Numerical Evolution of Dynamic 3D Black Holes: Extracting Waves

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Abstract

We consider the numerical evolution of dynamic black hole initial data sets with a full 3D, nonlinear evolution code. These data sets consist of single black holes distorted by strong gravitational waves, and mimic the late stages of coalescing black holes. Through comparison with results from well established axisymmetric codes, we show that these dynamic black holes can be accurately evolved. In particular, we show that with present computational resources and techniques, the process of excitation and ringdown of the black hole can be evolved, and one can now extract accurately the gravitational waves emitted from the 3D Cartesian metric functions, even though they may be buried in the metric at levels on the order of 10^{-3} and below. Waveforms for both the $\ell = 2$ and the much more difficult $\ell = 4$ modes are computed and compared with axisymmetric calculations. In addition to exploring the physics of distorted black hole data sets, and showing the extent to which the waves can be accurately extracted, these results also provide important testbeds for all fully nonlinear numerical codes designed to evolve black hole spacetimes in 3D, whether they use singularity avoiding slicings, apparent horizon boundary conditions, or other evolution methods.

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I. INTRODUCTION

As numerical relativity is empowered by ever larger computers, numerical evolutions of black hole data sets are becoming more and more common [1]. The need for such simulations is great, especially as gravitational wave observatories are gearing up to collect gravitational wave data over the next decade [2]. As black hole collisions are considered a most promising source of signals to be detected by these observatories, it is crucial to have a detailed theoretical understanding of the coalescence process that can only be achieved through numerical simulation. In particular, it is most important to be able to simulate accurately the excitation of the coalescing black holes, to follow the waves generated in the process, and to extract gravitational waveforms expected to be seen by detectors.

This is a very difficult calculation, as one must simultaneously deal with singularities inside the black holes, follow the highly nonlinear regime in the coalescence process taking place near the horizons, and also calculate the linear regime in the radiation zone where the waves represent a very small perturbation on the background spacetime metric. In axisymmetry this has been achieved for distorted black holes with rotation [3] and without [4], and for equal mass colliding black holes [5,6], but with difficulty. These 2D evolutions can be carried out to roughly $t = 100M$, where M is the ADM mass of the spacetime, although beyond this point large gradients related to singularity avoiding slicings usually cause the codes to become very inaccurate and crash.

In such simulations, it has been shown that, using a gauge-invariant radiation extraction technique developed originally by Abrahams [7,8], one can in principle extract the waveforms generated by the black holes. However, even in axisymmetric simulations, where the coordinate systems are naturally adapted to the black holes and the radiation, these waveforms can sometimes be difficult to compute cleanly. The waves are small perturbations buried in the metric functions actually being evolved, are generated in the strong field regime just outside the evolving horizon, and then propagate out to the wave zone where they must be extracted. The energy carried by these waves is typically found to be on the order of $10^{-6} - 10^{-2}M$. At such low amplitude, both the generation and propagation of these signals are susceptible to small numerical errors inherent in numerical simulations. For example, in the axisymmetric evolution of Misner data for two colliding black holes, although the $\ell = 2$ signals have been accurately computed, as verified by careful comparisons with perturbation theory in different regimes, the more difficult $\ell = 4$ signals are still rather uncertain [6].

In this paper we show that with current techniques and computational resources available to 3D numerical relativity, distorted black holes can be evolved through the initial relaxation and final ringdown period, and that the gravitational waveforms can be followed and accurately extracted from the numerical evolutions, even though they represent a small perturbation on the background spacetime which is also being evolved. Here we focus on the application of the 3D code to the evolution of an axisymmetric distorted black hole initial data set, so that careful comparisons can be made with results obtained with mature 2D codes. The evolutions can be carried out at present through about $t = 30 - 35M$, which provides time to study several wavelengths of the fundamental $\ell = 2$ and $\ell = 4$ modes present in the simulations. The extension to full 3D black hole initial data and wave modes, for which no testbeds exist at present, as well as more extensive comparisons with axisymmetric initial data, are in progress and will be published elsewhere. In particular, *nonaxisymmetric* modes, such as the $\ell = 2, m = 2$ mode expected to be important in the ringing radiation for rotating black holes at late times [9], and therefore an important signal for gravitational wave observations, can now be studied [10].

