Nonspherical perturbations of critical collapse and cosmic censorship

Carsten Gundlach

Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut), Schlaatzweg 1, 14473 Potsdam, Germany
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Choptuik has demonstrated that naked singularities can arise in gravitational collapse from smooth, asymptotically flat initial data, and that such data have codimension one in spherical symmetry. Here we show, for perfect fluid matter with an equation of state $p = \rho/3$, by perturbing around spherical symmetry, that such data have in fact codimension one in the full phase space, at least in a neighborhood of spherically symmetric data. [S0556-2821(98)50212-3]

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I. COSMIC CENSORSHIP AND CRITICAL COLLAPSE

Spacetime singularities are a ubiquitous feature of general relativity. In astrophysical situations, singularities only arise inside black holes. While no information (light signal) can leave the black hole region of spacetime, the singularity itself cannot emit any light signal at all: It is spacelike, that is, it can be thought of as a local end to time. Nevertheless, the field equations of general relativity also admit solutions with timelike singularities. Such a singularity may be loosely thought of as a point in space, just like a particle, and in many solutions, light signals could travel from such a singularity to infinity. The spacetime to the future of the singularity is then no longer uniquely determined by initial data in the past. It is therefore an interesting question to ask how generic or natural such spacetimes, or the initial data from which they evolve, are. A "cosmic censorship conjecture" has been formulated by various authors, stating roughly that naked singularities do not arise from asymptotically flat data for reasonable matter [1]. The only counterexamples existing until recently could be written off as either not asymptotically flat, involving matter models (dust) that form singularities even in the absence of gravity, or requiring high symmetry (spherical shells). Until recently, however, it could still be conjectured that no smooth, asymptotically flat data for a reasonable matter model would evolve into a naked singularity. That was disproved by the spacetimes constructed numerically by Choptuik [2], which we review now.

Consider a smooth one-parameter family of spherically symmetric, asymptotically flat, smooth initial data for a self-gravitating scalar field. The only condition on the family is that it contains both data sets which form a black hole (say, for large parameter p), and data sets which do not (say, for small p). Choptuik showed that by fine-tuning the value of p to a critical value p_*, one can obtain arbitrarily small black holes. For $p \gtrsim p_*$ the black hole mass scales approximately as $M_{\text{BH}} \simeq C(p - p_*)^{\gamma}$, with $\gamma \simeq 0.37$ a universal constant. A solution evolving from fine-tuned initial data approaches one and the same solution, independently of what the initial data looked like. The better the fine-tuning, the longer the solution follows this "critical solution." The critical solution is an attractor of codimension one, because it has precisely one growing perturbation mode. Its basin of attraction is the black hole threshold, because the eventual fate of the solution depends on the amplitude of the growing mode in the

following manner. If it is present with, say, positive amplitude, the final outcome will be a black hole. If it is present with negative amplitude, the final outcome will be dispersion of the scalar waves to infinity, without a black hole. If its amplitude is precisely zero, in the limit of infinitely good fine-tuning of the parameter p, the solution settles down to the critical solution and never leaves it.

The crucial point for cosmic censorship is that the critical solution has a naked curvature singularity. Any smooth, asymptotically flat initial data in which the growing mode is not present, approach the critical solution and its naked singularity. These data form a set of codimension one (in spherical symmetry) because only the amplitude of the one growing mode needs to be fine-tuned to zero. The spacetimes arising from data on the black hole threshold do not contain a "zero mass black hole," but contain a region of the universal critical solution surrounding the naked singularity, which is smoothly matched to an asymptotically flat region that depends on the initial data. Near the singularity, the deviation of the actual collapse spacetime from the universal critical solution decays as a (noninteger) power of geodesic distance to the singularity. For a recent review of critical collapse, see [3].

II. BEYOND SPHERICAL SYMMETRY

An obvious question is if critical collapse is tied to spherical symmetry. Abrahams and Evans [4] fine-tuned the collapse of axisymmetric gravitational waves, and found evidence for black hole mass scaling, universality, and a self-similar critical solution. This shows that critical phenomena can occur for axisymmetric, highly non-spherical initial data. In a complementary approach, we take here a known critical solution in spherical symmetry, and perturb it non-spherically. We find that the only growing mode is the previously known spherical one.

