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Existence of Constant Mean Curvature Foliations in Spacetimes with Two-Dimensional Local Symmetry

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Abstract: It is shown that in a class of maximal globally hyperbolic spacetimes admitting two local Killing vectors, the past (defined with respect to an appropriate time orientation) of any compact constant mean curvature hypersurface can be covered by a foliation of compact constant mean curvature hypersurfaces. Moreover, the mean curvature of the leaves of this foliation takes on arbitrarily negative values and so the initial singularity in these spacetimes is a crushing singularity. The simplest examples occur when the spatial topology is that of a torus, with the standard global Killing vectors, but more exotic topologies are also covered. In the course of the proof it is shown that in this class of spacetimes a kind of positive mass theorem holds. The symmetry singles out a compact surface passing through any given point of spacetime and the Hawking mass of any such surface is non-negative. If the Hawking mass of any one of these surfaces is zero then the entire spacetime is flat.

1. Introduction

There are a number of general results in the literature on the properties of foliations by compact spacelike hypersurfaces of constant mean curvature (CMC hypersurfaces) in spacetimes which admit a compact Cauchy hypersurface. (See [18, 1] and references therein.) In particular, a basic result of Gerhardt [9] shows that if there is a foliation whose mean curvature tends uniformly to infinity, there is a CMC foliation with the same property. However, the only results which give criteria in terms of Cauchy data for the existence of such foliations covering more than a small neighbourhood of a given CMC hypersurface are restricted to special classes of spacetimes, all of which have high symmetry. The results of this paper also apply only to certain spacetimes with symmetry but represent a significant generalization, since they include for the first time spacetimes containing both matter and gravitational waves. The method used suggests that there is

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a close connection between the question of global existence of CMC foliations and that of global existence of solutions of the Einstein-matter equations and such a connection helps to explain why it has up to now been necessary to make symmetry assumptions: we cannot understand the question of global existence of CMC foliations in a context where we do not understand the question of global existence for the Einstein-matter equations. When a CMC foliation exists in a spatially compact spacetime satisfying the strong energy condition it is unique. If the exceptional case of flat spacetime is excluded, the mean curvature of the leaves of this foliation varies in a strictly monotone manner. Thus it provides an invariantly defined preferred time coordinate on spacetime.

The spacetimes studied in the following are defined by two conditions. The first is that they be solutions of the Einstein equations coupled to certain matter fields and the second is that they admit a compact CMC Cauchy hypersurface which possesses a two-dimensional abelian group of local symmetries without fixed points. (The second condition, stated here informally, is made precise in the next section.) The simplest example of a symmetry of this kind is that where the compact Cauchy hypersurface is diffeomorphic to the torus $T^3 = S^1 \times S^1 \times S^1$, with the symmetries given by the action of $U(1) \times U(1)$ acting on two of the three S^1 factors by rotations. As shown below, the mean curvature of the Cauchy hypersurface must be non-zero, except in the trivial case where the spacetime is flat. Without loss of generality, reversing the time orientation if necessary, it can be assumed to be negative. The main result of this paper (Theorems 5.1 and 6.1) is that if the spacetime is the maximal globally hyperbolic development of data with local $U(1) \times U(1)$ symmetry on a CMC Cauchy hypersurface then the entire past of this hypersurface can be covered by a CMC foliation, with the mean curvature taking all values in the interval $(-\infty, H_0]$, where H_0 is the mean curvature of the initial hypersurface. In particular the initial singularity in these spacetimes is a crushing singularity in the sense of Eardley and Smarr[8]. The assumption made on the matter model is that it is either collisionless matter modelled by the Vlasov equation (Theorem 5.1) or a wave map with values in an arbitrary complete Riemannian manifold (Theorem 6.1). It is also shown that the CMC foliation can be extended so that the mean curvature takes on all values in the interval $(-\infty, 0)$. Unfortunately this does not by itself suffice to show that the CMC foliation covers the entire future of the initial hypersurface.