II. 3D EVOLUTION OF BLACK HOLES WITH RADIATION

A. Our code and prior 3D simulations

We have developed a 3D code to study black holes and gravitational waves in Cartesian coordinates. This code (known as the "G" code) was applied to Schwarzschild black holes, where we showed that using singularity avoiding time slicings, a spherical black hole could be evolved accurately to $t = 30 - 50M$, depending on the resolution, location of the outer boundary, and the slicing conditions [11]. Beyond that time, the code generally crashes due to the unbounded growth of metric functions generated by singularity avoiding slicings. However, the focus of [11] was on spherical black holes, so no studies were made of black hole oscillations and the waves that would be generated in the process. It was shown that with spherical initial data some nonspherical behavior could be introduced by the Cartesian mesh and boundary conditions, the numerics of which could in principle generate spurious gravitational waves.

The same code was simultaneously applied to the problem of pure gravitational waves [12], where many systems were studied, from pure linear quadrupole waves to nonlinear waves, and their propagation on a Cartesian mesh was studied. In that study it was shown that waves can be accurately evolved, although certain problems with gauge modes in the "near linear" regime that can confuse the results were identified, along with strategies to deal with them.

This 3D code was then applied to the collision of two axisymmetric black holes (Misner data) [13], where we showed by comparison to 2D results that one could accurately track the merging of the horizons, and that the radiation emitted was qualitatively the same, but at that time the waveforms were not studied extensively. Building on the work presented in this paper, a more detailed study of the Misner data in 3D, including the waveforms, is in preparation for publication elsewhere.

In this paper, we build on this prior work, focusing on the 3D evolution of distorted single black holes. In previous 2D axisymmetric studies, such data sets were shown to be similar to single black holes formed just after the merger of colliding axisymmetric black holes. The agreement between the waveforms produced with the 3D and 2D codes is the subject of this paper, while comparisons of metric functions and other quantities for a variety of axisymmetric initial data sets will be found in [10,14].

B. Initial data

The initial data we evolve in this paper consist of a single black hole that has been distorted by the presence of an adjustable torus of nonlinear gravitational waves which surround it. The amplitude and shape of the torus can be specified by hand, as described below, and can create very highly distorted black holes. Such initial data sets, and their evolutions in axisymmetry, have been studied extensively, as described in Refs. [4,15,16]. For our purposes, we consider them as convenient initial data that create a distorted black hole that mimics the merger, just after coalescence, of two black holes colliding in axisymmetry [6].

Following [15], we write the 3–metric as

$$d\ell^2 = \tilde{\psi}^4 \left(e^{2q} \left(d\eta^2 + d\theta^2 \right) + \sin^2 \theta d\phi^2 \right), \tag{1}$$

where η is a radial coordinate related to the Cartesian coordinates by $\sqrt{x^2+y^2+z^2} = e^\eta$. (We have set the scale parameter M in [15] to be 2 in this paper.) Given a choice for the "Brill wave" function q, the Hamiltonian constraint leads to an elliptic equation for the conformal factor $\tilde{\psi}$. The function q represents the gravitational wave surrounding the black hole, and is chosen to be

$$q\left(\eta, \theta, \phi\right) = a \sin^n \theta \left(e^{-\left(\frac{\eta+b}{w}\right)^2} + e^{-\left(\frac{\eta-b}{w}\right)^2} \right) \left(1 + c \cos^2 \phi \right). \tag{2}$$

Thus, an initial data set is characterized by the parameters (a, b, w, n, c), where, roughly speaking, a is the amplitude of the Brill wave, b is its radial location, w its width, and n and c control its angular structure. Note that we have generalized the original axisymmetric construction to full 3D by the addition of the parameter c, but in this paper we restrict ourselves to $c = 0$ for comparison with axisymmetric results. A study of full 3D initial data and their evolutions will be published elsewhere [10,14,17]. If the amplitude a vanishes, the undistorted Schwarzschild solution results, leading to

$$\tilde{\psi} = 2 \cosh \left(\frac{\eta}{2} \right). \tag{3}$$

We note that just as the Schwarzschild geometry has an isometry that leaves the metric unchanged under the operation $\eta \to -\eta$, our data sets also have this property, even in the presence of the Brill wave. As discussed in [11,16], this condition can also be applied during the evolution and in Cartesian coordinates as well. The evolution of the data set $(0.5, 0, 1, 2, 0)$ is considered in this paper. In what follows we solve the Hamiltonian constraint for this initial data set, interpolate it onto a 3D Cartesian grid, and study its evolution with a 3D evolution code.