This result is weaker than that of Abrahams and Evans in being only perturbative. But a linearized result establishes a result in an open neighborhood, up to linearization stability. In spherical symmetry the basin of attraction of the critical solution was empirically found to be the entire black hole threshold, even far from the critical solution [2]. This makes nonlinear instabilities seem extremely unlikely, and suggests that the open set may be in fact quite large. Generally it is remarkable how much in critical collapse can be understood

quantitatively through linear perturbation calculations: not only the critical exponent γ, but also universality with respect to initial data and matter models, and critical exponents for the black hole electric charge and angular momentum.

Our result is stronger than that of Abrahams and Evans in not being limited to axisymmetry. Choptuik's spherically symmetric results showed that a genericity condition on initial data is necessary in formulating cosmic censorship. Here we have demonstrated for the first time that this condition is sharp: The set of ''non-generic'' initial data that form a naked singularity can locally have codimension one in the full phase space.

III. THE BACKGROUND SOLUTION

We write the general spherically symmetric spacetime as a manifold $M = M^2 \times S^2$ with metric

$$g_{\mu\nu} = \mathrm{diag}(g_{AB}, r^2 \gamma_{ab}), \tag{1}$$

where g_{AB} is an arbitrary metric on M^2, r is a scalar on M^2, with $r = 0$ defining the boundary of M^2, and γ_{ab} is the unit curvature metric on S^2. This spacetime is continuously self-similar if there are coordinates x and τ on M^2 such that in these coordinates the rescaled metric coefficients $e^{2\tau} g_{AB}$ and $e^{2\tau} r^2$ do not depend on τ. It is discretely self-similar if they are periodic in τ. A linearized spherical perturbation of a continuously self-similar spacetime can be decomposed into modes of the schematic form $e^{\lambda\tau} f(x)$. The critical solution has precisely one mode with $\mathrm{Re}\lambda > 0$. This is the mode that has to be beaten down by fine-tuning, while all other modes die away naturally as the curvature singularity $\tau = \infty$ is approached.

As a matter model we choose the perfect fluid with equation of state $p = c_s^2 \rho$, with c_s^2 a constant. While our equations hold for $0 < c_s^2 < 1$, we have carried out the numerical calculations only for $c_s^2 = 1/3$, the equation of state of a radiation gas.

The coordinates x and τ adapted to self-similarity are not unique, and they need not be fixed for the purpose of the following discussion. In our numerical calculations, however, we make a coordinate choice that is based on the Schwarzschild-like form $ds^2 = -N^2 \, dt^2 + A^2 \, dr^2 + r^2 \, d\Omega^2$ of the metric. The coordinate transformation $t = -e^{-\tau}$, $r = sxe^{-\tau}$ brings this into the form

$$g_{AB} = e^{-2\tau} \begin{pmatrix} -N^2 + s^2 x^2 A^2 & -s^2 x A^2 \\ \mathrm{Symm} & s^2 A^2 \end{pmatrix}, \tag{2}$$

$$r^2 = e^{-2\tau} s^2 x^2. \tag{3}$$

As a final gauge condition, we impose $N = 1$ at $x = 0$, that is, t is central proper time. The spacetime in these coordinates is self-similar if $N(x, \tau)$ and $A(x, \tau)$ are functions of x only.

