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# New canonical variables for $d = 11$ supergravity

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**Abstract**

A set of new canonical variables for $d = 11$ supergravity is proposed which renders the supersymmetry variations and the supersymmetry constraint polynomial. The construction is based on the $SO(1,2) \times SO(16)$ invariant reformulation of $d = 11$ supergravity given by Nicolai [Phys. Lett. 187B (1987) 316], and has some similarities with Ashtekar's reformulation of Einstein's theory. The new bosonic variables fuse the gravitational degrees of freedom with those of the three-index photon $A_{MNP}$ in accordance with the hidden symmetries of the dimensionally reduced theory. Although $E_8$ is not a symmetry of the theory, the bosonic sector exhibits a remarkable $E_8$ structure, hinting at the existence of a novel type of "exceptional geometry". © 1998 Published by Elsevier Science B.V.

Recent advances in string theory (see e.g. [1]) have lent renewed support to the long held belief that $d = 11$ supergravity [2] has a fundamental role to play in the unification of fundamental interactions. In this letter, we present an unconventional formulation of this theory, developing further the results of Refs. [3,4] where new versions of $d = 11$ supergravity with local $SO(1,3) \times SU(8)$ and $SO(1,2) \times SO(16)$ tangent space symmetries, respectively, were presented. In both versions the supersymmetry variations were shown to acquire a polynomial form from which the corresponding formulas for the maximal supergravities in four and three dimensions can be read off directly and without the need for complicated duality redefinitions. Our reformulation can thus be regarded as a step towards the complete fusion of the bosonic degrees of freedom of $d = 11$ supergravity (i.e. the elfbein and the antisymmetric tensor $A_{MNP}$) in a way which is in harmony with the hidden symmetries of the dimensionally reduced theories [5,6]. The results are very suggestive of a novel kind of "exceptional geometry" for $d = 11$ supergravity (or some bigger theory containing it) that would be intimately tied to the special properties of the exceptional groups, and would be characterized by relations such as (1)–(4) below, which have no analog in ordinary Riemannian geometry.

The hamiltonian formulation of our results reveals surprising similarities with Ashtekar's reformulation of Einstein's theory [7] (for a conventional hamiltonian treatment of $d = 11$ supergravity, cf. [8] [1]). More specifically, the "248-bein" to be introduced below is the analog of the inverse densitized dreibein (or "triad") in [7]. Furthermore, in terms of the canonical variables proposed here the supersymmetry constraints become

[1] The Chern-Simons part of the $d = 11$ action was recently considered in [9]. The fusion of gravitational and matter (Yang Mills) degrees of freedom in the framework of canonical quantum gravity was also attempted in [10].

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polynomial; the polynomiality of the remaining canonical constraints is then implied by supersymmetry and the polynomiality of the canonical brackets. Unfortunately, not all sectors of the theory are as simple as one might have wished, and a further simplification will very likely require a better understanding of the exceptional structures alluded to above, as well as the further extension of the results of [3,4] to incorporate the even larger (infinite dimensional) symmetries arising in the dimensional reductions of $d=11$ supergravity to two and one dimensions, respectively.

Let us first recall the main results, conventions and notation of [4] (further details will be provided in a forthcoming thesis [11]). To derive the new version from the original formulation of $d=11$ supergravity, one first breaks the original tangent space symmetry SO(1,10) to its subgroup SO(1,2) × SO(8) through a partial choice of gauge for the elfbein, and subsequently enlarges it again to SO(1,2) × SO(16) by introducing new gauge degrees of freedom. This symmetry enhancement requires suitable redefinitions of the bosonic and fermionic fields, or, more succinctly, their combination into tensors w.r.t. the new tangent space symmetry. The basic strategy underlying this construction goes back to [5], but the crucial difference is that the dependence on *all eleven* coordinates is retained here. The construction thus requires a 3 + 8 split of the $d=11$ coordinates and indices, implying a similar split for all tensors of the theory. Accordingly, curved $d=11$ indices are decomposed as $M=(\mu,m), N=(\nu,n),\ldots$ with $\mu,\nu,\ldots=0,1,2$ and $m,n,\ldots=3,\ldots,10$; the associated flat indices are denoted by $\alpha,\beta,\ldots$ and $a,b,\ldots$, respectively. To distinguish the flat index 0 from its curved homolog, we will use the label $t$ for the latter. We will occasionally refer to the $d=8$ coordinates and field components as "internal" coordinates and components, respectively. We will also need SO(16) indices $I,J,\ldots$ for the 16-dimensional vector representation, and dotted and undotted indices $\dot{A},\dot{B},\ldots$ and $A,B,\ldots$ for the 128-dimensional left and right handed spinor representations, respectively; antisymmetrized index pairs $[IJ]$ label the 120-dimensional adjoint representation.