Special cases of this result are already known. The first is that of the Gowdy spacetimes on the torus. These are vacuum spacetimes with global $U(1) \times U(1)$ symmetry which satisfy the additional condition that the so-called twist constants vanish. (The meaning of this is explained in Sect. 3. For more information on its significance, see [6].) The result was proved in this case by Isenberg and Moncrief [14]. The second is that of solutions of the Einstein-Vlasov system with plane symmetry [19]. In the first of these cases there is no matter present, while in the second there are no gravitational waves. The results of this paper contain both these results as special cases. It should be noted that they go beyond previous results even in the vacuum case in two ways: they require only local, rather than global symmetry and they allow non-vanishing twist constants. The essential new element in comparison with the cases considered previously is the occurrence of nonlinear hyperbolic equations which are coupled to the matter equations. These are treated with the help of methods introduced by Gu [13] in the study of wave maps and by Glassey and Strauss [12] in the study of the Vlasov-Maxwell system.

As a by-product of this analysis, a theorem on the positivity of the Hawking mass in spacetimes with local $U(1)\times U(1)$ symmetry is obtained (Proposition 3.1). This says that the Hawking mass of any surface of symmetry in a spacetime of this type is non-negative and that if the Hawking mass vanishes for any one of these surfaces in a spacetime, then the spacetime is flat.

2. Local $U(1) \times U(1)$ Symmetry

The spacetimes considered in the following are defined on manifolds of the form $M = \mathbf{R} \times S$, where S is a bundle over the circle S^1 whose fibre is a compact orientable surface F. Let p be the projection of the universal cover $\tilde{F}$ onto F. Let $g_{\alpha\beta}$ be a globally hyperbolic metric on M for which each submanifold $\{t\} \times S$ is a Cauchy hypersurface. S is covered by its pull-back to a bundle over $\mathbf{R}$. Since $\mathbf{R}$ is contractible, the latter bundle is isomorphic to $\mathbf{R} \times F$. This in turn is covered in a natural way by $\mathbf{R} \times \tilde{F}$, which is simply connected. Hence the universal cover $\tilde{S}$ of S can be identified with $\mathbf{R} \times \tilde{F}$ and there is a natural fibre preserving projection corresponding to p. Let $\hat{p}$ be the associated projection of $\tilde{M} = \mathbf{R}^2 \times \tilde{F}$ onto M. Define $\hat{g}_{\alpha\beta}$ to be the pull-back of $g_{\alpha\beta}$ by $\hat{p}$. Suppose that a two-dimensional Lie group G acts effectively on $\tilde{M}$ by isometries of $\hat{g}_{\alpha\beta}$ in such a way that the orbits are the inverse images under $\hat{p}$ of the fibres of the bundle S. Each orbit with its induced metric is a simply connected Riemannian manifold of constant curvature and thus must be, up to a constant conformal rescaling, isometric to the standard metric on the sphere, the Euclidean plane or the hyperbolic plane. The isometry group of the sphere has no two-dimensional subgroups and thus the case $F = S^2$ is not possible. (If G is replaced by a three-dimensional Lie group then $F = S^2$ is possible and the spherically symmetric spacetimes studied in [19] are obtained.) The surfaces diffeomorphic to F which correspond to the fibres of S and whose inverse images are the group orbits will be referred to as surfaces of symmetry. These spacetimes will be said to have two-dimensional local symmetry. In the case where the orbits are isometric to the Euclidean plane, they will be said to have local $U(1) \times U(1)$ symmetry.

Consider now any spacelike hypersurface S in the spacetime $(M, g_{\alpha\beta})$ which is a union of surfaces of symmetry. Choose one of these surfaces of symmetry and call it F_0. Let γ be an affinely parametrized geodesic of the induced metric on S which starts orthogonal to F_0. It is also orthogonal to all the other surfaces of symmetry which it meets. Taking all geodesics of this type and following them until they meet F_0 again gives a smooth mapping ϕ from $F_0 \times I$ to S, where I is some interval. Let the parameter along the geodesics be chosen so that $I = [0, 2\pi]$. Let ψ denote the mapping which takes $\phi(x, 0)$ to $\phi(x, 2\pi)$ and let $\tilde{\psi}$ denote the lift of this mapping to a diffeomorphism between two inverse images of F_0 in $\tilde{M}$ defined by following geodesics in the covering space. The mapping $\tilde{\psi}$ maps the Killing vectors of F_0 corresponding to the group action on one of the inverse images of F_0 in $\tilde{M}$ to those corresponding to another inverse image. These two inverse images can be identified with each other by an isometry, which is uniquely determined up to an element of the isometry group of the Euclidean or hyperbolic plane respectively by their induced metrics. In the case that $\tilde{F}_0$ is isometric to the Euclidean plane $\tilde{\psi}_0$ is the composition of a linear mapping $\tilde{\psi}_1$ with a translation $\tilde{\psi}_2$. The mapping $\tilde{\psi}$ must preserve the lattice of inverse images in $\tilde{F}_0$ of a given point in F_0. These lattices are isometric, and so by using the freedom in identifying the two covering spaces, it can be assumed without loss of generality that they are identified with each other. It follows that $\tilde{\psi}_1$ can be represented as an element of $GL(2, \mathbf{Z})$. When this element is not the identity it means in general that the topology of S is that of a non-trivial torus bundle over the circle. Let y^A be periodic coordinates on F_0 corresponding to Cartesian coordinates on $\tilde{F}_0$. Let (x, y^A) be Gauss coordinates based on F_0 such that y^A restrict to the previously chosen coordinates on F_0. Now let $B^4 = \det(g_{AB})$, where upper case Roman indices take the values 2 and 3. The metric takes the form:

