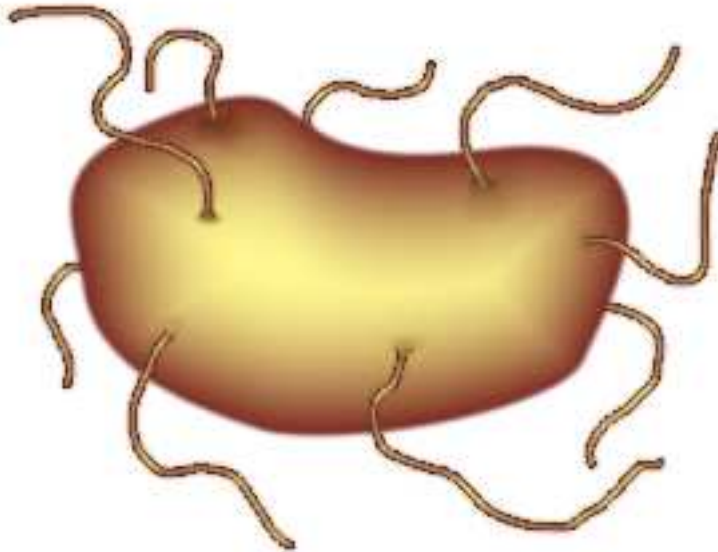


Fig. 2:A membrane growing tubes

with the kinetic energy T . The potential energy V is the integral of the density

$$\bar{g} = \det_{r,s}(\partial_r \vec{X} \cdot \partial_s \vec{X}) = (\epsilon^{rs} \partial_r X^a \partial_s X^b)^2$$

The potential density vanishes if the surface degenerates, which happens when the \vec{X} 's only depend on one linear combination of the σ 's, i.e. when the membrane grows infinitely thin (i.e. stringlike) spikes, see Fig. 2. This instability has important consequences. We will come back to it below to show that this degeneracy must be interpreted as manifestation of the second quantized nature of the quantum supermembrane.

The bosonic part of the above formula (4.1) can be generalized to arbitrary p , with the result that the degeneracies of the potential persist for higher p (generally, the zero energy configurations of a p -brane are those where the brane degenerates to an object of dimension less than p). It is only for $p = 1$, i.e. string theory, that the potential is confining when the zero mode contribution is removed. This is also the only case where the potential is quadratic $(\vec{X}'(\sigma))^2$. Then the Hamiltonian describes a free system, and we recover the well known result that the (super)string is an infinite set of (supersymmetric) harmonic oscillators. Thus, string theory is “easy” because it is a free theory, so that we

can not only find a complete set of solutions to its equations of motion subject to various boundary conditions, but also quantize the theory straightforwardly. By contrast, the membrane and the higher dimensional p -branes are non-linear theories (with potentials $\propto \vec{X}^{2p}$). So, for instance, only a very limited number of special solutions to the classical membrane equations of motion are known. A related observation is that, at a purely kinematical level, the type IIA string can be derived from the supermembrane by a “double dimensional reduction”.¹⁹ For this purpose, one compactifies the X^9 -dimension on a circle, letting the membrane wind around this dimension by making the identification

$$\sigma^2 = X^9.$$

When the circle is shrunk to a point, the σ^2 is gone as well and one is left with a string theory in ten dimensions. The resulting Hamiltonian is just the well known Green-Schwarz Hamiltonian

$$\mathcal{L} = \frac{1}{2P_0^+} \left(\vec{P}(\sigma)^2 + \vec{X}'(\sigma)^2 \right) - \bar{\theta} \Gamma^- \Gamma^9 \theta'(\sigma)$$

This is as expected if the supermembrane theory is a part of M-Theory, one of whose defining properties is that it must reduce to type IIA string theory upon compactification on a small circle. We would like to stress, however, that these considerations are by no means sufficient to establish that superstrings are contained in the supermembrane at the dynamical level. To recover the full dynamics of superstring scattering amplitudes, and in particular the full multistring vertex operators, is quite a different, and far more complicated task (see e.g. H. Verlinde’s lectures on matrix string theory at this School).

While these properties are for the most part almost self-evident, and also in accord with everything we know from superstring theory, we now return to the degeneracies of the Hamiltonian pointed out above. These constitute a very important physical property of the membrane, whose full significance has come to be fully appreciated only more recently. Namely, the spikes described above causing the instabilities need not have “free ends”. It can also happen that a membrane quenches and separates into several parts that are connected by infinitesimally thin tubes, see Fig. 3. Because these tubes do not carry any energy, such a configuration is physically indistinguishable from the multi-membrane configuration obtained by removing the strings connecting the various pieces. In this way, membranes are allowed to split and merge, and there is no conserved “membrane number”. In fact, the generic configuration occurring in a path integral formulation will be such that we cannot even assign a definite “membrane number” to a given configuration.

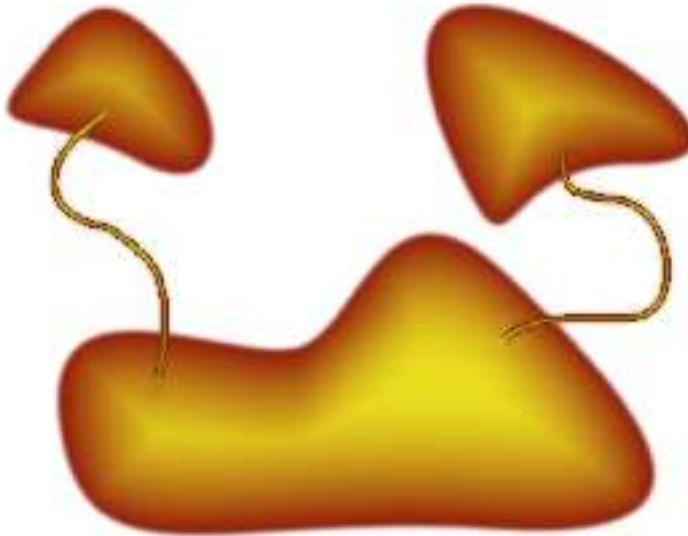


Fig. 3: Three membranes connected by tubes

Similar remarks apply to the treatment of different membrane topologies. For instance, one can build a toroidal membrane from a spherical membrane by connecting two points by a stringlike tube without any additional cost in energy. Hence it does not really make sense to talk about membranes of fixed topology, see Fig. 4.

This new interpretation represents an important change of viewpoint vis-à-vis the one widely accepted a few years ago, when people were trying to construct a first quantized version of the supermembrane. The futility of these attempts became glaringly obvious when it was shown that the spectrum of the ($SU(N)$ version of the) Hamiltonian (4.1) is continuous.²⁰ This looked like (and in fact was) a disaster for the interpretation of the quantum supermembrane as a first quantized theory, and prompted several attempts to argue the continuous spectrum away. We note that, of course, the instabilities would go away if the effective action of the quantum supermembrane contained terms depending on the world volume curvature, which would suppress spikes and stringlike tubes. However, such terms presumably would not be compatible with the (desired) renormalizability of the theory, but would also spoil the correspondence with the matrix model.

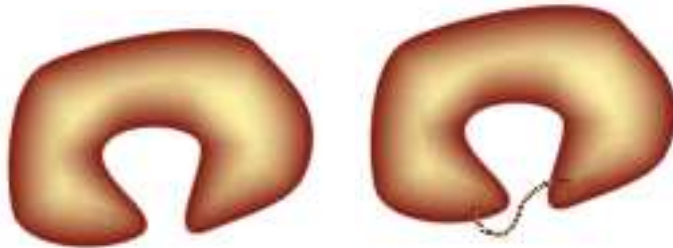


Fig. 4:Equivalent membranes with different topologies

To summarize: the old objection that membrane theories “*can’t be first quantized*” — meaning that the Hamiltonian is not quadratic unlike the superstring Hamiltonian — acquires a new and much deeper significance: the theory cannot be first quantized *because the quantum supermembrane is a second quantized theory from the very beginning!* This shows why membrane theory is so hard: we are dealing with a theory where the very notion of a one- (or multi-) particle state can only be extracted in certain asymptotic regimes, if it makes sense at all.

5 Area Preserving Diffeomorphisms

We now return to the canonical constraint (3.5) which expresses the invariance of the lightcone supermembrane under area preserving diffeomorphisms (or just APD, for short). To further analyze this residual symmetry, we introduce the following bracket on the space of functions A, B of the membrane coordinates

$$\{A, B\}(\sigma) := \frac{\epsilon^{rs}}{\sqrt{w(\sigma)}} \partial_r A(\sigma) \partial_s B(\sigma)$$

This is the Poisson bracket associated with the symplectic form

$$\omega = \frac{\epsilon_{rs}}{\sqrt{w(\sigma)}} d\sigma^r \wedge d\sigma^s$$

which, as any two form in two dimensions, is closed. The bracket is manifestly antisymmetric

$$\{A, B\} = -\{B, A\}$$

and obeys the Jacobi identity

$$\{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0.$$

and therefore has all the requisite properties of a Lie bracket. Consequently the functions on the membrane naturally form an infinite dimensional Lie algebra with the above bracket. The dimension $p = 2$ of the membrane is essential here, as these statements have no analog for $p > 2$ (at least not within the framework of ordinary Lie algebra theory).