The background stress-energy tensor is

$$t_{\mu\nu} = \mathrm{diag}(t_{AB}, c_s^2 \rho r^2 \gamma_{ab}), \tag{4}$$

where

$$t_{AB} = (1 + c_s^2) \rho u_A u_B + c_s^2 \rho g_{AB}, \tag{5}$$

and $u_\mu = (u_A, 0)$ is the fluid 4-velocity. The calculation of the background solution as a non-linear boundary value problem was carried out along the lines of [5–7]. For the density we make the ansatz $\rho = e^{2\tau} \bar{\rho}(x)$, and for the radial velocity, $u^A r_{,A} = v(x)$. Under this ansatz the Einstein and matter equations go over into a system of coupled ordinary differential equations (ODEs). These have a regular singular point (sonic point), where the surface $x = \mathrm{const}$ becomes an ingoing matter characteristic. The solution we want is the one uniquely specified by regularity at the origin $x = 0$ and at the sonic point. We define the constant s so that the sonic point is at $x = 1$, and solve the ODEs as a boundary value problem between $x = 0$ and $x = 1$, solving for s as a nonlinear eigenvalue.

IV. PERTURBATION METHOD

Now one could perturb the critical solution $Z(x)$ with an ansatz $\Delta Z(x, \tau) = e^{\lambda\tau} f(x)$ and solve the linearized boundary value problem for the discrete (complex) eigenvalues λ and mode functions $f(x)$. This program has been carried out both for continuously [6,7] and discretely [8] self-similar solutions, and has allowed a precise numerical calculation of the critical exponent γ, which can be shown by dimensional analysis to be simply the inverse of the one positive real eigenvalue λ. As a byproduct, one can determine the entire perturbation spectrum (which is discrete).

Here, we mainly want to know is if there is *any* eigenvalue with positive real part among the non-spherical perturbations, in addition to the one in the spherical perturbations. To answer this yes/no question, we write the perturbation equations as evolution equations in the time variable τ. We decompose into spherical harmonics, and consider each value of l and m separately. Because the background is spherically symmetric, the dynamics of perturbations are the same for all values of m (for given l). We evolve generic initial data for these equations for a sufficiently large interval of τ. Generic data, with no field vanishing anywhere, constitute a superposition of *all* the (unknown) perturbation modes.

In the time evolution, the mode with the largest $\mathrm{Re}\lambda$ takes over after a transition period, and both that λ and the corresponding $f(x)$ can be simply read off from the late-time data [9]. For $l = 0$ in particular, this allows us to check our procedure by reading off the critical exponent as $\gamma \approx 0.36$ in agreement with previous calculations [6,7].

On a numerical grid, the frequency of modes that can be represented is limited by the grid spacing, so that we are only probing a large finite-dimensional subspace of all possible modes. Nevertheless, one can rule out the existence of unstable modes at very high spatial frequency (with respect to the coordinate x) by the following argument. The perturbation equations form a system of linear wave equations with x-dependent coefficients and an x-dependent mass matrix. High-frequency modes propagate essentially by geometric optics, and the mass terms are irrelevant for them. Therefore all high-frequency modes have the same dynamics. If they are shown to be decaying at frequencies still resolved by the numerical grid, but at which the mass-like terms can already be neglected with respect to derivative terms, then we can be sure that even higher frequencies not resolved on the numerical grid will decay in the same way. The geometric optics
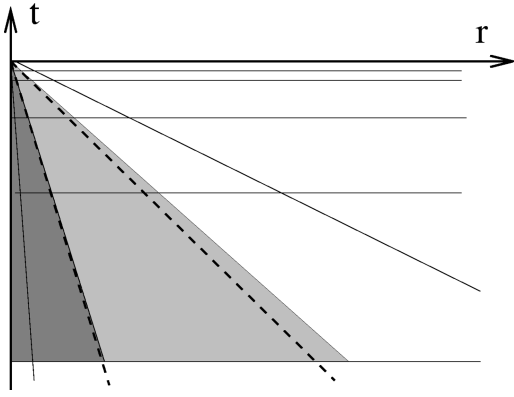
FIG. 1. The horizontal thin lines accumulating at $t=0$ are lines of constant τ. The fanning thin lines are lines of constant x. The inner dashed line is the ingoing sound cone. The outer dashed line is the ingoing light cone. The dark shaded region is the numerical domain of the background, spherical and odd perturbations calculations. The total shaded region is the numerical domain of the even perturbations calculation.

argument also applies to large values of l, which labels spatial frequencies in the angular coordinates θ and φ. However, high l modes actually decay faster in τ with increasing l, by a factor of $e^{-l\tau}$. Roughly speaking, this is because regular perturbations must be of $O(r^l)$ at the origin.