In the fermionic sector, the symmetry breaking induces a split of the 32 components of an SO(1,10) Majorana spinor in accordance with the subgroup SO(1,2) × SO(8); these fermions are then reassembled into $2\times 16$ component Majorana spinors transforming under the new local SO(1,2) × SO(16) symmetry. For notational simplicity, we will suppress the SO(1,2) spinor indices throughout. In this manner, the $d=11$ gravitino — the only fermionic field in $d=11$ supergravity — is decomposed into a "gravitino-like" vector spinor $\Psi_\mu^{\,I}$ belonging to the **16** vector representation of SO(16) and a "matter-like" fermion field $\chi_{\dot{A}}$ which is assigned to the (left-handed) $\mathbf{128}_c$ spinor representation; these are also the fermionic representations appearing in the dimensionally reduced theory, i.e. $N=16$ supergravity in three dimensions [12].

In the bosonic sector, the elfbein and the three-index photon are combined into new objects covariant w.r.t. to $d=3$ coordinate reparametrizations and the new SO(1,2) × SO(16) tangent space symmetry. The elfbein contains the (Weyl rescaled) dreibein $e_\mu{}^\alpha$ and the Kaluza–Klein vector $B_\mu{}^m$; the remaining "matter-like" degrees of freedom are merged into a rectangular "248-bein" $(e^m_{IJ}, e^m_A)$ obeying a generalization of the usual vielbein postulate. This 248-bein, which in the reduction to three dimensions contains all the propagating bosonic matter degrees of freedom of $d=3, N=16$ supergravity, is very much analogous to part of Ashtekar's variables: in a special SO(16) gauge, it is identified with the inverse densitized internal achtbein contracted with a $\Gamma$-matrix. Because $([IJ],A)$ label the 248-dimensional adjoint representation of $E_8$, it is really a soldering form that relates upper (internal) world indices to $E_8$ indices. In addition, we have the composite fields $(Q^{IJ}_\mu, P^A_\mu)$ and $(Q^{IJ}_m, P^A_m)$, which make up an $E_8$ connection in *eleven* dimensions and whose explicit expressions in terms of the $d=11$ coefficients of anholonomity and the four-index field strength $F_{MNPQ}$ can be found in [4].

There are various algebraic constraints between the vielbein components which ensure that the total number of bosonic physical degrees of freedom is the same as before, and which are most conveniently proven in the special SO(16) gauge of [4]. These relations are without analog in ordinary Riemannian geometry, because they rely in an essential way on special properties of the exceptional group $E_8$. We have

$$e^m_A e^n_A - \frac{1}{2} e^m_{IJ} e^n_{IJ} = 0 \tag{1}$$

and

$$\Gamma^{IJ}_{AB}\left(e^m_B e^n_{IJ} - e^n_B e^m_{IJ}\right) = 0, \quad \Gamma^{IJ}_{AB} e^m_A e^n_B + 4 e^m_{K[I} e^n_{J]K} = 0 \tag{2}$$

where, of course, $\Gamma^I_{A\dot{A}}$ are the standard SO(16) $\Gamma$-matrices and $\Gamma^{IJ}_{AB} \equiv (\Gamma^{[I}\Gamma^{J]})_{AB}$, etc. The identities (2) were already given in [13]; the minus sign in (1) reflects the fact that we are dealing with the non-compact form $E_{8(+8)}$. In addition we will need to make use of the more general relations

$$2 e^{(m}_A e^{n)}_B = -\frac{1}{16}(\Gamma^{IJ}\Gamma^{KL})_{AB} e^m_{IJ} e^n_{KL} + \frac{1}{8}\Gamma^{IJ}_{AC} e^m_C \, e^n_D \Gamma^{IJ}_{DB}$$

$$2 e^{(m}_{IJ} e^{n)}_{KL} = -4 e^{(m}_{I[K}\, e^{n)}_{L]J} + 2 e^m_{IM} e^n_{M[K}\,\delta_{L]J} - 2 e^n_{JM} e^m_{M[K}\,\delta_{L]I} - \frac{1}{4} e^m_A (\Gamma^{IJ}\Gamma^{KL})_{AB} e^n_B$$

$$2 e^{(m}_A e^{n)}_{IJ} = -\Gamma^{IK}_{AB} e^{(m}_B e^{n)}_{KJ} + \Gamma^{JK}_{AB} e^{(m}_B e^{n)}_{KI} - \frac{1}{8}(\Gamma^{IJ}\Gamma^{KL})_{AB} e^m_B e^n_{KL} \tag{3}$$

where (...) denotes symmetrization with strength one (the combinations antisymmetric in $[mn]$ contained in the above expressions all reduce to (2) and thus yield no new information). While the SO(16) covariance of these equations is manifest, it turns out, remarkably, that they are also covariant (i.e. transform into one another) under $E_8$. Obviously, (1) and (2) correspond to the singlet and the adjoint representations of $E_8$. The relations (3) are not irreducible as they stand, but can be made so by projecting out the singlet and the **3875** representation of $E_8$; the latter decomposes as **135** ⊕ **1920** ⊕ **1820** under SO(16):

$$e^{(m}_{IK} e^{n)}_{JK} - \frac{1}{16}\delta_{IJ} e^m_{KL} e^n_{KL} = 0, \quad \Gamma^K_{AB} e^{(m}_B e^{n)}_{IK} - \frac{1}{14}\Gamma^{IKL}_{AB} e^{(m}_B e^{n)}_{KL} = 0$$

$$e^{(m}_{[IJ}\, e^{n)}_{KL]} + \frac{1}{24} e^m_A \Gamma^{IJKL}_{AB} e^n_B = 0 \tag{4}$$

After this projection, one is left with the **27000** representation of $E_8$. There are presumably more algebraic relations of this type, but we have made no attempts towards a complete classification [2].