By means of this bracket we can rewrite the potential density as

$$\bar{g} = \det(\partial_r \vec{X} \cdot \partial_s \vec{X}) = (\{X^a, X^b\})^2$$

Similarly, the non-zero mode part of the Hamiltonian (4.1) can be cast into the form

$$\mathcal{M}^2 = \int d^2\sigma \left[\frac{1}{2\sqrt{w(\sigma)}} \left[\bar{P}^2(\sigma) \right]' + \sqrt{w(\sigma)} \left(\frac{1}{2} \{X^a, X^b\}^2 + \bar{\theta} \Gamma^{-1} \Gamma^a \{X^a, \theta\} \right) \right] \quad (5.1)$$

Again, the prime indicates that the zero modes have been left out. The advantage of rewriting the previous formulas in this peculiar way is that the similarities with Yang Mills theories become quite evident: the potential energy vanishes whenever $\{X^a, X^b\} = 0$, which is equivalent to the statement that the X^a belong to the Cartan subalgebra of the Lie algebra introduced above. When we truncate this infinite dimensional Lie algebra to a finite dimensional matrix Lie algebra, the zero energy configurations just correspond to the diagonal matrices.

Because

$$\{X^a, X^b\} = \frac{\epsilon^{rs}}{\sqrt{w(\sigma)}} \partial_r X^a \partial_s X^b$$

is nothing but the volume (or rather: area) element of the membrane pulled back into space time, the residual invariance consists precisely of those diffeomorphisms which leave the area density invariant. These correspond to diffeomorphisms

$$\sigma^r \mapsto \sigma^r + \xi^r(\sigma)$$

generated by divergence free vector fields

$$\partial_r (\sqrt{w(\sigma)} \xi^r(\sigma)) = 0.$$

This equation is solved locally by

$$\xi^r(\sigma) = \frac{\epsilon^{rs}}{\sqrt{w(\sigma)}} \partial_s \xi(\sigma)$$

For topologically non-trivial membranes, there will be further vector fields which cannot be expressed in this fashion, and which correspond to the harmonic one forms in the standard Hodge decomposition.

As a little exercise readers may check that the commutator of two APD vector fields is in one-to-one correspondence with the above Lie bracket in the sense that

$$[\xi_1^r \partial_r, \xi_2^s \partial_s] \longleftrightarrow \{\xi_1, \xi_2\}$$

This means that the above Lie algebra can be identified with the Lie algebra of divergence free vector fields which is the Lie algebra of area preserving diffeomorphisms. Using this correspondence, we can calculate the variation of any function f under an area preserving diffeomorphism as

$$\delta f = -\xi^r \partial_r f = \{\xi, f\}.$$

It is now straightforward to verify that the mass \mathcal{M} commutes indeed with the constraint generator (3.5). The latter can be reexpressed by means of the above Lie bracket as

$$\phi(\sigma) \equiv \{\vec{P}, \vec{X}\} - \{\bar{\theta}, \Gamma^- \theta\} \approx 0$$

This should be compared to closed string theory after the light cone gauge has been imposed: In that case, the length preserving diffeomorphisms are just the constant shifts

$$\sigma \mapsto \sigma + \text{const.}$$

This constraint implies that the oscillator levels of the left and the right movers are equal on physical states:

$$N_L = N_R$$

As is well known, already this single constraint has many non-trivial consequences!

6 Supercharges and Superalgebra

The supercharges expressing the global supersymmetry of the model are obtained by integrating the supercurrent (2.3) over the membrane:

$$Q = \int d^2\sigma J^0 \equiv Q^+ + Q^-$$

The chiral supercharges are given by

$$Q^+ \equiv \frac{1}{2} \Gamma^+ \Gamma^- Q = \int d^2\sigma \left(2P^a \Gamma_a + \sqrt{w(\sigma)} \{X^a, X^b\} \Gamma_{ab} \right) \theta$$

$$Q^- \equiv \frac{1}{2} \Gamma^- \Gamma^+ Q = \int d^2\sigma S = 2\Gamma^- \theta_0$$

We see that Q^- acts only on the fermionic zero modes. The zero mode part of Q^+

$$Q_0^+ = 2P_0^a \Gamma_a \theta_0 \equiv Q^+ - Q_1^+$$

is also conserved. Evidently, Q_0^+ is only relevant to the center of mass kinematics.

With the help of the canonical Dirac bracket (3.6) one can now determine the full superalgebra. This calculation, including possible surface contributions which could give rise to central charges was already performed in¹⁷. The most general $d = 11$ superalgebra has the form²¹

$$\{Q_\alpha, \bar{Q}_\beta\} = \Gamma_{\alpha\beta}^\mu P_\mu + \frac{1}{2} \Gamma_{\alpha\beta}^{\mu\nu} Z_{\mu\nu} + \frac{1}{5!} \Gamma_{\alpha\beta}^{\mu\nu\rho\sigma\tau} Z_{\mu\nu\rho\sigma\tau}.$$

The corresponding Lie-superalgebra, including the bosonic commutation relations of the bosonic operators, is known as $OSp(1|32)$ in mathematical terminology. We recognize that besides the membrane charge $Z_{\mu\nu}$ the algebra also admits a 5-brane charge $Z_{\mu\nu\rho\sigma\tau}$. However, a recent investigation of the lightcone superalgebra for the supermembranes with winding^{22,23,24,25,26} has shown that this 5-brane charge is, in fact, absent despite the fact that $d = 11$ supergravity admits both 2-brane and 5-brane-like solutions.^{27,28} This little puzzle should not come as a total surprise in view of the following fact. While the 2-brane couples to the 3-index field of 11d supergravity, the 5-brane charge would couple to a “dual” 6-index field; one would therefore expect the 5-brane charge to be related to another version of 11d supergravity with a 6-index field. However, for all we know such a version of 11d supergravity does not exist,²⁹ but see³⁰ for more recent references and a reformulation containing both a 3-index *and* a 6-index tensor field).

The remaining part Q_1^+ of Q^+ obtained upon removing all zero-mode contributions contains the non-trivial information about the supermembrane dynamics. It gives rise to the superalgebra

$$\{Q_{1\alpha}^+, \bar{Q}_{1\beta}^+\} = \Gamma_{\alpha\beta}^+ \mathcal{M}^2$$

The existence of massless states at threshold is equivalent to finding a normalizable state Ψ_0 obeying $\mathcal{M}^2 \Psi_0 = 0$. Once such a state is found, a whole massless multiplet of $d = 11$ supergravity is generated by acting with the zero mode supercharges given above. If \mathcal{M}^2 does not annihilate the state, one gets many more states, and the supermultiplet is a massive (long) supermultiplet of $d = 11$ supersymmetry. If $d = 11$ supergravity is to emerge as a low energy limit from the supermembrane there must be normalizable states that are annihilated by the quantum operator for \mathcal{M}^2 . Let us also note that the winding states of minimal mass are BPS states, and hence belong to short multiplets, too^{24,25}.

7 Supermembranes and Matrix Models

We are now ready to establish the connection between the supermembrane Hamiltonian and the large N limit of supersymmetric $SU(N)$ matrix model,¹⁷ which also underlies the recent proposal for a concrete formulation of M-Theory.³¹ To this aim, we need to truncate the supermembrane theory to a supersymmetric matrix model with finitely many degrees of freedom. The idea is then to *define* the quantum supermembrane as the limit where the truncation is removed, taking into account possible renormalizations. The truncated model can be alternatively obtained by dimensional reduction of the maximally supersymmetric $SU(N)$ Yang Mills theory from 1+9 to 1+0 dimensions, a reduction which had been originally investigated in^{32,33,34}). Upon quantization it becomes a model of supersymmetric quantum mechanics with extended ($\mathcal{N} = 16$) supersymmetry.

To proceed, we expand all superspace coordinates in terms of some complete orthonormal set of functions $Y_A(\sigma)$ on the membrane

$$\vec{X}(\sigma) = \vec{X}_0 + \sum_A \vec{X}^A Y_A(\sigma)$$

We next define a metric on this function space by

$$\int d^2\sigma \sqrt{w(\sigma)} Y_A(\sigma) Y_B(\sigma) = \eta_{AB}.$$

by means of which indices A, B, \dots can be raised and lowered, such that we have the orthogonality relations

$$\int d^2\sigma \sqrt{w(\sigma)} Y_A(\sigma) Y^B(\sigma) = \delta_A^B.$$

For instance, for spherical membranes, we can take for the Y^A the standard spherical harmonics $Y_{lm}(\theta, \varphi)$, in which case raising and lowering corresponds to complex conjugation such that $Y^{lm} = (Y_{lm})^* = Y_{l,-m}$. For a real basis of orthogonal functions, we can choose $\eta_{AB} = \delta_{AB}$. For the toroidal membrane, the indices are of the form $A = (m_1, m_2)$, where each index labels a Fourier mode. It is convenient but not strictly necessary to take the Y_A to be eigenfunctions of the corresponding Laplace-Beltrami operator (w.r.t. the background metric w_{rs}). We also record the general completeness relation

$$\sum_A Y^A(\sigma) Y_A(\sigma') = \frac{1}{\sqrt{w(\sigma)}} \delta(\sigma - \sigma')$$