The spherical perturbation equations are obtained simply by linearizing the spherical field equations. There is no difficulty in writing them as evolution equations in time coordinate τ and radial coordinate x. Boundary conditions at $x=0$ arise from demanding regularity at the center of spherical symmetry (all fields must be either even or odd in x). The surface $x=1$ is an ingoing characteristic of the spherical perturbation equations, that is, a sound cone. No physical boundary conditions for perturbations are required there. Numerically, this absence of boundary conditions is implemented by a finite difference scheme that is aware of the characteristics. Then we can bound the numerical domain by $x=0$ on one side and $x=1$, or any larger constant x, on the other and evolve to arbitrary values of τ. If one allows for non-spherical perturbations, one encounters gravitational as well as sound waves. The numerical domain for the entire system of perturbations then has to be extended to the ingoing light cone, that is to $x=x_c$ defined by $sx_cA(x_c)=N(x_c)$. The background solution is easily extended from $x=1$ to $x=x_c$ and beyond. Figure 1 illustrates the coordinates, the characteristics, and the numerical domain.

V. GAUGE-INVARIANT PERTURBATIONS

In going beyond spherical symmetry, one also has to deal with gauge-dependence and the presence of constraints in the linearized Einstein and matter equations. Throughout this letter, we use the formalism and notation of Gerlach and Sengupta (GS) [10]. Any linear perturbation around spherical symmetry can be decomposed into scalar, vector and tensor fields on M^2 times spherical harmonics Y_l^m on S^2. Different l,m decouple. In the following we consider one value of l,m at a time, and no longer write these indices on Y. Spherical harmonic vector fields on S^2 are $Y_{:a}$ and $S_a = \epsilon_a{}^b Y_{:b}$, where a colon indicates the covariant derivative on S^2, $\gamma_{ab:c}=0$,

and ϵ_{ab} is the covariantly constant totally antisymmetric tensor on S^2, $\epsilon_{ab:c}=0$. Tensor perturbations based on Y and its derivatives (even or polar perturbations) decouple from tensor perturbations based on S_a and its derivatives (odd or axial perturbations). In the following we consider even and odd perturbations separately.

GS express the 10 metric perturbations in the $2+2$ split. They calculate their transformation to linear order under the 4 infinitesimal coordinate changes, and find 6 linear combinations of metric perturbations (with coefficients depending on the background) that are gauge-invariant to this order. The 10 stress-energy perturbations can be combined to form 10 gauge-invariant ones. The linearized Einstein equations can then be expressed in terms of those 16 variables alone. The remaining 4 variables are pure gauge in the sense that one can give them arbitrary values, and reconstruct all 20 metric and stress-energy perturbations in the particular gauge determined that way.

Calculation with, and interpretation of, the GS variables is simplified by the fact that there is a gauge in which they can be identified directly with 16 gauge-dependent perturbations, while the remaining 4 gauge-dependent metric perturbations vanish. This gauge is the Regge-Wheeler gauge. In order to keep the notation simple, we present the 16 gauge-invariant perturbations in this manner. The split between the odd and even sectors is as follows. There are three odd metric perturbations, and one odd infinitesimal coordinate transformation, leaving two odd gauge-invariant perturbations, in the form of a vector field on M^2. There are $7-3=4$ even gauge-invariant metric perturbations, in the form of a symmetric tensor and a scalar. There are three odd and seven even gauge-invariant matter perturbations. The general odd metric and matter perturbations, in Regge-Wheeler gauge, but expressed through the gauge-invariants, are

$$\Delta g_{\mu\nu}=\begin{pmatrix} 0 & k_A Y_{:b} \\ \text{Symm} & 0 \end{pmatrix},\tag{6}$$

$$\Delta t_{\mu\nu}=\begin{pmatrix} 0 & L_A Y_{:b} \\ \text{Symm} & L(S_{a:b}+S_{b:a}) \end{pmatrix}.\tag{7}$$