C. Evolution and Analysis

Using the techniques described in [11], we evolve the initial data sets described above in 3D Cartesian coordinates. The present evolution code is based on the one detailed in [11,12], using the same finite difference algorithms, having the same convergence properties, *etc.*, but having been rewritten to take advantage of newer parallel computers.

Although the 3D evolution code is written without making use of any symmetry assumptions, the initial data we evolve in this paper have both equatorial plane symmetry and axisymmetry. Hence we save on the memory and computation required by evolving only one octant of the system. As shown in [11], this has no effect on the simulations except to reduce the computational requirements by a factor of eight. Even with such computational savings, these are extravagant calculations. The results presented in this paper were computed on a 3D Cartesian grid of 300^3 numerical grid zones, which is about a factor three larger than the largest production relativity calculations of which we are aware (which were about 200^3 zones). With our new code, these take about 12 Gbytes of memory, and require about a day on a 128 processor, early access SGI/Cray Origin 2000 supercomputer.

Given a choice of lapse and shift, the Cartesian metric functions γ_{xx}, γ_{xy}, *etc.*, are evolved using the ADM formulation of the Einstein equations. In this paper we use a lapse which is initially maximal, with antisymmetric conditions across the throat of the black hole, defined by the isometry surface $\eta = 0$, or $r = 1$. The initial data are then evolved with the "1+log" algebraic lapse condition [11], an isometry operator in Cartesian coordinates, and with zero shift. These choices have been made for computational efficiency, and are not unique choices for successful evolution. For example, we have performed similar calculations with maximal slicing and no isometry with similar results, except that the computational time needed to solve the elliptic maximal slicing equation can double or triple the computational time needed to perform these simulations.

As in the case of a spherical black hole [11], singularity avoiding slicings lead to large gradients in metric functions that cannot presently be resolved in 3D and eventually cause the code to crash. The same problem occurs with distorted black holes. In Fig. 1 we show the radial metric function γ_{rr}/ψ^4, with its large round peak, at time $t = 27.2M$, reconstructed from the Cartesian metric functions that are actually evolved. The spike developing near the origin is inside the throat, and is a result of the application of the isometry condition. One expects that the region near the metric function peaks needs to be accurately computed in order to produce the correct waveform, because the ringing radiation is produced by scattering off the Zerilli potential, which is located just outside of the peak as we know from studies of horizon location [18]. Although this potential is never explicitly computed in the calculations, it is implicitly built into the Cartesian metric functions being evolved.

D. Radiation Extraction

Although in black hole simulations we evolve directly the metric and extrinsic curvature, for applications to gravitational wave astronomy we are particularly interested in computing the waveforms emitted. One measure of this radiation is the Zerilli function, ψ, which is a gauge-invariant function that obeys the Zerilli wave equation [19]. The Zerilli function can be computed by writing the metric as the sum of a spherically symmetric part and a perturbation: $g_{\alpha\beta} = \overset{o}{g}_{\alpha\beta} + h_{\alpha\beta}$, where the perturbation $h_{\alpha\beta}$ is expanded in tensor spherical harmonics. To compute the elements of $h_{\alpha\beta}$ in a numerical simulation, one integrates the numerically evolved metric components $g_{\alpha\beta}$ against appropriate spherical harmonics over a coordinate 2–sphere surrounding the black hole. The resulting functions can then be combined in a gauge-invariant way, following the prescription given by Moncrief [20]. This procedure was originally developed by Abrahams [7], and was applied to the same class of distorted black hole initial data sets discussed here, but evolved in 2D spherical–polar coordinates and with a different gauge, as discussed in [4].