Because of the completeness relation we can express the Lie bracket in terms of the new basis:

$$\{Y_A, Y_B\} = f_{AB}{}^C Y_C$$

One finds the structure constants to be

$$f_{AB}{}^C = \int d^2\sigma \epsilon^{rs} \partial_r Y_A \partial_s Y_B Y^C.$$

Exploiting these observations we now wish to truncate the theory to another one with only finitely many degrees of freedom. This “regularization” of the supermembrane is achieved by introducing a cut-off on the number of modes, such that the mode indices $A, B \dots$ are restricted to a finite range of values $1, \dots, \Lambda$. Consistency then demands that the group of APDs must be approximable by a finite Lie group G_Λ with $\dim G_\Lambda = \Lambda$ in the sense that

$$\lim_{\Lambda \rightarrow \infty} f_{AB}{}^C(G_\Lambda) = f_{AB}{}^C(\text{APD})$$

for any fixed triple A, B, C . The crucial result

$$G_\Lambda = SU(N) \quad \text{with} \quad \Lambda = N^2 - 1 \quad (7.1)$$

was first established for spherical membranes in^{14,15}. Later on, it was extended to toroidal membranes^{35,36,37}, and finally to membranes of arbitrary genus³⁸. The APD Lie algebra is thereby replaced by a finite dimensional Lie algebra, namely the Lie algebra of $SU(N)$ matrices. To emphasize the matrix character of this regularization, we will thus replace the Lie bracket by a commutator:

$$\{\cdot, \cdot\} \longrightarrow [\cdot, \cdot].$$

Before continuing, we would like to point out one essential difference between this “regularization” and the lattice regularization of gauge theories, which look very similar at first sight: Unlike in lattice QCD, there is no “small parameter” here analogous to the lattice spacing (or the inverse momentum cutoff). To see this, we note that large N does not necessarily mean “large energy”, because for any given N we can find configurations of arbitrarily small energy by taking commuting $N \times N$ matrices.

Although there is no room here to describe the construction in full detail, we would like to give readers at least an idea how the correspondence between $SU(N)$ and APD works for toroidal membranes. This is the simplest case because we can make use of a double Fourier expansion. Choosing coordinates $0 \leq \sigma_1, \sigma_2 < 2\pi$, we have the orthonormal basis

$$Y_{\vec{m}}(\vec{\sigma}) = \frac{1}{\sqrt{4\pi}} e^{i\vec{m} \cdot \vec{\sigma}}$$

The APD structure constants follow from

$$\{Y_{\vec{m}}, Y_{\vec{n}}\} = -4\pi^2(\vec{m} \times \vec{n})Y_{\vec{m}+\vec{n}}. \quad (7.2)$$

To see the relation with $SU(N)$, we employ the 't-Hooft clock and shift matrices

$$U = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} \quad V = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix}$$

where ω is an N^{th} root of unity $e^{2\pi ik/N}$. They commute up to a phase factor:

$$UV = \omega VU$$

Any traceless $N \times N$ matrix can be written as a linear combination of matrices $U^{m_1}V^{m_2}$. One finds the matrix commutator to be

$$[U^{m_1}V^{m_2}, U^{n_1}V^{n_2}] = (\omega^{m_2n_1} - \omega^{m_1n_2})U^{m_1+n_1}V^{m_2+n_2}$$

If we now take N to infinity keeping \vec{m} and \vec{n} fixed this approaches

$$\lim_{N \rightarrow \infty} [U^{m_1}V^{m_2}, U^{n_1}V^{n_2}] \rightarrow \frac{2\pi ik}{N}(\vec{m} \times \vec{n})U^{m_1+n_1}V^{m_2+n_2}.$$

So, in this limit this is the same Lie-algebra as the one we found for the $Y_{\vec{m}}$ in (7.2). This corroborates our claim that the Lie algebra of area preserving diffeomorphisms on the torus can be approximated by $su(N)$.

At this point we should emphasize that the question of how to rigorously define the notion of “limit” here is very subtle. Namely, the above statements hold only for particular bases of $SU(N)$ matrices, which must be specially and differently chosen for every given membrane topology. In other words, the limits are highly basis dependent. This explains the — at first sight paradoxical — fact that the area preserving diffeomorphisms for *different* membrane topologies can be approximated by the *same* group: while all bases are equivalent for finite N , this is no longer true in the limit $N \rightarrow \infty$ because the large N limit of the corresponding equivalence transformations will not exist. So for instance we have $\text{Diff}_0(S^2) \neq \text{Diff}_0(T^2)$, and neither of the associated Lie algebras is isomorphic to $su(\infty)$ (defined as the set of $\infty \times \infty$ matrices with only finitely many non-vanishing entries), see e.g. ^{39,38}. Observe also that the APDs generated by harmonic vector fields (whose number depends on the membrane topology) have no finite N analogs. On the other hand, in the quantum theory, where we are mainly interested in the large N limit of gauge invariant

correlators, and not in approximating C^∞ diffeomorphisms, these issues are no longer so prominent.

Our main point here is that the restriction to finite N should be regarded as a non-perturbative regularization that can be used to give a rigorous and non-perturbative definition of the quantum supermembrane.^b This interpretation is similar in spirit to lattice gauge theory whose continuum limit is *by definition* the quantum Yang Mills theory (provided the limit exists). We have seen in the above discussion that this quantum matrix model in principle describes membranes of all possible topologies in the large N limit.

After these preliminary remarks, we can now write down the truncated Hamiltonian. Before doing so, it is convenient to switch from the $SO(1,10)$ basis to a basis of $SO(9)$ γ matrices, as this is the residual symmetry of the lightcone gauge theory. The precise relation is

$$\Gamma^+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1} \quad \Gamma^- = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} \otimes \mathbb{1}$$

and

$$\Gamma^a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \gamma^a$$

where the γ^a are the standard $SO(9)$ γ -matrices. Accordingly, we will henceforth work with 16 component (real) spinors of $SO(9)$, eliminating all Γ^\pm from the equations. As we already mentioned, the matrix Hamiltonian is then nothing but the dimensional reduction of the Hamiltonian of maximally extended super Yang Mills theory with gauge group $SU(N)$ from 9+1 down to 0+1 dimensions:

$$H = \frac{1}{2} P_a^A P_{aA} + \frac{1}{4} (f_{AB}{}^C X_a^A X_b^B)^2 - \frac{i}{2} f_{ABC} X_a^A \theta^B \gamma^a \theta^C \quad (7.3)$$

In writing this Hamiltonian, we have suppressed the Yang Mills coupling constant which can be identified with the longitudinal momentum P_0^+ , see (4.1). The Gauss constraint reads

$$\phi_A = f_{ABC} (X_a^B P_a^C - \frac{i}{2} \theta_\alpha^B \theta_\alpha^C) \approx 0$$

now simply expresses the $SU(N)$ gauge invariance of the model. It is straightforward to check that it commutes with the Hamiltonian, i.e.

$$\{\phi^A, H\} = 0$$

^bHowever, it is conceivable that the quantum mechanical model with infinite dimensional gauge group APD can be made sense of *eo ipso*, in which case we cannot even rule out the possibility that it is actually different from the large N limit of the matrix model.

The global supersymmetry is reflected in the existence of the supercharges

$$Q_\alpha = (P_A^a \gamma_a + \frac{1}{2} f_{ABC} X_a^B X_b^C \gamma^{ab})_{\alpha\beta} \theta_\beta^A.$$

As usual, the superalgebra relation

$$\{Q_\alpha, Q_\beta\} \approx 2\delta_{\alpha\beta} H$$

implies the lower boundedness

$$H \geq 0.$$

of the energy spectrum.

8 Spectrum

The supersymmetric Hamiltonian (7.3) has a continuous spectrum, unlike the bosonic Hamiltonian contained in it. We will illustrate the idea of the proof by means of a simpler toy model. The proof in this case is much simpler but, apart from technicalities, the same as for the supermembrane. The model is a two dimensional supersymmetric quantum mechanical system with flat valleys. The supercharges are

$$Q = Q^+ = \begin{pmatrix} -xy & i\partial_x + \partial_y \\ i\partial_x - \partial_y & xy \end{pmatrix}$$

They square to yield the Hamiltonian

$$H = \frac{1}{2} \{Q, Q^+\} = \begin{pmatrix} -\Delta + x^2 y^2 & x + iy \\ x - iy & -\Delta + x^2 y^2 \end{pmatrix}$$

which acts on two component wave functions

$$\Psi = \begin{pmatrix} \psi_1(x, y) \\ \psi_2(x, y) \end{pmatrix}.$$

The potential $V(x, y) = x^2 y^2$ has valleys along the coordinate axes. Consider, for example, the point $(x, 0)$ for some large x . If we take into account only the bosonic part given by the diagonal values of the Hamiltonian, we would find a harmonic oscillator potential in the y direction with frequency proportional to x . The zero point fluctuations therefore induce an effective $|x|$ potential for the slower motion in the x direction which confines the wave function and prevents it from leaking out to $x \rightarrow \pm\infty$. So the Born-Oppenheimer