The general even metric and matter perturbations are

$$\Delta g_{\mu\nu}=\begin{pmatrix} k_{AB} & 0 \\ 0 & kr^2\gamma_{ab} \end{pmatrix},\tag{8}$$

$$\Delta t_{\mu\nu}=\begin{pmatrix} T_{AB} & T_A Y_{:b} \\ \text{Symm} & T^1 r^2 \gamma_{ab}+T^2 Y_{:ab} \end{pmatrix}.\tag{9}$$

We now define gauge-invariant fluid perturbations, and present them once more by giving the equivalent gauge-dependent perturbations in Regge-Wheeler gauge. The odd perturbation of the fluid 4-velocity is

$$\Delta u_\mu=(0,\beta S_a).\tag{10}$$

β parametrizes axial fluid rotation. The even perturbation of the fluid 4-velocity is

$$\Delta u_\mu=(\Delta u_A,\alpha Y_{:a}),\quad\text{where }\Delta u_A=\gamma n_A+k_{AB}u^B.\tag{11}$$

Here $n^A = \epsilon^{AB} u_B$, with ϵ_{AB} the totally antisymmetric covariant unit tensor on M^2. Note that $u^A \Delta u_A = 0$. α parametrizes axial fluid motion, while γ parametrizes perturbations of the radial fluid motion. The density perturbation also belongs to the even sector, and we parametrize it as $\Delta \rho = \omega Y \rho$. (By virtue of our equation of state, $\Delta p = c_s^2 \Delta \rho$.) The general gauge-invariant stress-energy perturbations of GS are related to the gauge-invariant fluid perturbations as follows. In the odd sector, we have

$$L_A = \beta(1+c_s^2)\rho u_A, \quad L=0, \tag{12}$$

and in the even sector,

$$T_A = \alpha(1+c_s^2)\rho u_A, \quad T^1 = (\omega+k)c_s^2\rho, \quad T^2 = 0, \tag{13}$$

$$T_{AB} = \omega t_{AB} + 2\Delta u_{(A}u_{B)}(1+c_s^2)\rho + k_{AB}c_s^2\rho. \tag{14}$$

VI. ODD PERTURBATIONS

We must now extract a well-posed initial value problem from the gauge-invariant perturbation equations. For the odd sector this is straightforward. The one nontrivial matter conservation equation is $(r^2 L_A)^{|A} = 0$ (where $g_{AB|C} = 0$), or

$$(\beta r^2 \rho u^A)_{|A} = 0. \tag{15}$$

This equation is an advection equation for β, and can be solved independently of all other perturbations, from β given on an initial spacelike hypersurface of M^2. GS have shown that by defining the scalar $\Pi = \epsilon^{AB}(r^{-2}k_A)_{|B}$, the Einstein equations for k_A can always be reduced to the scalar wave equation

$$[r^{-2}(r^4\Pi)_{|A}]^{|A} - (l+2)(l-1)\Pi = 16\pi\epsilon^{AB}L_{A|B}, \tag{16}$$

which they call the odd-parity master equation. The full k_A can be reconstructed by quadratures, once this equation has been solved for Π. (Note that the source L_A is already known in the case of perfect fluid matter.)

VII. EVEN PERTURBATIONS

The even perturbations are more entangled. For α we once again have an advection equation, but now with sources. The first-order equations for the density perturbation ω and radial velocity perturbation γ can be combined to form a single wave equation at the speed of sound for either ω or γ. For fixed metric perturbation k_{AB} and k, the initial value problem would then be clear. Unfortunately, one apparently cannot extract a master equation for the metric perturbations from the linearized Einstein equations for arbitrary matter, but one always has a system with constraints. Instead, we follow Seidel [11] in first focussing attention on those components of the linear Einstein equations with vanishing matter sides. Four such components exist, because the seven even stress-energy perturbations are linear in only the three even matter perturbations. One of these is

$$k_A{}^A = -16\pi T^2 = 0. \tag{17}$$

It is natural to decompose the trace-free tensor k_{AB} covariantly into two scalars, via