We have developed numerical methods based on the same ideas to extract the waves in a full 3D Cartesian setting. The method used is essentially that used in the axisymmetric case, except that the metric functions and their spatial derivatives need to be interpolated onto a two-dimensional surface, which we choose to have constant coordinate radius. The projections of the perturbed metric functions $h_{\alpha\beta}$, and their radial derivatives, are then computed by numerically performing two-dimensional surface integrals for each $\ell - m$ mode desired. Then, for each mode, the Zerilli function is constructed from these projected metric functions, according to Moncrief's gauge-invariant prescription. This is a complicated but straightforward procedure. Both the numerical interpolations and integrations involved in this extraction procedure were chosen to be second order accurate, and both have been shown to converge to second order in the relevant grid spacing. As in Ref. [4], we choose to normalize the Zerilli function so that the asymptotic energy flux in each mode is given by $\dot{E} = (1/32\pi)\dot{\psi}^2$. While previously only axisymmetric simulations have been studied, we can now study all non-trivial wave modes, including those with $m \neq 0$.

We extracted the $\ell = 2$ and $\ell = 4$ Zerilli functions during an evolution of the distorted black hole initial data set $(a, b, w, n, c) = (0.5, 0, 1, 2, 0)$, using the extraction method described above. In Figure 2a we show the $\ell = 2$ Zerilli function extracted at a radius $r = 8.7M$ as a function of time. Superimposed on this plot is the same function computed during the evolution of the same initial data set with a 2D code, based on the one described in detail in [4,16]. The agreement of the two plots over the first peak is a strong affirmation of the 3D evolution code and extraction routine. It is important to note that the 2D results were computed with a different slicing (maximal), different coordinate system, and a *different spatial gauge*. Yet the physical results obtained by these two different numerical codes, as measured by the waveforms, are remarkably similar (as one would hope). This is the principal result of this paper. A full evolution with the 2D code to $t = 100M$, by which time the hole has settled down to Schwarzschild, shows that the energy emitted in this mode at that time is about $4 \times 10^{-3}M$.

In Fig. 2b we show the $\ell = 4$ Zerilli function extracted at the same radius, computed during evolutions with 2D and 3D codes. This waveform is more difficult to extract, because it has a higher frequency in both its angular and radial dependence, and it has a much lower amplitude: the energy emitted in this mode is three orders of magnitude smaller than the energy emitted in the $\ell = 2$ mode, *i.e.*, $10^{-6}M$, yet it can still be accurately evolved and extracted.

Small differences between the 2D and 3D results can be seen. Resolution studies of the 3D results indicate that the differences are not completely due to resolution of the 3D evolution code. The small differences in phase can be understood as a result of the different shift and slicings being used in the two simulations. The radiation is extracted at a constant *coordinate* location, and the coordinates fall towards the black hole at different rates with different slicings and shifts. By measuring the physical radial position of the wave extraction in these simulations, we determined that the difference between the 2D and 3D phases at late time is consistent with the slightly different extraction locations in the two cases. The additional differences in the $\ell = 4$ waveforms could be related to slight differences in the initial data, which were generated in independent ways, or even differences in gauge (the waveforms are gauge-invariant, meaning they are unaffected only at first order under gauge transformations). As $\ell = 4$ has a much smaller amplitude than $\ell = 2$, it will be more sensitive to such details. The differences are very small, and do not affect the conclusions of this paper, but they will be studied in detail and discussed elsewhere.

III. SUMMARY AND CONCLUSIONS

We have shown that, in 3D numerical relativity, given sufficient resolution, distorted black holes can be accurately evolved. Furthermore the gravitational waveforms generated by the black hole, consisting of small perturbations on the evolving black hole background, can be accurately propagated and extracted from the numerically generated metric, on a 3D Cartesian grid. We have demonstrated this by comparing results from a mature 2D code, showing good agreement not only for the $\ell = 2$, but also the $\ell = 4$ modes of the radiation.