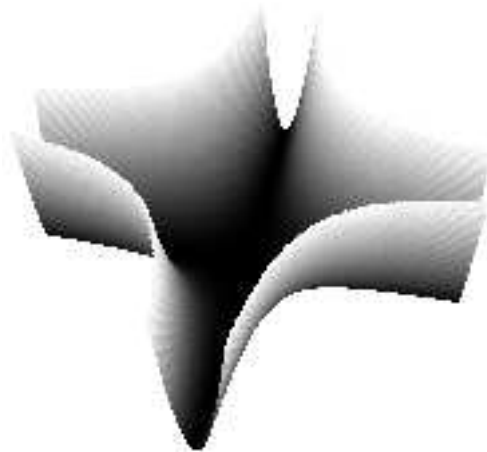


Fig. 5: The potential of the toy model

approximation that is valid for large values of x lets us expect a normalized ground state with finite $E > 0$ which is located near the origin (nowadays also called “the stadium”). Because of this confinement the spectrum is discrete despite the presence of flat directions.^{40,41}

To make these qualitative considerations a little more precise, we write the bosonic Hamiltonian as

$$\begin{aligned} H_B &= \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(p_x^2 + x^2 y^2) + \frac{1}{2}(p_y^2 + x^2 y^2) \\ &\geq \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}|y| + \frac{1}{2}|x| \end{aligned}$$

The inequality is meant as an inequality of operators. Take the expectation value of the bosonic Hamiltonian in any state: It is bounded from below by the expectation value of the second operator. But for the latter, the valleys have disappeared and we are left with a confining potential (which looks like the inverted pyramid in the Paris Louvre). Therefore, there can only be a finite number of states with an energy less than this expectation value. Thus we have shown that the spectrum of the bosonic model is indeed discrete.

But things change dramatically when supersymmetry is turned on: Now, the motion in the y direction is described by the supersymmetric analogue of a harmonic oscillator whose bosonic zero point energy is cancelled by the fermionic

contribution. Thus, there is no confining potential any more and the wave function can escape to infinity. To give a rigorous proof of this fact for any given energy $E \geq 0$, we will construct a trial wave function Ψ such that $\|(H - E)\Psi\|$ is less than any given positive number.

To this aim, we first define a function $\chi(x)$ of one real variable for any given $E \geq 0$ as follows:

$$\chi(x) := e^{ikx} \chi_0(x)$$

where $k = \sqrt{E}$ and χ_0 is a slowly varying real function of compact support in \mathbb{R} normalized such that

$$\|\chi\|^2 = \int \chi^2 dx = 1$$

Observe that χ_0 can be chosen in such a way that its derivatives can become arbitrarily small while it is still normalized to one. Then we define, for $\lambda \in \mathbb{R}$,

$$\Psi_\lambda(x, y) := \chi(x - \lambda) \sqrt{\frac{|x|}{4\pi}} e^{-\frac{1}{2}|x|y^2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (8.1)$$

This is a harmonic oscillator wave function in the variable y transversal to the valley along the x axis, but with a frequency proportional to x , and judiciously chosen such that the bosonic and fermionic zero point energies cancel to better and better accuracy as we push this wave function further into the valley by picking λ sufficiently large. In this way, given any $\epsilon > 0$, we can find λ such that

$$\|(H - E)\Psi_\lambda\| < \epsilon,$$

This proves that the spectrum of H consists of all non-negative numbers.

The results on the continuity of the spectrum for the supermembrane are entirely analogous.²⁰ Again, the $SU(N)$ truncated bosonic membrane has a discrete spectrum, but the spectrum of its supersymmetric extension is continuous, extending all the way down to $E = 0$. This result is quite contrary to what one would expect from the superstring, whose discrete excitations are interpreted as one-particle excitations of a higher dimensional target space theory. Had we expected the model to describe a first quantized membrane, this would have been the end of the story. With this interpretation, it would be impossible to find particle-like excitations in the low energy theory the way it is possible in string theory.

However, the result acquires a completely different significance in view of the multi-particle interpretation of the dimensionally reduced super-Yang-Mills Hamiltonian^{42,31} and our previous remarks in section 5. Namely, the continuous spectrum is now *required* for consistency, since scattering states of (super)membranes connected by tubes should come with a continuum of energies.

Turning the argument around, we can now even claim that the discreteness of the bosonic membrane spectrum indicates the inconsistency of the quantized bosonic membrane! In order to perform the scattering calculations just mentioned to leading order, it is fortunately not necessary to know the detailed structure of the ground state: It is sufficient to know the asymptotic state when the two gravitons are largely separated, in which case the Born-Oppenheimer approximation becomes better and better. ⁴³

9 Renormalizability vs. $N \rightarrow \infty$ Limit

The key question is whether the $N \rightarrow \infty$ limit of the supersymmetric matrix model exists. As we emphasized already, this is related to the question of whether the supermembrane makes sense as a full fledged quantum field theory, and whether it is renormalizable or even finite as a 2+1 dimensional quantum field theory. A priori there is, of course, no reason to expect a complicated Lagrangian such as (2.1) to be so well-behaved. Indeed, the bosonic membrane is not renormalizable and thus most likely not viable as a quantum theory. However, maximal supersymmetry could make all the difference here!

Treating the X 's and the θ 's as quantum fields on the world volume, one can in principle calculate Feynman loop diagrams. For the supermembrane this has not yet been done, not least because of technical problems such as the absence of an off-shell formulation, which makes any calculation extremely cumbersome. As in any quantum field theory, the higher order amplitudes are potentially divergent. This means that some regulator must be introduced, which possibly can be removed only at the expense of certain renormalizations or higher order counterterms respecting all symmetries. The only parameter appearing in the supermembrane Lagrangian is the membrane tension. Hence, introducing some cutoff Λ , we would set $T_2 = T_2(\Lambda)$ and try to adjust this dependence in such a way that the limit exists for all physically relevant correlators. The supermembrane would be finite if all these limits existed with T_2 kept fixed.

If on the other hand the theory is non-renormalizable, the higher loop diagrams would necessitate an infinite number of counterterms of the form R, R^2, \dots , where R is the world volume curvature tensor (actually expressible through the Ricci tensor in three dimensions), just like perturbatively treated matter coupled Einstein gravity in 2+1 dimensions. If this were the case, not only would the theory be ill-defined as a continuum theory, but also the connection with matrix theory would be lost: there seems to be no way to express the induced world volume curvature R in terms of the APD Lie-bracket $\{X^a, X^b\}$ (for the membrane this was possible because the Lagrangian depends only on

the metric determinant, not the curvature tensor).

The hope is therefore that maximal supersymmetry is strong enough to rule out possible counterterms for the $d = 11$ supermembrane. Indeed, to date no counterterm respecting the full $\mathcal{N} = 16$ supersymmetry and κ symmetry is known (see e.g. ⁴⁴ for an early discussion). However, what is required to definitely settle this issue is an off-shell formulation, or some equally powerful new mathematical tool that would allow us to prove or disprove these statements. For the time being, however, we have to be optimistic and simply assume that the full $d = 11$ supersymmetry does not admit higher order counterterms. A further issue here is whether the regularization procedure can be made to respect all symmetries.

How are these possibilities mirrored in the matrix model? Since the finite N matrix model is a perfectly well defined model of ordinary quantum mechanics (and as such in principle easier to deal with than a model of quantum field theory), the crucial question concerns the $N \rightarrow \infty$ behavior. This limit is presumably different from 't Hooft's large N limit with $g^2 N$ kept fixed as $N \rightarrow \infty$. Moreover, there it is the $N = \infty$ theory that is better defined (being based on planar diagrams only), whereas it is the finite N theory which is "harder", and one tries to work one's way from $N = \infty$ to finite N (with $N = 3$ for QCD). Here, we are trying to do the opposite.

The first possibility is that this limit simply does not exist, no matter how the parameters are varied. This might be due to large N divergences, which would be analogous to the divergences underlying the potential non-renormalizability of the supermembrane. But it might also arise from a non-analytic N dependence even in the absence of genuine large N divergences. We will present an example of such behavior in section 12. Again, since we expect the large N matrix model to be a description of M-Theory, the non-existence of a suitable large N limit would be the end for this conjecture.

The second possibility is that a limit does exist if we vary and renormalize the Yang-Mills coupling constant $g = g(N)$ as we go to large N . More precisely, the function $g = g(N)$ would have to be *universal* in the sense that the limit

$$\lim_{N \rightarrow \infty} \frac{1}{f(N)} \langle \dots \rangle_{g(N)}$$

exists simultaneously for all physically relevant correlators. Here, $f(N)$ is an appropriate wave function renormalization that might depend on the correlator under consideration. This means that the large N limit would be a *weak limit*, similar to the asymptotic limit relating the in- and out-fields of LSZ quantum field theory to the interacting fields. This would be the matrix analog of renormalizability. Finally, if we could leave g fixed independently of N , this

would correspond to finiteness of the supermembrane. In any case, the question of how to properly define the limits relevant to the approximation of C^∞ area preserving diffeomorphisms on membranes of different topology appears to be of no great relevance in the quantum mechanical context.

