SELF-CONSISTENT SOLUTIONS FOR LOW-FREQUENCY GRAVITATIONAL BACKGROUND RADIATION

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We study in a Brill-Hartle type of approximation the back-reaction of a superposition of linear gravitational waves in an Einstein-de Sitter background up to the second order in the small wave amplitudes h_{ik}. The wave amplitudes are assumed to form a homogeneous and isotropic stochastic process. No restriction for the wavelengths is assumed. The effective stress-energy tensor $T^e_{\mu\nu}$ is calculated in terms of the correlation functions of the process. We discuss in particular a situation where $T^e_{\mu\nu}$ is the dominant excitation of the background metric. Apart from the Tolman radiation universe, a solution with the scale factor of the de Sitter universe exists with $p = -\rho$ as effective equation of state.

While the description of cosmic gravitational radiation in linearized relativity is fairly well known, a study of its nonlinear aspects is much harder. Numerical relativity[3] is one way, but there are also some analytical and semianalytical approaches. Already in 1964 Brill and Hartle[4] proposed a scheme which takes the back-reaction of linear gravitational waves into account. As it stands, the Brill-Hartle method is considered as a high-frequency approximation for gravitational radiation. Thus its application to early inflationary stages of the Universe is questionable, since low-frequency modes may be present, which turn into high-frequency modes (with wavelengths smaller than the temporary horizon) only at later time. As shortly discussed in this note, a slight modification of the Brill-Hartle approach can remove this shortcoming. The detailed calculations will be published elsewhere. The wave perturbations are considered as random variables, forming a stochastic process, which shares the symmetries of the background metric. The method is similar to the Monin-Yaglom[7] approach to statistical fluid dynamics, and is also used in optical coherence theory[6]. For simplicity, an Einstein-de Sitter model is chosen as background metric. Tensor perturbations are added in synchronous gauge, assuming $g_{00} = -a^2$, $g_{0i} = 0$, $g_{ik} = a^2\delta_{ik} + h_{ik}$, as well as the gauge conditions $h_{ii} = 0, h_{ik,k} = 0$. $a(\eta)$ is the scale factor. The Einstein tensor may be expanded in powers of h_{ik}. Retaining terms up to second order and performing a stochastic (ensemble) average on the field equations, they split into linear wave equations for the h_{ik} and the back-reaction equations. The back-reaction equations relate the Einstein tensor for the background metric to the effective stress-energy tensor of the waves, which is represented as stochastic average over bilinear expressions in h_{ik} and derivatives of h_{ik}. As well known, the effective stress-energy tensor is not gauge invariant in general . However, as shown by Abramo, Brandenberger and Mukhanov[1] (see also [2], [5]), the gauges change the background geometry to second order, and these changes just compensate the change of the stress-energy tensor. Representing the h_{ik} as stochastic Fourier integrals $\int \gamma_{ik}(\mathbf{k},\eta)e^{i\mathbf{kx}}d\mathbf{k} + conj.compl.$, the amplitudes γ_{ik} satisfy an ordinary differential equation. The symmetry properties of the problem allow a simple representation of expressions which are bilinear in h_{ik}, e.g.

$$\langle h_{ik}h_{lm}\rangle = \frac{16\pi a^2}{15}(3\delta_{il}\delta_{km}+3\delta_{im}\delta_{kl}-2\delta_{ik}\delta_{lm})\int dkk^2 f, \tag{1}$$

in terms of a single spectral density $f(k,\eta)$. f satisfies a nonlinear differential equation (ϵ_0 depends only on $k=|\mathbf{k}|$, and a prime denotes the differentiation with respect to η)

$$2ff''-f'^2+4f^2(k^2-\frac{a''}{a})-4\epsilon_0=0. \tag{2}$$

For high-frequency waves $k\eta\gg 1$, $f=\sqrt{\epsilon_0}/k$ follows as solution ("high-frequency approximation"). It is convenient to write ρ and p in the effective stress-energy tensor in terms of four frequency-independent, but in general time-dependent integrals ("moments") over the spectral density $f(k,\eta)$:

$$f_0=\int dk\ k^2\frac{\epsilon_0(k)}{f},\ f_1=\int dk\ k^2\frac{f'^2}{f},\ f_2=\int dk\ k^2 f,\ f_4=\int dk\ k^4 f. \tag{3}$$