Although we regard this as an important step in establishing numerical relativity as a viable tool to compute waveforms from black hole interactions, the calculations one can presently do are limited. With present techniques, the evolutions can only be carried out for a fraction of the time required to simulate the 3D orbiting coalescence. Many techniques to handle this more general case are under development, such as hyperbolic formulations of the Einstein equations and the advanced numerical methods they bring [21], adaptive mesh refinement that will enable placing the outer boundary farther away while resolving the strong field region where the waves are generated, and apparent horizon boundary conditions that excise the interiors of the black holes, thus avoiding the difficulties associated with singularity avoiding slicings.

All of these techniques, and others, may be needed to handle the more general, long term evolution of coalescing black holes. Our purpose in this paper has been to show that *(a)* given present resources one can evolve simpler distorted black hole systems and accurately extract the waveforms, even when they carry only $10^{-6}M$ in energy, and *(b)* to establish testbeds for the techniques under development for the more general case. Each of these techniques may introduce numerical artifacts, even if at very low amplitude, to which the waveforms may be very sensitive. As new methods are developed and applied to numerical black hole simulations, they can now be tested on evolutions such as those presented here to ensure that the waveforms are accurately represented in the data.

In future papers we address the wave extraction in more detail; work is presently in progress to apply it to more extensive axisymmetric initial data [14], to full 3D initial data sets where nonaxisymmetric modes can be extracted for the first time [10], and to the evolution of colliding black holes in 3D, extending the work in [13]. Once this has been fully developed and tested on full 3D data sets, it will be important to apply it to true 3D black hole collision simulations, such as those recently reported by Brügmann [22].

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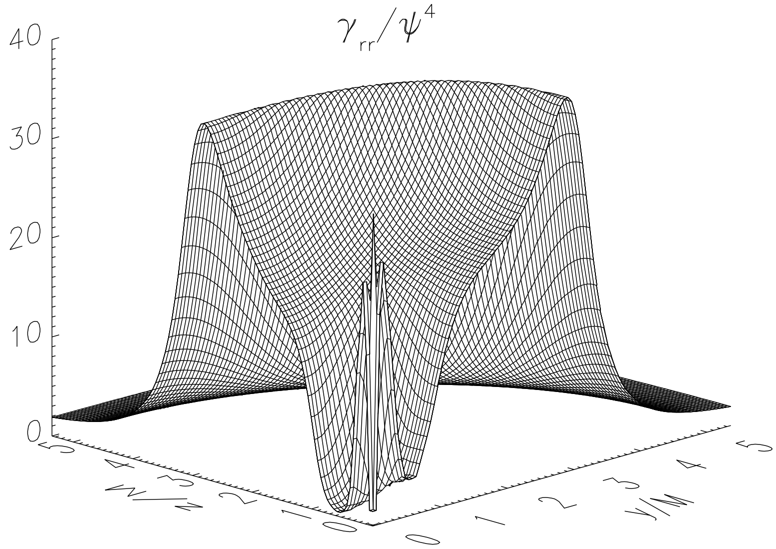
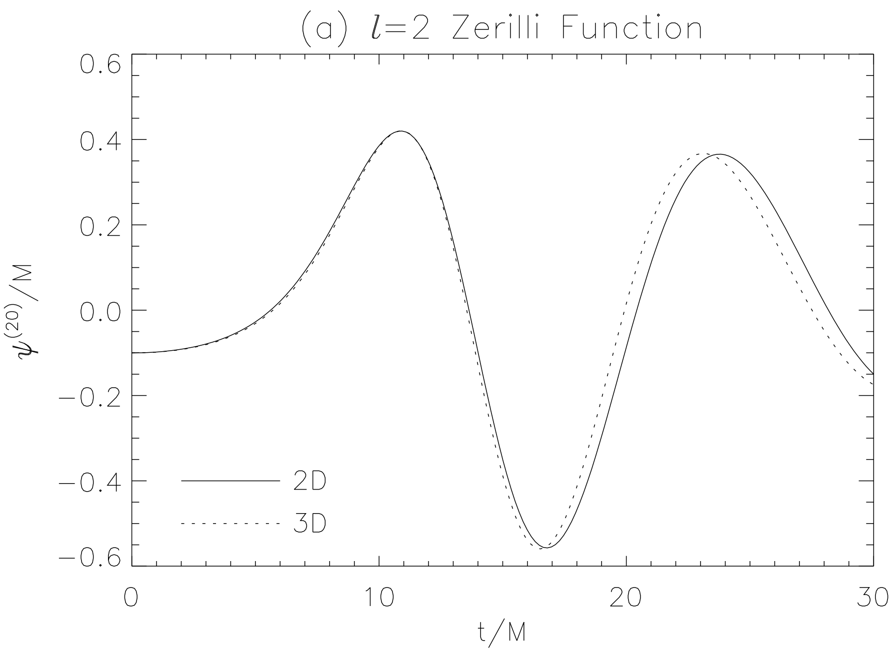
FIGURES