10 Relation to D-Branes

Over the last two years, the supersymmetric matrix model has attracted a great deal of attention and stirred up quite some excitement. The proposal according to which the matrix model for $N \rightarrow \infty$ is M-Theory on a flat background³¹, had its roots in the discovery of D-branes and their role in the description of non-perturbative string excitations⁴⁵ and the realization that the dynamics of these objects is governed by dimensionally reduced Yang Mills theories.⁴² These developments were independent of supermembrane theory, and it is therefore all the more remarkable that one ends up with the same model: $SU(\infty)$ super-Yang Mills theory reduced to one dimension. Here, we would like to briefly review how the matrix model arose in the context of D0 branes, referring readers to C. Johnson's lectures at this School for a more detailed treatment and many further references.

One distinction between the M(atr)ix theory and the quantum supermembrane is the treatment of the longitudinal momentum: in the proposal of³¹ it is identified with the number of longitudinal quanta for gauge group $SU(N)$:

$$P_0^+ = \frac{N}{R_s}$$

where R_s is the compactification radius, whereas in the supermembrane the longitudinal momentum is treated as a canonical variable having non-trivial commutation relations. The above identification may be viewed as an indication that the matrix model is physically meaningful already for finite N , and should be identified with M-Theory compactified on a light-like circle.⁴⁶ This proposal was made more precise in^{47,48}: One should think of the light-like circle as a small space-like circle with radius R_s that is boosted by a large amount. Then one should take the R_s to zero while compensating with further boosts in order to keep the Hamiltonian finite. One finds that, in this limit, the ten dimensional Planck mass goes to infinity. But M-Theory, when compactified on a small space-like circle, is (almost by definition) type IIA string theory, when one identifies the Kaluza Klein excitations with the non-perturbative (BPS type) string states. As lightest solitonic objects, type IIA string theory contains D0-branes. These are particles on which open strings can end. Since the energies of the excited string modes scale with the Planck mass, which

goes to infinity, these excited modes become infinitely heavy. Only the zero mode dynamics of the strings survive. Since the open strings are attached to the D0-branes the gauge fields live only on the world lines of the D0-branes. If there are N different D0-branes, strings can have N different positions for each of their two end points. This is the D0-brane explanation for the emergence of Chan-Paton labels, and the resulting theory is ten dimensional $SU(N)$ super-Yang Mills theory dimensionally reduced to one time dimension (the world line of the D0-branes).

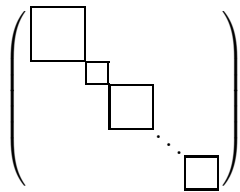
Unfortunately, the story is a bit more complicated than it seemed in the beginning: It has been known for quite some time that the treatment of the zero modes is quite subtle when one attempts to compactify a light-like direction. Furthermore, the low energy limit of M-Theory that is used for this proposal, namely $d = 11$ supergravity, is strictly valid only as long as the D-branes are far apart and the curvature is small. On the other hand, the regime in which the Yang-Mills description of the D-brane dynamics is valid is the limit where the D-branes approach each other at substringy distances. The possibility that the supergravity description is also valid for some short range processes was indicated by the agreement of scattering amplitudes calculated in supergravity and in the D-brane/matrix picture^{49,50} for two graviton scattering and more recently for three graviton scattering.⁵¹ It has been shown in⁵² that at least the two graviton term is protected by supersymmetry at all distance scales.

There are also arguments, due to⁵³, that in curved supergravity backgrounds the truncation to finitely many degrees is not possible because the excited string modes do not decouple. But this problem might be overcome in the full supermembrane theory since it is also consistent for arbitrary backgrounds that satisfy the supergravity equations of motion. It is not at all clear how these backgrounds could be incorporated into the D0-brane picture, although an obvious guess would be to repeat the procedure of¹⁷ to such a curved background.

M(atr)ix theory naturally leads to a kind of non-commutative geometry at short distances because the coordinates are represented by non-commuting matrices. We noted earlier that the potential energy vanishes when all the matrices commute with each other. But this means we can diagonalize them simultaneously. So for zero energy the only remaining degrees of freedom are the N diagonal entries of the nine matrices plus their fermionic partners. According to⁴² we can interpret these eigenvalues as the (commuting) coordinates of N free particles moving in nine dimensional transversal space. The degrees of freedom corresponding to the off-diagonal entries become heavy and can be integrated out giving rise to effective interactions which resemble strings stretching between the D0-particles. The masses of these stretched strings are

proportional to the distance between the D0-branes. They become important only when the particles are close together: only in this case the off-diagonal matrix elements become light and the matrices become non-commutative. On the other hand, this non-commutativity means that the notion of positions makes less and less sense for short distances. So M(atric) theory can in principle explain the emergence of a commutative space-time at long distances (small energies) from a non-commutative “spacetime foam” at short distances.

Finally, let us underline once more the multi-particle nature of the quantum supermembrane. For finite N , a multi-membrane configuration is approximated by a set of block diagonal matrices, while in the D0-brane interpretation this matrix configuration describes a bunch of coincident D0-particles. When the limit $N \rightarrow \infty$ is taken in such a way that each block becomes infinite dimensional, each block matrix is thereby related to a membrane of its own. The effective interactions between these blocks arise can then be computed by integrating over the off-diagonal elements. And indeed, the effective action at the one loop level can be interpreted as being due to strings stretching between the different bunches of D0-branes, or alternatively between the separate membranes as in Fig. 3.



As we pointed out already, this multi-particle interpretation is now also in complete accord with the continuous spectrum of the matrix model Hamiltonian.

11 Light-cone gauge and Lorentz invariance

The remaining two sections of these lectures are devoted to some more specific technical issues, on which there has been some progress recently. The first of these is the question of Lorentz invariance of the matrix model in the large N limit. As we will see the truncation to finite N breaks the Lorentz invariance even at the classical level, but the symmetry is restored as we take the limit $N \rightarrow \infty$.^{37,54,55} The crucial question, which still awaits answer, is whether this remains true at the quantum level. Since the matrix model is well defined for any finite N , we can meaningfully address this problem at least in principle.

Recall that after imposing light cone gauge we found an equation (3.4) for X^- (where we have redefined the spinors by factors of $w(\sigma)^{\frac{1}{4}}$)

$$\partial_r X^- = V_r = -\frac{1}{P_0^+} \left(\frac{1}{\sqrt{w(\sigma)}} \vec{P} \cdot \partial_r \vec{X} + \frac{i}{2} \theta \partial_r \theta \right)$$

We have to solve this differential equation and substitute the solution into the expression for the Lorentz generators to check if the algebra closes.

Following^{14,37} we introduce a Greens function and write

$$X^-(\sigma) = \int d\sigma' \sqrt{w(\sigma')} G^r(\sigma, \sigma') V_r(\sigma'), \quad (11.1)$$

always keeping in mind that this formula only works if V_r obeys the integrability constraint $\partial_{[r} V_{s]} = 0$. The requisite Greens function can be given rather explicitly. Up to this point, the Y^A 's were an arbitrary complete set of orthonormal functions. We can now impose that they should also be eigenfunctions of the Laplace-Beltrami operator

$$\Delta = \frac{1}{\sqrt{w(\sigma)}} \partial_r (\sqrt{w(\sigma)} w^{rs} \partial_s)$$

(recall that w^{rs} is the inverse of the fiducial background metric on the membrane)

$$\Delta Y^A = -\omega^A Y^A$$

On the complement of the constant mode $Y^0 = \text{const}$, the operator $-\Delta$ is positive definite. The ω^A are therefore positive numbers. From the completeness relation

$$\sum_A Y^A(\sigma) Y_A(\sigma') = \frac{1}{\sqrt{w(\sigma)}} \delta(\sigma, \sigma')$$

we immediately see that the scalar Green's function

$$G(\sigma, \sigma') = -\sum_A \frac{1}{\omega^A} Y^A(\sigma) Y_A(\sigma').$$

satisfies

$$\Delta G(\sigma, \sigma') = \frac{1}{\sqrt{w(\sigma)}} \delta^{(2)}(\sigma, \sigma') - 1, \quad (11.2)$$

the -1 being there because we have taken out the zero mode. The requisite Green's function is given by

$$G^r(\sigma, \sigma') = \sum_A \frac{1}{\omega^A} Y^A(\sigma) w^{rs}(\sigma') \partial_s Y_A(\sigma')$$

The above Green's function enables us to write down the explicit expressions for the Lorentz generators in the lightcone gauge. To do so, one simply works out the Noether charges associated with the original $SO(1, 10)$ symmetry, and as in lightcone gauge string theory, subsequently replaces the X^- by (11.1) everywhere. This procedure yields the following expressions³⁷

$$\begin{aligned}
M^{ab} &= \int d^2\sigma (-P^a X^b + P^b X^a - \frac{i}{4}\theta\gamma^{ab}\theta) \\
M^{+-} &= \int d^2\sigma (-P^+ X^- + P^- X^+) \\
M^{+a} &= \int d^2\sigma (-P^+ X^a + P^a X^+) \\
M^{-a} &= \int d^2\sigma (-P^- X^a + P^a X^- - \frac{i}{4P_0^+}\theta\gamma^{ab}\theta P_b - \frac{i\sqrt{w(\sigma)}}{8P_0^+}\{X_b, X_c\}\theta\gamma^{abc}\theta)
\end{aligned}$$