$$k_{AB} = \phi(u_A u_B + n_A n_B) + \psi(u_A n_B + n_A u_B), \tag{18}$$

and to introduce the frame derivatives

$$\dot{f} = u^A f_{|A}, \quad f' = n^A f_{|A}. \tag{19}$$

The three remaining source-free Einstein equations can then be written as

$$-(\dot{\chi})\dot{} + (\chi')' = S_\chi, \tag{20}$$

$$-(\dot{k})\dot{} + c_s^2(k')' = S_k, \tag{21}$$

$$-\dot{\psi} = S_\psi, \tag{22}$$

where $\chi = \phi - k$ replaces ϕ. The source terms S_χ, S_k and S_ψ are linear in χ, k, ψ and their first derivatives χ', k', ψ', $\dot{\chi}$, and $\dot{k}$, but do not contain $\dot{\psi}$. While the highest derivatives of χ form a wave equation with characteristics given by the metric $g_{AB} = -u_A u_B + n_A n_B$, k obeys a wave equation with characteristics given by the "fluid metric" $-u_A u_B + c_s^{-2} n_A n_B$. These characteristics have speed c_s relative to the fluid. Finally, ψ is advected with the fluid. Therefore χ characterizes gravitational waves, k sound waves, and ψ polar fluid flow. The metric perturbation k "knows about" the speed of sound because we have used $\Delta p = dp/d\rho \Delta \rho = c_s^2 \Delta \rho$ in finding the source-free linearized Einstein equations. If we add to these three equations the identity $(f'u^A)_{|A} = (\dot{f}n^A)_{|A}$ and the definitions (19), we have a complete first-order system of equations. The variables χ, $\dot{\chi}$, k, $\dot{k}$ and ψ can be set freely on a spacelike hypersurface in M^2. From the equations of motion, one can see that any regular solution must scale at $r=0$ as $k \sim r^l$, $\psi \sim r^{l+1}$, and $\chi \sim r^{l+2}$. This follows also from the requirement that the metric perturbation (8) be a regular tensor in four dimensions at $r=0$. Furthermore, the perturbed metric remains continuously self-similar if k, ψ and χ are independent of τ. The perturbed metric is discretely self-similar if these fields are periodic in τ.

The Einstein equations we have not used yet give the matter perturbations directly in terms of the metric perturbations. As a check of the correctness of our equations and their numerical implementation, we have numerically differentiated the numerical solution and verified that the perturbed matter equations of motion (or Bianchi identities) are obeyed.

The case of $l=1$ even perturbations is not covered by the framework of GS and has to be treated separately. Clearly $l=1$ does not admit gravitational waves, so that we can use Newtonian intuition. The $l=1$ matter perturbations are pure gauge, corresponding to an initial displacement and velocity of the spherical background solution.

VIII. NUMERICAL METHOD

We conclude with a remark on the numerical implementation. The linearized field equations are of the form

$$\frac{\partial u}{\partial \tau} = A\frac{\partial u}{\partial x} + Bu. \tag{23}$$

In order to make the numerical evolution stable even though the lines of constant x change from timelike to spacelike in our computational domain, it is essential to use a characteristic scheme [12]. Let V be the matrix of (column) eigenvectors of A. Let Λ be the diagonal matrix composed of the corresponding eigenvalues. Then $A = V\Lambda V^{-1}$. Let Λ_+ be the diagonal matrix with zeros in the place of the negative eigenvalues. Define Λ_-, A_+ and A_- in the obvious manner, so that $A = A_+ + A_-$. We have used the numerical scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta\tau} = A_+ \frac{u_{j+1}^n - u_j^n}{\Delta x} + A_- \frac{u_j^n - u_{j-1}^n}{\Delta x} + Bu_j^n, \quad (24)$$

which is first-order accurate, and stable even for superluminal shift. The matrix A is just sparse enough for its eigenvalues and eigenvectors to be calculated in closed form. As expected, the characteristics are the fluid world lines, light cones and sound cones.

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