FIG. 1. We show the radial metric function γ_{rr}/ψ^4 for the evolution of the distorted black hole data set $(a, b, w, n, c) = (0.5, 0, 1, 2, 0)$ at time $t = 27.2M$. The evolution was performed with 150^3 grid points, although data from only the inner 100^3 grid points are shown to bring out detail. The resolution was $\Delta x = 0.0544M$.

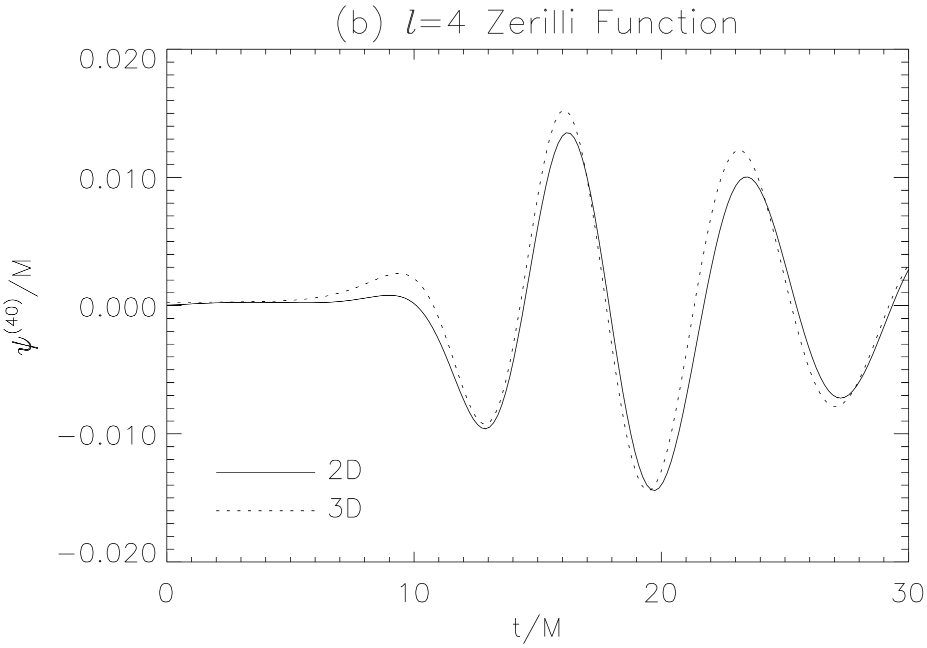
FIG. 2. We show the (a) $\ell = 2$ and (b) $\ell = 4$ Zerilli functions vs. time, extracted during 2D and 3D evolutions of the data set $(a, b, w, n, c) = (0.5, 0, 1, 2, 0)$. The functions were extracted at a radius of $8.7M$. The 2D data were obtained with 202×54 grid points, giving a resolution of $\Delta\eta = \Delta\theta = 0.03$. The 3D data were obtained using 300^3 grid points and a resolution of $\Delta x = 0.0816M$.



γ_{rr}/ψ^4
40
30
20
10
0
5
4
3
2
1
0
z/M
0
1
2
3
4
5
y/M



(a) l=2 Zerilli Function
$\psi^{(20)}/M$
t/M
2D
3D
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
0
10
20
30


(b) l=4 Zerilli Function
0.020
0.010
0.000
−0.010
−0.020
$\psi^{(40)}/M$
0
10
20
30
t/M
2D
3D