When we substitute our expression for X^- we find that M^{+-} only involves the zero modes. Expanding the expression for M^{-a} one encounters new overlap integrals that were not needed up to this point. We have already made use of the APD structure constants

$$f_{ABC} = \int d^2\sigma \sqrt{w} Y_A \{Y_B, Y_C\}$$

In addition, we now need the symmetric invariants

$$d_{ABC} = \int d^2\sigma \sqrt{w} Y_A Y_B Y_C$$

which are APD analogs of the symmetric $SU(N)$ structure constants

$$\text{Tr}(T^A T^B T^C).$$

Both f_{ABC} and d_{ABC} are manifestly invariant under area preserving diffeomorphisms, because the overlap integrals only involve the metric determinant. This corresponds to the invariance under $SU(N)$ of their matrix analogs. However, for the full transcription of the lightcone gauge Lorentz generators into matrix language, we need yet another overlap integral, viz.

$$c_{ABC} = \int d^2\sigma \frac{\sqrt{w(\sigma)}}{\omega^A} w^{rs} \partial_r Y_A Y_B \partial_s Y_C.$$

To find its proper $SU(N)$ counterpart, we would have to introduce an analog of the Laplace Beltrami operator on the $SU(N)$ Lie algebra, such as for instance the expression $\sum_A [t^A, [t_A, \cdot]]$ (where the sum runs over a complete basis of the Lie algebra); see e.g. ^{56,55}. Another potentially worrisome feature is that c_{ABC} depends explicitly on the metric w_{rs} , and therefore is no longer APD invariant. It is only by a “miracle” of supermembrane theory that the concomitant ambiguities in the calculation of Lorentz algebra will drop out at the end. This may for instance be seen by working out the APD transformation on X_A^- , which also lacks manifest covariance³⁷:

$$\delta X_A^- = \xi^B f_{AB}{}^C X_C^- + \frac{\xi^B}{2P_0^+} (c_{ABC} + c_{ACB}) \phi^C$$

where ξ^A is are the orthonormal basis coefficients of the APD transformation parameter. Happily, the non-covariant terms in this variation vanish precisely on the physical subspace where $\phi^A \approx 0$.

As is well known from superstring theory (see e.g. ¹³), the crucial part in the check of Lorentz invariance is the bracket of two boost generators:

$$\{M^{-a}, M^{-b}\} \approx 0 \tag{11.3}$$

which must vanish on the physical states. For the supermembrane, this can be shown indeed after a lot of algebra.^{54,55} Let us just mention that in course of the calculation one has to employ identities such as

$$\begin{aligned} f^E{}_{[AB} f_{C]DE} &= 0 \\ f^E{}_{A(B} d_{CD)E} &= 0 \\ d_{ABC} f^A{}_{[DE} f^B{}_{F]G} &= 0 \\ c_{DE}{}^{[A} f^{BC]E} &= 0 \end{aligned}$$

These identities are valid for the APD tensors, but some of them are violated in the finite N truncation. The first is just the usual Jacobi identity, which is clearly also valid for $SU(N)$. Likewise, the second can be shown to hold also for the finite N matrix approximation. By contrast, the third holds only up to terms of order $O(1/N^2)$; the fourth depends on what expression is used as the finite N analogue of c_{ABC} and will be violated for any such choice. This shows that the finite N matrix model is not Lorentz invariant, as was to be expected (this conclusion is also in line with the interpretation of the finite N model as a lightlike compactification of M-Theory). But let us emphasize again that at

the classical level Lorentz invariance is restored in the large N limit, where the identities are valid again.

String theory teaches us that the closure of the classical Lorentz algebra by no means guarantees the Lorentz invariance of the quantum theory. Quite to the contrary, (super)string theories are only quantum consistent in certain critical dimensions where the anomalies cancel. Let us recall the procedure there: one first defines the physical Hilbert space by means of the Virasoro constraints. Then the ordering ambiguities in the Lorentz generators are removed by requiring that they be well-defined, i.e. have finite matrix elements between any two physical states. This is achieved by the normal ordering prescription. Only after this step does it make sense to actually compute the algebra, and to determine the anomaly — with the well known result that the superstring can only live in ten dimensions.

For the membrane Lorentz algebra, the situation is considerably more complicated. The algebra of the $M^{\mu\nu}(N)$ for finite N does not close even classically, as we just explained, but only in the limit $N \rightarrow \infty$. Replacing the classical phase space quantities by quantum operators, one immediately runs into quantum mechanical ordering ambiguities already for $N < \infty$ (where everything is still well defined). To proceed one would have to first establish the existence of the $N \rightarrow \infty$ limit, possibly with some N dependent coupling $g(N)$. Besides the obvious ordering ambiguities, which may contribute $\mathcal{O}(N)$ or $\mathcal{O}(N^2)$ terms to the algebra and thereby yield unwanted extra contributions in the large N limit, the construction of the quantum Lorentz generators may require extra renormalizations as is the case for composite operators in any renormalizable quantum field theory such as QCD. It would be very encouraging if one could show that this type of renormalization was, in fact, unnecessary for the $d = 11$ supermembrane. Remarkably, a recent calculation⁵⁷ shows that the lowest non-trivial terms induced by quantum mechanical operator ordering do cancel!^c

12 Massless states?

Our second special topic where there has been considerable work and some progress over the past two years concerns the question of massless states. Although there is still a scarcity of rigorous results for the $d = 11$ supermembrane, there are hints that a normalizable groundstate wavefunction does exist, while none is expected to exist for the $d = 4, 5$ or $d = 7$ supermembranes.

^cAmazingly, this calculation seems to work for all classically allowed dimensions, so there is no indication of a critical dimension $d = 11$ at the order considered.

For pedagogical reasons (and because the relevant equations are hardly ever written out in the literature!), we would first like to give some more details on the realization of the various operators. Straightforward replacement of the canonical brackets by quantum (anti)commutators immediately leads to

$$[P_a^A, X_b^B] = -i\delta_{ab}\delta^{AB}$$

$$\{\theta_\alpha^A, \theta_\beta^B\} = \delta_{\alpha\beta}\delta^{AB}$$

A technical inconvenience is that the reality of the fermions forces us to break the manifest $SO(9)$ invariance down to $Spin(7) \times U(1)$, if we want an operator realization of the fermionic brackets. This can be accomplished by singling out the 8 and 9 directions, and by combining the 16 real $SO(9)$ spinors into two complex 8-component spinors, which are complex conjugate to one another. Consequently, we set

$$\lambda_\alpha^A := \frac{1}{\sqrt{2}}(\theta_\alpha^A + i\theta_{\alpha+8}^A)$$

$$\bar{\lambda}_\alpha^A := \frac{1}{\sqrt{2}}(\theta_\alpha^A - i\theta_{\alpha+8}^A) \equiv \frac{\partial}{\partial \lambda_\alpha^A}$$

where α now runs over the values $1, \dots, 8$ appropriate for $Spin(7)$. The coordinates split into

$$X_a^A = (X_i^A, Z^A, \bar{Z}^A)$$

with

$$Z^A := \frac{1}{\sqrt{2}}(X_8^A + iX_9^A)$$

$$\bar{Z}^A := \frac{1}{\sqrt{2}}(X_8^A - iX_9^A)$$

The center of mass (zero mode) coordinates are split similarly. As already explained, the zero mode part of the supercharge is completely separated from the (dynamical) rest of the theory. In the $Spin(7) \times U(1)$ the corresponding supercharges become

$$Q_{0\alpha} = -i\Gamma_{\alpha\beta}^i \frac{\partial}{\partial X_{0i}} \lambda_{0\beta} + \sqrt{2} \frac{\partial}{\partial Z_0} \frac{\partial}{\partial \lambda_{0\alpha}}$$

$$Q_{0\alpha}^+ = i\Gamma_{\alpha\beta}^i \frac{\partial}{\partial X_{0i}} \frac{\partial}{\partial \lambda_{0\beta}} + \sqrt{2} \frac{\partial}{\partial Z_0} \lambda_{0\alpha}$$

They act on wave functions depending on the zero mode variables (X_0, Z_0, \bar{Z}_0) and $\lambda_{0\alpha}$. Since the fermionic variables are treated as generating elements of a Grassmann algebra, the wave function is a polynomial over this Grassmann algebra. The bosonic part of the wave function is made up of the even elements

$$1, \lambda_{0\alpha}\lambda_{0\beta}, \lambda_{0\alpha}\lambda_{0\beta}\lambda_{0\gamma}\lambda_{0\delta}, \dots$$

multiplied by plane wave functions of the bosonic coordinates (X_0, Z_0, \bar{Z}_0) . Under $SO(9)$, these states transform as the $\mathbf{44} \oplus \mathbf{84}$ representations, corresponding to the on-shell degrees of freedom of the metric $g_{\mu\nu}$ and the three index field $A_{\mu\nu\rho}$, respectively, of $d = 11$ supergravity. The odd elements (again multiplied by plane wave functions)

$$\lambda_{0\alpha}, \lambda_{0\alpha}\lambda_{0\beta}\lambda_{0\gamma}, \lambda_{0\alpha}\lambda_{0\beta}\lambda_{0\gamma}\lambda_{0\delta}\lambda_{0\epsilon}, \dots$$

transform as the $\mathbf{128}$ (i.e. traceless vector spinor) of $SO(9)$ and thus comprise the on-shell degrees of freedom of the $d = 11$ gravitino. Thus we recover precisely the representations corresponding to a massless multiplet of $d = 11$ supergravity, i.e. the on-shell states of $d = 11$ supergravity (also referred to as “supergraviton”).