For a general scale factor one obtains ($g=f_0+f_1/4$)

$$\rho_g = \frac{1}{2Ga^4}(f_4+g+3\frac{a'}{a}f_2'-7\frac{a'^2}{a^2}f_2), \tag{4}$$

$$3p_g = \frac{1}{2Ga^4}(7f_4-5g+5\frac{a'}{a}f_2'-5\frac{a'^2}{a^2}f_2). \tag{5}$$

In the high-frequency approximation $3Ga^4p_g=Ga^4\rho_g=\int dkk^3\sqrt{\epsilon_0}$, energy density and pressure are positive. If low-frequency modes are present, their contribution can be negative, and also the equation of state can deviate considerably from the high-frequency relation $p=\rho/3$. If pressure and density of gravitational waves cannot be neglected compared to other forms of matter, the back-reaction on the scale factor must be taken into account. Taking a pure gravitational radiation universe, one has to solve the field equations

$$6\frac{a'^2}{a^4} = 16\pi G\rho_g, \tag{6}$$

$$\frac{a''}{a}+\frac{a'^2}{a^2} = 4\pi Ga^2(\rho_g-p_g), \tag{7}$$

with ρ_g,p_g taken from (4) and (5). Note the further equations

$$g' = -f_4'+\frac{a''}{a}f_2', \tag{8}$$

$$f_2'' = 2g+2\frac{a''}{a}f_2-2f_4, \tag{9}$$

which follow from differentiating f_1,f_2 and using the differential equation for f. The four equations (6)-(9) give the relation

$$(2a'f_2-af_2')\ (aa''-2a'^2)=0. \tag{10}$$

Vanishing of the first factor leads to the Tolman radiation universe, vanishing of the second factor gives the scale factor of the de Sitter cosmos. It is easy to find the moments f_2, f_4, g from the corresponding differential equations. Self-consistency however requires, that the moments found in this way must be compatible with the expressions following directly from (3), if the solution of the wave equation is inserted. Compatibility can be achieved indeed, it however requires singular infrared components for some spectral quantities. The general solution of (2) for the radiation ($s = 0, a \sim \eta$) and de Sitter ($s = 1, a \sim 1/\eta$) cosmos is known to be

$$f = 2npp^* + (l + im)p^2 + (l - im)p^{*2}, \tag{11}$$

where l, m, n are three functions of k, connected with ϵ_0 by $\epsilon_0 = 4k^2(n^2 - l^2 - m^2)$, $p(x) = (1+is/x)e^{ix}$ is a complex function of $x = k\eta$. Whereas $n(k)$ is not restricted, the spectral functions $l(k)$ and $m(k)$ should be understood as generalized functions[8] and have the form (b is a constant)[a]

$$l(k) = -\frac{b}{2}\frac{\delta''(k)}{k^2} - 2n_2\frac{\delta(k)}{k^2}, \tag{12}$$

$$m(k) = 0, \tag{13}$$

where $\delta(k)$ is the Dirac delta function and $n_2 = \int dkk^2 n(k)$. The spectral densities for the energy density and the pressure are then (again in the Tolman case)

$$a^4 G\rho(k,\eta) = n(k)(k^2 - \frac{7}{2\eta^2}) + \delta(k)(\frac{7n_2}{\eta^2} - b), \tag{14}$$

$$3a^4 Gp(k,\eta) = n(k)(k^2 - \frac{5}{2\eta^2}) + \delta(k)(\frac{5n_2}{\eta^2} - b). \tag{15}$$

These quantities show infrared singularities, part of the effective energy density and pressure resides in infrared ($k = 0$) modes. It is so far not clear whether the singularities result from using only a second-order approximation. The integrated quantities are finite, however.

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[a]for the Tolman cosmos, the condition is different for the de Sitter case.