$$dx^2 + B^2 \tilde{g}_{AB}(x) dy^A dy^B, \tag{2.1}$$

where $\det \tilde{g}_{AB} = 1$. Let L be the length of a geodesic which starts normal to an orbit and ends when it intersects the same orbit again. Define $a = 2\pi/(\int_0^L B^{-1}(x)dx)$ and

$$x' = a \int_0^x B^{-1}(y)dy. \tag{2.2}$$

If x' is used as a coordinate and the primes omitted from the notation the metric takes the form:

$$A^2(dx^2 + a^2\tilde{g}_{AB}dy^A dy^B), \tag{2.3}$$

where A is $a^{-1}B$, reexpressed as a function of x'. The new coordinate x runs from 0 to 2π. The functions $\tilde{g}_{AB}(x)$ satisfy the relation $\tilde{g}_{AB}(x + 2\pi) = n_A^C n_B^D \tilde{g}_{CD}(x)$, where n_B^A are the components of a matrix in $GL(2, \mathbf{Z})$. A satisfies $A(x + 2\pi) = A(x)$. A similar analysis for the case that $\tilde{F}_0$ is the hyperbolic plane would no doubt be more complicated and is not attempted here. However it appears that, due to the fact that vector fields and tracefree symmetric rank two tensors on a surface of genus higher than one must have zeroes, in that case nothing will be obtained which goes beyond the spacetimes with hyperbolic symmetry already studied in [19].

The metric $\tilde{g}_{AB}$ can be parametrized in terms of two functions W and V in the following way:

$$\tilde{g}_{22} = e^W \cosh V, \quad \tilde{g}_{33} = e^{-W} \cosh V, \quad \tilde{g}_{23} = \sinh V. \tag{2.4}$$

The values of V and W at $x = 0$ and $x = 2\pi$ are related by a diffeomorphism N which does not have a simple explicit form. There are several special cases which are of interest. Consider first the case where the matrix N with components n_B^A is the identity. Then there is a natural action of $U(1) \times U(1)$ on the spacetime and we have the case of (global) $U(1) \times U(1)$ symmetry. A further specialization is given by the assumption that the reflections $y^A \mapsto -y^A$ are isometries of the spacetime metric for $A = 2, 3$. Spacetimes satisfying this condition will be called polarized $U(1) \times U(1)$-symmetric spacetimes. They have the property that $V = 0$. The vacuum spacetimes of this class are the polarized Gowdy spacetimes [7]. The plane symmetric spacetimes studied in [19] have the property that $W = V = 0$. When N is not the identity, there are two qualitatively different cases. If N is diagonalizable and not the identity, then either it is minus the identity, or the two eigenvalues are distinct. If it is minus the identity then S has a two-fold cover which is a torus and for our purposes is essentially the same as when N is the identity. When the eigenvalues are distinct the manifold S admits a geometric structure of type Sol in the sense of Thurston [20]. The metrics obtained in that case include ones which are of Bianchi type VI_0. There is a polarized case, where reflections in the eigendirections of N are supposed to be isometries of the metric $\hat{g}_{\alpha\beta}$ on the universal covering space. If N has a non-standard Jordan form then, by passing to a two-fold cover if necessary, we can assume that these eigenvalues are equal to unity. The resulting manifold S admits a geometric structure of type Nil [20]. The metrics obtained in that case include those of Bianchi type II.