Readers should keep in mind that these considerations are purely kinematical and by themselves provide no evidence for or against the existence of a massless $d = 11$ supermultiplet for either supermembrane or M(atrrix) Theory. In order to find the supergravity states in the Hilbert space of the supermembrane, one must identify the non-zero mode part of the wave function which is annihilated by the non-zero mode part of the full supercharge. Before we state what the problem is, it is instructive to have a look at the equations that must be solved for this purpose.

In the $Spin(7) \times U(1)$ formalism, the non-zero mode supercharges are given by¹⁷

$$\begin{aligned} Q_\alpha &= \left\{ -i\Gamma_{\alpha\beta}^i \frac{\partial}{\partial X_i^A} + \frac{1}{2} f_{ABC} X_i^B X_j^C \Gamma_{\alpha\beta}^{ij} - f_{ABC} Z^B \bar{Z}^C \delta_{\alpha\beta} \right\} \lambda_\beta^A + \\ &\quad + \sqrt{2} \left\{ \delta_{\alpha\beta} \frac{\partial}{\partial Z^A} + i f_{ABC} X_i^B \bar{Z}^C \Gamma_{\alpha\beta}^i \right\} \frac{\partial}{\partial \lambda_{\beta A}}, \\ Q_\alpha^+ &= \left\{ i\Gamma_{\alpha\beta}^i \frac{\partial}{\partial X_i^A} + \frac{1}{2} f_{ABC} X_i^B X_j^C \Gamma_{\alpha\beta}^{ij} + f_{ABC} Z^B \bar{Z}^C \delta_{\alpha\beta} \right\} \frac{\partial}{\partial \lambda_{\beta A}} + \\ &\quad + \sqrt{2} \left\{ -\delta_{\alpha\beta} \frac{\partial}{\partial Z^A} + i f_{ABC} X_i^B Z^C \Gamma_{\alpha\beta}^i \right\} \lambda_\beta^A. \end{aligned}$$

The Gauss constraint is realized by the operator

$$\phi^A = f^{ABC} \left\{ X_{iB} \frac{\partial}{\partial X_C^C} + Z_B \frac{\partial}{\partial Z^C} + \bar{Z}_B \frac{\partial}{\partial \bar{Z}^C} + \lambda_{\alpha B} \frac{\partial}{\partial \lambda_\alpha^C} \right\}$$

The supercharges satisfy the following anticommutation relations:

$$\{Q_\alpha, Q_\beta\} = 2\sqrt{2}\delta_{\alpha\beta}\bar{Z}^A\phi_A \approx 0$$

$$\{Q_\alpha^+, Q_\beta^+\} = 2\sqrt{2}\delta_{\alpha\beta}Z^A\phi_A \approx 0$$

$$\{Q_\alpha, Q_\beta^+\} = 2\delta_{\alpha\beta}H - 2i\Gamma_{\alpha\beta}^i X_i^A \phi_A$$

The last anticommutator is thus weakly equal to the Hamiltonian alias the mass operator \mathcal{M}^2 .

The Hamiltonian can be split into two contributions, namely

$$H = H_B + H_F.$$

with the positive bosonic part

$$\begin{aligned} H_B &= -\frac{1}{2} \frac{\partial^2}{\partial X_i^A \partial X_i^A} - \frac{\partial^2}{\partial Z^A \partial \bar{Z}^A} + \\ &+ \frac{1}{4} f_{AB}{}^E f_{CDE} \left\{ X_i^A X_j^B X_i^C X_j^D + 4X_i^A Z^B X_j^C \bar{Z}^D + 2Z^A \bar{Z}^B Z^C \bar{Z}^D \right\} \\ &\geq 0 \end{aligned}$$

and the fermionic part

$$H_F = if_{ABC} X_i^A \lambda_\alpha^B \Gamma_{\alpha\beta}^i \frac{\partial}{\partial \lambda_{\beta C}} + \frac{1}{\sqrt{2}} f_{ABC} \left\{ Z^A \lambda_\alpha^B \lambda_\alpha^C - \bar{Z}^A \frac{\partial}{\partial \lambda_\alpha^B} \frac{\partial}{\partial \lambda_\alpha^C} \right\}.$$

These operators act on the non-zero mode wave functions

$$\Psi = \sum_k \sum_{\substack{\alpha_1, \dots, \alpha_k \\ A_1, \dots, A_k}} \Phi_{A_1 \dots A_k}^{\alpha_1 \dots \alpha_k} (x_i^A, Z^A, \bar{Z}^A) \lambda_{\alpha_1}^{A_1} \dots \lambda_{\alpha_k}^{A_k}.$$

where k assumes only even values for bosonic wave functions; we here anticipate that this wave function must be an $SO(9)$ singlet. Since $\alpha_i \in \{1, \dots, 8\}$ and the $SU(N)$ indices take values in $\{1, \dots, N^2 - 1\}$ these wave functions have $2^{8(N^2-1)}$ components.

Let us now state the requirements that Ψ must satisfy in order to be an acceptable non-zero mode wave function for the groundstate describing a massless state that combines with the zero mode wave function to yield a short (massless) $d = 11$ supermultiplet.

First of all, Ψ must be annihilated by the Hamiltonian or by the supercharges:

$$H\Psi = 0 \iff Q_\alpha\Psi = Q_\alpha^+\Psi = 0 \quad (12.1)$$

The equivalence of these equations is ensured by the physical state constraints

$$\phi^A\Psi = 0$$

for $A \in \{1, \dots, N^2 - 1\}$. Inspection of the explicit operator realization of the supercharges immediately shows that the equations (12.1) relate all $2^{8(N^2-1)}$ components of Ψ . Therefore we are dealing with a highly coupled system of equations.

Secondly, the non-zero mode wave function must be a singlet of the rotational $SO(9)$. This is necessary because the zero mode part of the full wave function by itself already provides the desired $SO(9)$ representations for $d = 11$ supergravity, so that the remainder of the wave-function must obey

$$J_{ab}\Psi \equiv (L_{ab} + S_{ab})\Psi = 0$$

where L_{ab} and S_{ab} are the ‘‘orbital’’ and ‘‘spin’’ parts of the angular momentum operator, respectively. The singlet requirement is unique to eleven dimensions: in lower dimensions the center of mass supercharges do not yield the full supergravity multiplet, and therefore the associated groundstate wave functions could not be singlets.

Finally, Ψ must be square integrable

$$\|\Psi\|^2 = \sum_k \sum_{\substack{\alpha_1, \dots, \alpha_k \\ A_1, \dots, A_k}} \|\Phi_{A_1 \dots A_k}^{\alpha_1 \dots \alpha_k}\|^2 < \infty$$

where $\|\dots\|$ denotes the usual L^2 norm. So altogether we must check the square integrability of $2^{8(N^2-1)}$ component functions!

An important observation now is that, unlike in superstring theory, Ψ *can not* be written as a tensor product

$$\Psi = \Psi_B(X, Z, \bar{Z}) \otimes \Psi_F(\lambda, \bar{\lambda}),$$

where both factors are separately invariant under $SO(9)$, i.e. $L_{ab}\Psi_B = 0$ and $S_{ab}\Psi_F = 0$. To see this, recall that for the superstring, the supercharges are

of the form¹³

$$Q_\alpha \propto \sum_{n \neq 0} \alpha_{-n}^i \Gamma_{\alpha\beta}^i S_{n\beta}.$$

The associated groundstate wavefunction is

$$\Psi(\text{superstring}) \propto \left(\prod_{n \geq 1} e^{-n \bar{X}_n^2} \right) \left(\prod_{n,\alpha} \lambda_{n\alpha} \right) \quad (12.2)$$

with a somewhat schematic, but hopefully self-explanatory, notation: the Dirac sea is “half filled” such that the state is either annihilated by the bosonic oscillators (for $n < 0$), or by the fermionic oscillators (for $n > 0$). Both factors in (12.2) are obviously $SO(8)$ singlets. To see that this factorization does not work for a normalizable supermembrane (or matrix model) groundstate, assume otherwise. Then, since H_F is linear in the coordinate variables, its expectation value $(\Psi, H_F \Psi)$ contains a factor $(\Psi_B, X_i^A \Psi_B)$ which vanishes by the assumed $SO(9)$ invariance of Ψ_B . Consequently, we would also have

$$(\Psi, H_F \Psi) = 0$$

whence

$$(\Psi, H \Psi) = (\Psi, H_B \Psi) = 0$$

But H_B is a positive operator, and together with the assumed square integrability of Ψ_B this implies the vanishing of Ψ_B , and hence of Ψ , which is a contradiction. (If we did not assume normalizability of Ψ_B here, we would not be able to conclude $\Psi_B = 0$ from this argument).