The 248-bein and the new connection fields are subject to the so-called "generalized vielbein postulates"

$$D_\mu e^m_{IJ} + \partial_n B_\mu{}^n e^m_{IJ} + \partial_n B_\mu{}^m e^n_{IJ} + P^A_\mu \Gamma^{IJ}_{AB} e^m_B = 0$$

$$D_\mu e^m_A + \partial_n B_\mu{}^m e^n_A + \partial_n B_\mu{}^n e^m_A - \frac{1}{2}\Gamma^{IJ}_{AB} P^B_\mu e^m_{IJ} = 0 \tag{5}$$

$$D_m e^n_{IJ} + P^A_m \Gamma^{IJ}_{AB} e^n_B = 0, \quad D_m e^n_A - \frac{1}{2}\Gamma^{IJ}_{AB} P^B_m e^n_{IJ} = 0 \tag{6}$$

[2] The 56-bein $(e^m_{AB}, e^{mAB})$ of [3] obeys analogous relations, which can be similarly assigned to irreducible representations of $E_7$. For instance, the analog of (1) reads

$$e^m_{AB} e^{nAB} - e^{mAB} e^n_{AB} = 0$$

where the combination on the l.h.s. is just the symplectic second order invariant for the 56 dimensional fundamental representation of $E_7$. Furthermore, we have

$$e^m_{AC} e^{nBC} + e^n_{AC} e^{mBC} - \frac{1}{4}\delta^B_A e^m_{CD} e^{nCD} = 0, \quad e^m_{[AB} e^n_{CD]} - \frac{1}{24}\varepsilon_{ABCDEFGH} e^{mEF} e^{nGH} = 0$$

corresponding to the **133** representation of $E_7$. The first of these relations was referred to as the "Clifford property" in [3] (cf. Eq.(3.9)).

where the SO(16) covariant derivatives on the 248-bein are defined by

$$D_\mu e^n_{IJ} = \mathscr{D}_\mu e^n_{IJ} + 2Q_\mu{}^K{}_{[I} e^n_{J]K}\,, \quad D_\mu e^n_A = \mathscr{D}_\mu e^n_A + \frac{1}{4} Q^{IJ}_\mu \Gamma^{IJ}_{AB} e^n_B \tag{7}$$

$$D_m e^n_{IJ} = \partial_m e^n_{IJ} + 2Q_m{}^K{}_{[I} e^n_{J]K}\,, \quad D_m e^n_A = \partial_m e^n_A + \frac{1}{4} Q^{IJ}_m \Gamma^{IJ}_{AB} e^n_B \tag{8}$$

with the modified derivative

$$\mathscr{D}_\mu := \partial_\mu - B_\mu{}^m \partial_m \tag{9}$$

Like (1)–(4), these relations are $E_8$ covariant; in fact, (5) and (6) simply state the covariant constancy of the 248-bein w.r.t. to a fully $E_8$ covariant derivative. For the spatial components (i.e. $\mu \neq t$), this feature is the analog of the covariant constancy of the inverse densitized dreibein in $d=4$ (canonical) gravity w.r.t. a covariant derivative involving only the Ashtekar connection.

Unlike the previous relations, the following duality constraint has no analog in the SO(1,3) × SU(8) version of [3]:

$$e^m_A P^A_\mu = \varepsilon_\mu{}^{\nu\rho} \mathscr{D}_\nu B_\rho{}^m \tag{10}$$

In the reduction to $d=3$, the relation (10) shows how the Kaluza–Klein vectors $B_\mu{}^m$ are dualized into scalar fields, but a straightforward dualization is obviously no longer possible if the dependence on all eleven coordinates is retained, due to the explicit appearance of the field $B_\mu{}^m$ without a derivative in this and other formulas.