Lemma 2.1. *Let (M, g) be a non-flat spacetime with local $U(1)\times U(1)$ symmetry having a symmetric constant mean curvature Cauchy hypersurface and satisfying the dominant and strong energy conditions. Then given any point p on the Cauchy surface there exists an open neighbourhood U of p and a smooth local diffeomorphism ϕ of $I \times [0, 2\pi] \times T^2$ onto U for some interval I such that:*
(i) If $\phi(t, x_1, y_1) = \phi(t, x_2, y_2)$ then $x_1 = 0$ and $x_2 = 2\pi$ or vice versa.

(ii) For each $t \in I$ the set $\phi(\{t\} \times [0, 2\pi] \times T^2)$ is a hypersurface of constant mean curvature t.
(iii) The pull-back of the metric under ϕ has the form

$$-\alpha^2 dt^2 + A^2[(dx + \beta^1 dt)^2 + a^2 \tilde{g}_{AB}(dy^A + \beta^A dt)(dy^B + \beta^B dt)]. \tag{2.5}$$

The functions α, β^a, A and $\tilde{g}_{AB}$ depend on t and x and $\tilde{g}_{AB}$ has unit determinant. They satisfy $\alpha(t, 2\pi) = \alpha(t, 0)$, $A(t, 2\pi) = A(t, 0)$, $\beta^1(t, 2\pi) = \beta^1(t, 0) = 0$, $\beta^B(t, 0) = 0$, $\tilde{g}_{AB}(t, 2\pi) = n_A^C n_B^D \tilde{g}_{CD}(t, 0)$, where n_B^A is an element of $GL(2, \mathbf{Z})$. The quantity a depends only on t.

Proof. It is a standard fact that, in a non-flat spacetime satisfying the strong energy condition, a neighbourhood of a compact CMC hypersurface with non-zero mean curvature can be foliated by constant mean curvature hypersurfaces and that the mean curvature of these hypersurfaces can be used as a time coordinate. If the $U(1) \times U(1)$-symmetry is global then it follows from the uniqueness of CMC hypersurfaces that they are unions of surfaces of symmetry. If the symmetry of the data is only local then some more care is needed, but since an almost identical argument has been given in [19] the details are omitted here. The mean curvature of the Cauchy hypersurface in the assumption of the lemma cannot be zero. To see this consider the Hamiltonian constraint

$$R - k^{ab} k_{ab} + (\mathrm{tr} k)^2 = 16\pi\rho. \tag{2.6}$$

When the mean curvature is zero this implies that the scalar curvature R is non-negative. The topology of the Cauchy hypersurface is such that any metric with non-negative scalar curvature must be flat. For its universal cover is diffeomorphic to $\mathbf{R}^3$. This implies ([16], p. 324) that the Cauchy hypersurface admits no metric of positive scalar curvature and it is well-known that a compact 3-manifold satisfying the latter condition admits no non-flat metrics of non-negative scalar curvature. Hence the induced metric on the Cauchy hypersurface is flat and, from the Hamiltonian constraint the second fundamental form and the energy density are zero. It follows from the dominant energy condition that the spacetime is vacuum everywhere and uniqueness in the Cauchy problem for the vacuum Einstein equations shows that the spacetime is flat. Since the spacetime is by hypothesis non-flat, it follows that a maximal hypersurface is impossible. Let U be an open neighbourhood of the Cauchy hypersurface covered by a CMC foliation and let t be the function on U which is equal to the mean curvature of the leaf of the foliation on which the point lies. Choose a surface of symmetry F_0 in the initial hypersurface t =const. and identify this with surfaces in the other hypersurfaces t =const. by means of geodesics which start on F_0 orthogonal to the Cauchy hypersurface. Construct a mapping on each hypersurface t =const. from $[0, 2\pi] \times T^2$ in the way described above. Putting together these mappings for all values of t occurring in the foliation of U gives the mapping whose existence is asserted by the lemma. □

Remark. If the hypotheses of the lemma are weakened to allow the spacetime to be flat then almost all the conclusions remain true. The only property which is lost is that the hypersurfaces of constant t, while still CMC, cannot always be chosen to have mean curvature t.