This means that Ψ is annihilated by the total angular momentum operators J_{ab} that generate the $SO(9)$ rotations but the orbit and spin parts L_{ab} and S_{ab} do not annihilate Ψ separately. The difficulties are also illustrated by the following very rough argument. For superstring theory, the equation $Q\Psi = 0$ is schematically of the form $(\frac{\partial}{\partial X} + X)\psi = 0$, which has the square integrable solution $\psi = e^{-\frac{1}{2}X^2}$, whereas for the supermembrane it has the schematic form $(\frac{\partial}{\partial X} + X^2)\psi = 0$ which suggests a non-normalizable solution of the type $\psi = e^{-\frac{1}{3}X^3}$. Of course, this is by no means a counterargument against the existence of a normalizable groundstate, precisely because the above equations do *not* factorize, and all the components are coupled. But the argument does indicate that the wave function must be very clever indeed if it is to evade this apparent obstruction to square integrability.

Even if it is not possible to find the ground state wavefunction explicitly, evidence for (or against) its existence can be supplied by calculating the Witten

index^{58,59} (see also⁶⁰ for earlier work in this direction). This index is formally defined as the difference between the number of bosonic and the number of fermionic states:

$$I_W = \#(\text{bosonic states}) - \#(\text{fermionic states})$$

For a well behaved theory with a discrete spectrum every bosonic state with nonzero energy can be transformed into a fermionic state by acting on it with the supercharge and vice versa. So the contribution to the index from the states of nonzero energy vanishes. This reasoning does not hold for states with zero energy since they are annihilated by the supercharge. So a non-vanishing Witten index indicates the existence of zero energy states, i.e. ground states. A regularized formula handy for calculations is

$$I_W = \lim_{\beta \rightarrow \infty} \text{Tr} [(-1)^F e^{-\beta H}].$$

Here, F is the fermion number operator. The exponential factor makes the trace well defined but does not disturb the result since it only affects positive energies. In fact, for well behaved theories with a discrete spectrum (where the Hamiltonian H is a Fredholm operator) the expression is independent of the “inverse temperature” β and can be evaluated at small β with the help of field theory perturbative methods.

If, however, the spectrum is continuous and if there is no mass gap things are quite a bit more tricky: The above argument does not imply that the bosonic and fermionic densities of states cancel for positive energies. So we really have to evaluate the index for large β , a region that is not accessible to perturbation theory. Following^{58,59} we therefore write the index as

$$I_W = \lim_{\beta \rightarrow 0} \text{Tr} [(-1)^F e^{-\beta H}] + \delta I$$

where so-called defect is defined as

$$\begin{aligned} \delta I &= \int_0^\infty d\beta \frac{d}{d\beta} \text{Tr} (-1)^F e^{-\beta H} \\ &= - \int_0^\infty d\beta \text{Tr} (-1)^F H e^{-\beta H} \\ &= - \int_0^\infty dE [\rho_B(E) - \rho_F(E)]. \end{aligned}$$

with the bosonic and fermionic spectral densities ρ_B and ρ_F , respectively. The difference of these two spectral densities would vanish for Fredholm operators. The defect is difficult to compute and few rigorous results are so far available.

With the help of the heat-kernel representation we write the bulk term as a finite dimensional integral: Defining

$$\begin{aligned}\mathcal{Z}_{D,N} &= \int \prod_{\mu,A} \frac{dX_\mu^A}{\sqrt{2\pi}} \prod_{\alpha,A} d\Psi_\alpha^A \exp \left\{ \frac{1}{2} \text{tr}[X_\mu, X_\nu]^2 + \text{tr} \Psi \Gamma^\mu [X_\mu, \Psi] \right\} \\ &= \int \prod_{\mu,A} \frac{dX_\mu^A}{\sqrt{2\pi}} \mathcal{P}_{D,N}(X) \exp \left\{ \frac{1}{2} \text{tr}[X_\mu, X_\nu]^2 \right\}\end{aligned}$$

we have

$$I_W = \mathcal{F}_N^{-1} \mathcal{Z}_{D,N} + \delta I.$$

where the normalization factor \mathcal{F}_N is given by^{61,62}

$$\mathcal{F}_N = \frac{2^{\frac{N(N+1)}{2}} \pi^{\frac{N-1}{2}}}{2\sqrt{N} \prod_{k=1}^{N-1} k!}$$

and $\mathcal{P}_{D,N}(X)$ is the Pfaffian

$$\mathcal{P}_{D,N}(X) = \text{Pf}(-if_{ABC} \Gamma_{\alpha\beta}^\mu X_\mu^C).$$

It is the square root of the determinant of an antisymmetric $\mathcal{N}(N^2 - 1) \times \mathcal{N}(N^2 - 1)$ matrix and therefore a homogeneous and $SO(D) \times SU(N)$ invariant polynomial of degree $(D - 2)(N^2 - 1)$ in the X 's, which is explicitly known only in a very limited number of cases.

The analytic results for $SU(2)$ read:

$$\mathcal{Z}_{D,2} = \sqrt{8\pi} \begin{cases} 0 & \text{for } D = 3 \\ 1/4 & \text{for } D = 4 \\ 1/4 & \text{for } D = 6 \\ 5/4 & \text{for } D = 10 \end{cases}$$

The normalization is $\mathcal{F}_2 = \sqrt{8\pi}$. This suggests

$$\delta I = -\frac{1}{4}$$

which was indeed found in⁵⁹. This means that for $SU(2)$ the index is

$$I_W = \begin{cases} 0 & \text{for } D = 4, 6 \\ 1 & \text{for } D = 10 \end{cases}.$$

implying the existence of a normalizable groundstate for the maximally supersymmetric $SU(2)$ matrix model, but not for the lower dimensional ones.

Instanton calculations suggest the following generalization for $SU(N)$.⁶³

$$\mathcal{Z}_{D,N} = \mathcal{F}_N \begin{cases} 0 & \text{for } D = 3 \\ 1/N^2 & \text{for } D = 4, 6 \\ \sum_{m|N} \frac{1}{m^2} & \text{for } D = 10 \end{cases} \quad (12.3)$$

and

$$\delta I = \begin{cases} 0 & \text{for } D = 3 \\ -1/N^2 & \text{for } D = 4, 6 \\ -\sum_{\substack{m|N \\ m \geq 2}} \frac{1}{m^2} & \text{for } D = 10 \end{cases} \quad (12.4)$$

A proof of the first part of this conjecture was given very recently in⁶⁴, while an independent test using Monte-Carlo integration has been performed in⁶¹. If the second part of the above conjecture could also be verified this would indicate that the above result for $SU(2)$ generalizes to any N , such that a normalizable ground state exist for all N when $D = 10$, but none for $D = 4, 6$. A curious feature of (12.3) is the non-analytic nature of the $N \rightarrow \infty$ limit in ten dimensions: Because for all N

$$1 < \sum_{m|N} \frac{1}{m^2} < \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}$$

we can find increasing subsequences $\{N_j\}$ of the natural numbers, such that any given real number $1 \leq c \leq \pi^2/6$ can be obtained as a limiting value:

$$\lim_{j \rightarrow \infty} \sum_{m|N_j} \frac{1}{m^2} = c$$

This shows that, even in the absence of large N divergences, the $N \rightarrow \infty$ limit could be quite subtle. The non-analytic behavior may also be a special feature of the supersymmetric theory and $D = 10$, whereas there are indications⁶⁵ that the bosonic matrix model exhibits a more regular behavior^d. Incidentally, the validity of the formulas (12.3) and (12.4) indicates that one should be able

^dThe non-supersymmetric bosonic model (obtained simply by dropping the Pfaffian from the above integral) was recently investigated in⁶², where numerical evidence was presented that the integrals exist if

$$N > \frac{D}{D-2}.$$

to analytically solve the so-called IKKT model.⁶⁶

$$\begin{aligned} \mathcal{Z}_{IKKT}(\beta) &= \sum_{N=1}^{\infty} \sum_{m|N} \frac{1}{m^2} e^{-\beta N} \\ &= \sum_{m,n=1}^{\infty} \frac{1}{m^2} e^{-\beta mn} = \sum_{m=1}^{\infty} \frac{1}{m^2} \frac{1}{e^{m\beta} - 1} \end{aligned}$$

Apart from the fact that analyticity has been restored, the main interest of this model resides in the fact that it treats all N simultaneously in a kind a grand canonical partition function, albeit only for the completely reduced model. It would be remarkable if one could extend these considerations to the full (1+0)-dimensional matrix model in terms of a chemical potential for the D0 particles.

Besides the work on Witten indices, there is now a growing body of rigorous results, although so far mostly for certain truncations. For the $SU(2)$ model of ¹⁷, a rigorous proof of non-existence is now available⁶⁷; for further extensions of these results, see ⁶⁸. In ⁶⁹ the question of normalizable states was investigated by means of a deformed version of the equations defining the supersymmetric groundstate (where the deformation in particular introduces “mass terms” modifying the continuous spectrum to a discrete one). Yet another approach based on an investigation of the asymptotic nature of the groundstate for $SU(2)$ was initiated in ⁷⁰, and further pursued in ⁷¹.

However, even if we were able to establish the existence of massless states for arbitrary N we would not yet be done: after all, we would also like to know *what they look like!* If M(atr)ix theory is really the fundamental theory it is claimed to be, knowledge of its groundstate wave function would profoundly alter our outlook on the world.

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