The invariance of $d=11$ supergravity under local supersymmetry in its original form implies an analogous local supersymmetry for the new formulation as well. Deriving the new transformation laws requires some tedious calculations, which are most conveniently done "backwards" as explained in [3,4]. In particular, attention must be paid to various compensating rotations, and the final variations differ from those of the original theory by a local SO(16) rotation. Modulo this compensating rotation, the supersymmetry variations of the bosonic fields assume a rather simple form [4], viz. [3]

$$\delta e_\mu{}^\alpha = \frac{1}{2} \bar\varepsilon^I \gamma^\alpha \Psi^I_\mu\,, \quad \delta B_\mu{}^m = \frac{1}{2} e^m_{IJ} \bar\varepsilon^I \Psi^J_\mu + \frac{1}{4} e^m_A \Gamma^I_{A\dot A} \bar\varepsilon^I \gamma_\mu \chi_{\dot A}$$

$$\delta e^m_{IJ} = e^m_A \Gamma^{IJ}_{AB} \omega^B\,, \quad \delta e^m_A = \frac{1}{2} e^m_{IJ} \Gamma^{IJ}_{AB} \omega^B \tag{11}$$

where the last two variations have been written in the form of a local $E_8/\mathrm{SO}(16)$ rotation with parameter

$$\omega^A := \frac{1}{4} \bar\varepsilon^I \Gamma^I_{A\dot A} \chi_{\dot A} \tag{12}$$

The fact that the variations on the 248-bein take this form ensures the compatibility of the algebraic constraints (1)–(4) with local supersymmetry. The fermionic fields transform as

$$\delta\Psi^I_\mu = \left( D_\mu - \frac{1}{2} \partial_m B_\mu{}^m \right) \varepsilon^I + \frac{1}{2} e^m_{IJ} \left( \gamma_\mu D_m \varepsilon^J + D_m \left( \gamma_\mu \varepsilon^J \right) \right) + \frac{1}{2} \gamma_\mu e^m_A \Gamma^{IJ}_{AB} P^B_m \varepsilon^J$$

$$\delta\chi_{\dot A} = 4 \Gamma^I_{A\dot A} \gamma^\mu \varepsilon^I P^A_\mu + 2 \Gamma^I_{\dot A A} e^m_A D_m \varepsilon^I + \frac{1}{2} e^m_{IJ} \left( \Gamma^{IJ} \Gamma^K + 4 \Gamma^I \delta^{JK} \right)_{\dot A A} P^A_m \varepsilon^K \tag{13}$$

[3] We have changed a few normalizations in comparison with [4].

modulo higher order fermionic contributions. The SO(1,2) × SO(16) covariant derivatives on the spinors are given by

$$D_\mu \varepsilon^I = \left( \mathscr{D}_\mu + \frac{1}{2} A_\mu{}^\alpha \gamma_\alpha \right) \varepsilon^I + Q_\mu^{IJ} \varepsilon^I \tag{14}$$

$$D_m \varepsilon^I = \left( \partial_m + \frac{1}{2} A_m{}^\alpha \gamma_\alpha \right) \varepsilon^I + Q_m^{IJ} \varepsilon^I \tag{15}$$

$$D_\mu \chi_{\dot{A}} = \left( \mathscr{D}_\mu + \frac{1}{2} A_\mu{}^\alpha \gamma_\alpha \right) \chi_{\dot{A}} + \frac{1}{4} Q_\mu^{IJ} \Gamma_{\dot{A}\dot{B}}^{IJ} \chi_{\dot{B}} \tag{16}$$

$$D_m \chi_{\dot{A}} = \left( \partial_m + \frac{1}{2} A_m{}^\alpha \gamma_\alpha \right) \chi_{\dot{A}} + \frac{1}{4} Q_m^{IJ} \Gamma_{\dot{A}\dot{B}}^{IJ} \chi_{\dot{B}} \tag{17}$$

where we have introduced the dualized connections

$$A_m{}^\alpha := -\frac{1}{2} \varepsilon^{\alpha\beta\gamma} e_\beta^\nu \partial_m e_{\nu\gamma} \tag{18}$$

and

$$A_\mu{}^\alpha := \frac{1}{2} \varepsilon^{\alpha\beta\gamma} \left( \omega_{\mu\,\beta\gamma} + 2 e_{\mu\beta} e_\gamma{}^\nu \partial_m B_\nu{}^m \right) \tag{19}$$

The SO(1,2) spin connections

$$\omega_{\mu\,\alpha\beta} = \frac{1}{2} e_\mu{}^\gamma \left( \Omega_{\alpha\beta\gamma} - \Omega_{\beta\gamma\alpha} - \Omega_{\gamma\alpha\beta} \right) \tag{20}$$

and the anholonomity coefficients

$$\Omega_{\alpha\beta\gamma} = 2 e_{[\alpha}{}^\mu e_{\beta]}{}^\nu \mathscr{D}_\mu e_{\nu\gamma} \tag{21}$$

differ from the standard $d=3$ expressions by their extra dependence on $B_\mu{}^m$ and the internal coordinates. In addition to their manifest covariance properties the above variations are evidently polynomial in the new fields.

We have also calculated the variations of the connection fields, which were not given in [3,4],

$$\begin{aligned} \delta P_\mu^A = {} & D_\mu \omega^A - \delta B_\mu{}^m P_m^A + e_A^m X_{m\mu} - \frac{1}{32} e_{IJ}^m \left( \Gamma^{IJ} \Gamma^K \right)_{A\dot{B}} D_m \left( \bar{\varepsilon}^K \gamma_\mu \chi_{\dot{B}} \right) \\ & + \frac{1}{32} \Gamma_{AB}^{IJ} e_B^m P_m^C \left( \Gamma^{IJ} \Gamma^K \right)_{C\dot{D}} \bar{\varepsilon}^K \gamma_\mu \chi_{\dot{D}} - \frac{1}{8} \Gamma_{AB}^{IJ} e_B^m D_m \left( \bar{\varepsilon}^I \Psi_\mu^J \right) - \frac{1}{16} \left( \Gamma^{IJ} \Gamma^{KL} \right)_{AB} P_m^B e_{IJ}^m \bar{\varepsilon}^K \Psi_\mu^L \end{aligned} \tag{22}$$

$$\begin{aligned} \delta Q_\mu^{IJ} = {} & 2 P_\mu^A \Gamma_{AB}^{IJ} \omega^B - \delta B_\mu{}^m Q_m^{IJ} + 2 e_{IJ}^m X_{m\mu} - \frac{1}{8} e_A^m \left( \Gamma^{IJ} \Gamma^K \right)_{A\dot{B}} D_m \left( \bar{\varepsilon}^K \gamma_\mu \chi_{\dot{B}} \right) \\ & - \frac{1}{4} e_{IK}^m P_m^A \left( \Gamma^{KJ} \Gamma^L \right)_{A\dot{B}} \bar{\varepsilon}^L \gamma_\mu \chi_{\dot{B}} + \frac{1}{4} e_{JK}^m P_m^A \left( \Gamma^{KI} \Gamma^L \right)_{A\dot{B}} \bar{\varepsilon}^L \gamma_\mu \chi_{\dot{B}} \\ & - e_{IK}^m D_m \left( \bar{\varepsilon}^{[J} \Psi_\mu^{K]} \right) + e_{JK}^m D_m \left( \bar{\varepsilon}^{[I} \Psi_\mu^{K]} \right) - \frac{1}{4} e_A^m \left( \Gamma^{IJ} \Gamma^{KL} \right)_{AB} P_m^B \bar{\varepsilon}^K \Psi_\mu^L \end{aligned} \tag{23}$$

with the parameter (12). These transformations have been computed by varying (5) and making use of the identities (3) to solve for $\delta P_\mu^A$ and $\delta Q_\mu^{IJ}$. The contributions here which are not of the form of a local $E_8$ rotation originate without exception from the variation of $B_\mu{}^m$; in particular, the terms $\delta B_\mu{}^m Q_m^{IJ}$ and $\delta B_\mu{}^m P_m^A$

are needed to maintain covariance. The remaining ambiguity is contained in the terms proportional to $X_{m\mu}$; these contributions drop out in (5) and must be determined directly by comparison with the original variations of $d=11$ supergravity. A subtlety here, not encountered up to now, is that the variations agree with those of the original $d=11$ theory only upon use of the Rarita-Schwinger equation, i.e. on shell. This feature can be directly traced to the occurrence of dualized bosonic field strengths in the explicit expressions for $P_\mu^A$ and $Q_\mu^{IJ}$.

By contrast, the supersymmetry variations of the "internal" components $\delta P_m^A$ and $\delta Q_m^{IJ}$ that follow from the generalized vielbein postulate (6) take a much simpler form

$$\delta P_m^A = D_m \omega^A + e_A^n X_{mn}, \quad \delta Q_m^{IJ} = 2\, P_m^A \Gamma_{AB}^{IJ} \omega^B + 2 e_{IJ}^n X_{mn} \tag{24}$$

Again there is an ambiguity: the terms proportional to $X_{mn}$ drop out of the variation of (6) (to see this, use must be made of (2)). However, inspection shows that no terms with this index structure and the correct dimensionality can be manufactured out of the SO(1,2) $\times$ SO(16) covariant fields, and we therefore conclude that $X_{mn}$ must vanish.

With the new SO(16) fields at hand it is possible to rewrite the fermionic part of the $d=11$ Lagrangian in this environment [11]. This is sufficient for the derivation of the supersymmetry constraint, which is just the time component of the Rarita Schwinger equation expressed in terms of the canonical variables. To determine the full SO(1,2) $\times$ SO(16) invariant Lagrangian is much harder: because of the explicit occurrence of the three index field $A_{MNP}$ in the original Lagrangian of [2], one cannot directly rewrite the latter, but must go back to the SO(1,2) $\times$ SO(16) covariant bosonic equations of motion. Since we are here solely concerned with the supersymmetry constraint we refer readers to [11] for a detailed discussion.

In setting up the hamiltonian formulation we follow the standard procedure (see e.g. [14]), which requires amongst other things that we foliate the $d=11$ space time by spatial slices. This entails in particular that the latter must be assumed to be of the global form $\boldsymbol{R} \times \Sigma_{10}$. The ten-dimensional spatial manifold $\Sigma_{10}$ is locally parametrized by coordinates $\boldsymbol{x},\boldsymbol{y},\ldots$, so that the $d=11$ coordinates are represented as $x^M = (t,\boldsymbol{x})$, etc.; $d=3$ indices will be split as $\mu = (t,i)$ with $i,j,\ldots = 1,2$. In order to determine the canonical variables "from scratch" we would again need to know the full SO(1,2) $\times$ SO(16) invariant Lagrangian. Since only its fermionic part is available, we will proceed in a more pragmatical fashion, requiring that the bosonic brackets lead to the correct supersymmetry variations on all canonical fields. The correctness of the ensuing brackets can be tested in alternative ways, for instance by inspection of the composite connections and their field content in terms of the original $d=11$ fields. Furthermore, in the dimensional reduction to $d=3$, the canonical brackets must match with those derived in [13].

The fermionic brackets are easily deduced from the fermionic Lagrangian. Because of the Majorana conditions, we have second class constraints such as

$$\pi_{\dot A} = \frac{1}{4} e e_\alpha{}^t \gamma^\alpha \chi_{\dot A} \tag{25}$$

for the "matter fermions" (note that $ee_\alpha{}^t \equiv \varepsilon_{\alpha\beta\gamma} \varepsilon^{ij} e_i{}^\beta e_j{}^\gamma$). These constraints are dealt with in the usual fashion, and after a little algebra we arrive at the following equal time (Dirac) brackets

$$\left\{ \Psi_i^I(\boldsymbol{x}), \overline{\Psi}_j^J(\boldsymbol{y}) \right\} = \varepsilon_{ij} \delta^{IJ} \delta(\boldsymbol{x},\boldsymbol{y}), \left\{ \chi_{\dot A}(\boldsymbol{x}), \overline{\pi}_{\dot B}(\boldsymbol{y}) \right\} = \delta_{\dot A \dot B} \delta(\boldsymbol{x},\boldsymbol{y}) \tag{26}$$

Owing to the complications described above, the canonical structure of the bosonic sector is considerably more involved. First of all, because of the algebraic and differential constraints on the 248-bein and the connections given before, our new bosonic variables constitute a redundant set, and thus cannot be grouped into canonically conjugate pairs. In principle, one could eliminate these redundancies by solving the constraints, but this would force us to abandon the new local symmetries, and thereby obscure the new geometrical structures we are about to expose. For this reason, we will keep the constraints; their consistency with the canonical brackets essentially hinges on the $E_8$ structure of the bosonic sector.

Secondly, in the original version of $d=11$ supergravity, all fields are gauge connections associated with the local symmetries. Their time components thus serve as Lagrange multipliers for the associated constraint generators. Of these, the time components of the dreibein and the gravitino survive as Lagrange multipliers of the Hamiltonian and diffeomorphism constraints, and the supersymmetry constraint, respectively. The time components $B_t{}^m$ correspond to local reparametrizations of the "internal" coordinates, and their role is somewhat obscured in the present framework. Since the original local SO(1,10) symmetry has been traded for SO(1,2) × SO(16), of the original $d=11$ spin connection, only the component $A_t{}^\alpha$ remains as a Lagrange multiplier for local SO(1,2). Instead of the "internal" part of the spin connection, we now have the SO(16) constraint $\phi^{IJ}$ multiplying the time component $Q_t^{IJ}$ of the SO(16) gauge connection. The former generates local SO(16) rotations on all fields, viz.

$$\delta_\omega \varphi(x) = \left\{ \varphi(x), \frac{1}{2} \int dy\, \omega^{IJ}(y) \phi^{IJ}(y) \right\} \tag{27}$$

where $\omega^{IJ}$ is a local SO(16) parameter.

Thirdly, in terms of the original $d=11$ fields, the set of bosonic canonical variables consists of both "elementary" and "composite" objects. Because of the dualizations implicit in our reformulation, the $\mu = i$ components of the $E_8$ connections contain time derivatives of the original fields. Consequently, there is not much point in singling out particular components as canonical momenta; however, we find it useful to define

$$\tilde{P}^A := 4 e g^{t\mu} P_\mu{}^A \tag{28}$$

which in the reduction to three dimensions are just the canonical momenta associated with the scalar fields, cf. [13]. We emphasize that bosonic brackets may vanish only up to bilinear fermionic contributions which we have neglected here; this may necessitate redefinitions by fermionic bilinears, such as e.g. for $A_i{}^\alpha$ in order to achieve $\{A_i{}^\alpha, A_j{}^\beta\} = 0$ [15].

Apart from the brackets involving the connections, we have

$$\left\{ e_i{}^\alpha(x), A_j{}^\beta(y) \right\} = \varepsilon_{ij} \eta^{\alpha\beta} \delta(x,y) \tag{29}$$

$$\left\{ B_i{}^m(x), P_j^A(y) \right\} = \frac{1}{4} \varepsilon_{ij} e_A^m(x)\, \delta(x,y), \quad \left\{ B_i{}^m(x), Q_j^{IJ}(y) \right\} = \frac{1}{2} \varepsilon_{ij} e_{IJ}^m(x)\, \delta(x,y) \tag{30}$$

$$\left\{ e_A^m(x), \tilde{P}^B(y) \right\} = \frac{1}{2} e_{IJ}^m(x) \Gamma_{AB}^{IJ}\, \delta(x,y), \quad \left\{ e_{IJ}^m(x), \tilde{P}^B(y) \right\} = e_A^m(x) \Gamma_{AB}^{IJ}\, \delta(x,y) \tag{31}$$

In the $d=3$ sector, the conjugate pairs $\varepsilon^{ik} e_k{}^\alpha$ and $A_j{}^\beta$ correspond to Ashtekar's variables in three dimensions [16] and become identical with them upon dimensional reduction to $d=3$. The second and third lines extend this analogy to the Kaluza–Klein components $B_i{}^m$. The non-vanishing result of the bracket between $B_i{}^m$ and $(Q_j^{IJ}, P_j{}^A)$ is explained by the fact that latter fields contain time derivatives of $B_i{}^m$ when expressed in terms of the original $d=11$ fields [4], as is also obvious from (10) by putting $\mu = i$. The consistency of the algebraic identities (1)–(4) with the above brackets follows from their invariance under $E_8$ and the fact that (31) effectively corresponds to an $E_8$ rotation of the 248-bein.

The remaining brackets involving $\tilde{P}^A$ are given by

$$\begin{aligned}
&\left\{ \tilde{P}^A(x), \tilde{P}^B(y) \right\} = \Gamma_{AB}^{IJ} \phi^{IJ}(x) \delta(x,y) \\
&\left\{ \tilde{P}^A(x), P_i^B(y) \right\} = \left( \delta^{AB} (\mathscr{D}_i - \partial_m B_i{}^m(x)) + \frac{1}{4} Q_i^{IJ}(x) \Gamma_{AB}^{IJ} \right) \delta(x,y) \\
&\left\{ \tilde{P}^A(x), Q_i^{IJ}(y) \right\} = 2\, \Gamma_{AB}^{IJ} P_i^B(x)\, \delta(x,y) \\
&\left\{ \tilde{P}^A(x), P_m^B(y) \right\} = \left( \delta^{AB} \partial_m + \frac{1}{4} Q_m^{IJ}(x) \Gamma_{AB}^{IJ} \right) \delta(x,y) \\
&\left\{ \tilde{P}^A(x), Q_m^{IJ}(y) \right\} = 2\, \Gamma_{AB}^{IJ} P_m^B(x)\, \delta(x,y)
\end{aligned} \tag{32}$$

where the derivative always acts on the first argument of the $\delta$-function. The brackets between the internal components $P_m^A$ and $Q_m^{IJ}$ all vanish (this is not true for the $d=3$ components). In the dimensional reduction to three dimensions, where $\partial_m \equiv 0$ and $Q_m^{IJ} = P_m^A \equiv 0$, these brackets coincide with the brackets predicted by the $\sigma$-model formulation [13,15] (modulo different normalizations).

Varying the fermionic part of the action w.r.t. the time component of the gravitino and expressing the result in terms of the canonical variables yields the supersymmetry constraint

$$\begin{aligned}\mathcal{S}^I = {} & \varepsilon^{ij}\left(D_i - \frac{1}{2}\partial_m B_i{}^m\right)\Psi_j^I + \frac{1}{4}\tilde{P}^A\Gamma^I_{A\dot{B}}\chi_{\dot{B}} + \varepsilon^{ij}e_{j\alpha}\gamma^\alpha P_i^A\Gamma^I_{A\dot{B}}\chi_{\dot{B}} - 2e_A^m\Gamma^I_{A\dot{B}}D_m\pi_{\dot{B}} \\ & + \frac{1}{2}e^m_{JK}P_m^A(\Gamma^I\Gamma^{JK} + 12\delta^{IK}\Gamma^J)_{A\dot{B}}\pi_{\dot{B}} - \varepsilon^{ij}e_{j\alpha}\gamma^\alpha e^m_{IJ}D_m\Psi_i^J - \frac{1}{2}\varepsilon^{ij}e_{j\alpha}\gamma^\alpha e^m_A\Gamma^{IJ}_{AB}P^B_m\Psi_i^J \\ & - \frac{1}{2}\varepsilon^{ij}\gamma^\alpha D_m e_{j\alpha}e^m_{IJ}\Psi_i^J\end{aligned} \tag{33}$$

The local supersymmetry of the theory is implemented by the constraint $\mathcal{S}^I(x) \approx 0$ in accordance with the general theory [17]. As is well known, the supersymmetry constraint is the key constraint because all other constraints can be obtained from it by commutation. In this sense it is the "square root" of the bosonic constraints, enabling us to determine them without explicit knowledge of the bosonic part of Lagrangian. However, the necessary calculations are not very illuminating and quite tedious already for the dimensionally reduced theory [13]. We also note that (33) is polynomial in the canonical variables, just like the constraints in Ashtekar's formulation of $d=4$ gravity.

The canonical brackets given above are sufficient to verify that the supersymmetry variations (11) and (13) of the basic fields are recovered by means of the formula

$$\delta_\varepsilon \varphi(x) = \{\varphi(x), \mathcal{S}[\varepsilon]\} \tag{34}$$

where we have introduced the integrated constraint

$$\mathcal{S}[\varepsilon] := \int dx\, \bar{\varepsilon}^I(x)\mathcal{S}^I(x) \tag{35}$$

with $\varepsilon^I(x)$ an arbitrary spinorial test function. The verification is equally straightforward for (24) (with $X_{mn}=0$), but much more tedious for the $\mu = i$ components of (22) and (23), where have performed only partial checks. In particular, we are led to postulate non-vanishing brackets between $P_i{}^A$, $Q_i^{IJ}$ and $A_i{}^\alpha$ that have a rather complicated structure due to the occurrence of dualized field strengths "inside" the connections. However, these complications are restricted to the components with $\mu = i$ and disappear altogether in the reduction to $d=3$.

The $E_8$ structure exhibited by the bosonic sector remains an ill understood feature. Of course, $E_8$ is not a symmetry of the theory in eleven dimensions, but it does become a (rigid) symmetry upon dimensional reduction to three dimensions [6,12]. Although the relation of $E_8$ with the internal $d=8$ coordinate transformations, which are no longer manifest in the present formulation, remains somewhat mysterious, we can offer the following hints. Returning to the constraint (10) (its $\mu = t$ component, to be precise) and its hamiltonian analog, we recall that such constraints need only hold weakly, i.e. on the constraint surface. Indeed, checking the compatibility of the brackets (30) and (31) with the duality constraint (10), a little calculation reveals that this constraint is modified by a term proportional to the SO(16) generator $\phi^{IJ}$:

$$e_A^m\tilde{P}^A - \frac{1}{2}e^m_{IJ}\phi^{IJ} = 4\varepsilon^{ij}\mathcal{D}_i B_j{}^m \tag{36}$$

Now we recall from [13] the identification of the 248-bein with the $\sigma$-model field $\mathscr{V} \in E_8$ appearing in the reduction to $d = 3$,

$$e_{IJ}^m = \frac{1}{60} \mathrm{Tr}\left( Z^m \mathscr{V} X^{IJ} \mathscr{V}^{-1} \right), \quad e_A^m = \frac{1}{60} \mathrm{Tr}\left( Z^m \mathscr{V} Y^A \mathscr{V}^{-1} \right) \tag{37}$$

where $X^{IJ}$ and $Y^A$ are the compact and non-compact generators of $E_8$, respectively, and where the $Z^m$ for $m = 3, \ldots, 10$ are eight nilpotent, hence non-compact, commuting generators [4]. On the other hand, the $E_8$ Noether charge density of the dimensionally reduced theory is given by [13,15]

$$\mathscr{J} = \mathscr{V} \left( \tilde{P}^A Y^A - \frac{1}{2} \phi^{IJ} X^{IJ} \right) \mathscr{V}^{-1} \tag{38}$$

But in view of (36), this implies that

$$\varepsilon^{ij} \mathscr{D}_i B_j{}^m = \frac{1}{60} \mathrm{Tr}( Z^m \mathscr{J} ); \tag{39}$$

Consequently, the projection of the charge density onto the nilpotent subalgebra spanned by the $Z^m$'s survives the decompactification and is given by $\varepsilon^{ij} \mathscr{D}_i B_j{}^m$. While the eight internal coordinates are thus associated with the nilpotent subalgebra of $E_8$, the remaining part of the $E_8$ Lie algebra presumably corresponds to some kind of non-commutative geometry.

It does not appear that the present version of $d = 11$ supergravity can be quantized any more easily than the original one of [2]. However, it is anyhow very unlikely that this theory can be consistently quantized all by itself: rather, to achieve consistency it must be embedded in some as yet unknown, but bigger theory (M-theory?). We would thus expect that the remaining open problems can only be resolved in such a larger framework. While the "internal" sector of SO(1,2) × SO(16) invariant $d = 11$ supergravity exhibits a certain conceptual simplicity, complications persist in the $d = 3$ sector, as can be immediately seen for instance by comparing (22),(23) with (24). Since the same might have been said about the $d = 7$ and $d = 4$ sectors of the SO(1,3) × SU(8) invariant version of $d = 11$ supergravity given in [3], the natural next step is to search for yet another version of $d = 11$ supergravity based on a $2 + 9$ split, which we would expect to possess local SO(1,1) × SO(16)$^\infty$ invariance, where SO(16)$^\infty$ is the maximal compact subgroup of $E_9$ defined by means of the generalized Cartan Killing metric on the affine Lie algebra $E_9$. The embedding into a bigger theory with even larger symmetries might not only explain the emergence of space time symmetries from a pre-geometrical theory, but should also provide a simplifying principle that might help to avoid some of the cumbersome calculations of [3,4,11].

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