3. Estimates for the Hawking Mass and Area Radius

In this section certain general estimates for spacetimes with local $U(1) \times U(1)$ symmetry are derived. A solution of the Einstein constraint equations consists of a 3-dimensional manifold S and a Riemannian metric h_{ab}, a symmetric tensor k_{ab}, a real-valued function ρ and a covector j_a on M which satisfy the Hamiltonian constraint (2.6) and the momentum constraint

$$\nabla^a k_{ab} - \nabla_b(\mathrm{tr}k) = 8\pi j_b. \tag{3.1}$$

In this paper it is always assumed that the dominant energy condition holds and this implies that $\rho \geq |j_a|$. Suppose now that S is covered by a Gaussian foliation. In other words, if F_0 is a fixed leaf of the foliation, any other leaf is obtained by going a fixed distance along the geodesics which start normal to F_0. If we think of S as being embedded in spacetime, then the resulting embedding of each leaf F in this spacetime defines various geometrical objects on F, as is always the case for an embedding of pseudo-Riemannian manifolds. We present the definition of these objects in terms of such an embedding, but in fact they are uniquely defined by the initial data. There is a preferred orthonormal basis of the normal bundle of F in spacetime, where the first vector is normal to S and the second vector tangent to S. These vectors are defined uniquely up to sign by this condition. The geometric objects defined on F by the embedding are then the induced metric, a second fundamental form associated to each normal vector and a 1-form, which is the representation of the normal connection in the given normal basis. The two second fundamental forms will be denoted by κ_{AB} and λ_{AB} respectively and the 1-form representing the normal connection will be denoted by η_A. (Here upper case Roman indices are used for objects intrinsic to F. Indices of objects of this kind are raised and lowered using the induced metric g_{AB} and its inverse.) In a vacuum spacetime the freedom in η_A consists of just two spacetime constants. These are the twist constants referred to in the introduction, whose vanishing is one of the defining conditions of Gowdy spacetimes. If u is an arc length parameter along the normal geodesics, the constraints can be written in the following form:

$$\begin{aligned}
&\partial_u(\mathrm{tr}\lambda + \mathrm{tr}\kappa) = H(\mathrm{tr}\lambda + \mathrm{tr}\kappa) + \nabla^A\eta_A + K \\
&\quad - 8\pi(\rho + J) - \tfrac{3}{4}(\mathrm{tr}\lambda + \mathrm{tr}\kappa)^2 - \tfrac{1}{2}(\tilde\lambda^{AB} + \tilde\kappa^{AB})(\tilde\lambda_{AB} + \tilde\kappa_{AB}) - \eta^A\eta_A, \qquad (3.2)\\
&\partial_u(\mathrm{tr}\lambda - \mathrm{tr}\kappa) = -H(\mathrm{tr}\lambda - \mathrm{tr}\kappa) - \nabla^A\eta_A + K \\
&\quad - 8\pi(\rho - J) - \tfrac{3}{4}(\mathrm{tr}\lambda - \mathrm{tr}\kappa)^2 - \tfrac{1}{2}(\tilde\lambda^{AB} - \tilde\kappa^{AB})(\tilde\lambda_{AB} - \tilde\kappa_{AB}) - \eta^A\eta_A, \qquad (3.3)\\
&\partial_u\eta_A = -(\mathrm{tr}\lambda)\eta_A - \nabla^B\kappa_{AB} - 8\pi j_A. \qquad (3.4)
\end{aligned}$$

Here H is the trace of the second fundamental form k_{ab} (i.e. the mean curvature), K is the Gaussian curvature of F, $\tilde\kappa_{AB}$ and $\tilde\lambda_{AB}$ are the trace-free parts of κ_{AB} and λ_{AB} respectively and J is the contraction of the unit normal vector to F in S with j_a. This way of writing the constraints generalizes an approach used by Malec and Ó Murchadha [17], for spherically symmetric asymptotically flat spacetimes, by the author [19] for spatially compact surface symmetric spacetimes and by Chruściel [5] for vacuum spacetimes with $U(1)\times U(1)$ symmetry. In the following these equations are only used in the case of local $U(1)\times U(1)$ symmetry. It should, however, be noted that the form of the equations suggests that there may exist an analogue in the general case. The terms which cause difficulties in general are those containing derivatives tangential to the foliation by surfaces F. These are of the form $\nabla^A\eta_A$ and K. The integral of the first of these over F is zero, while the integral of the second is a constant only depending on the topology as a consequence of the Gauss-Bonnet theorem. Thus integrating over F eliminates the tangential derivatives from Eqs. (3.2) and (3.3).

Consider now the case of local $U(1)\times U(1)$ symmetry, with the Gaussian foliation being that by surfaces of symmetry. Let $\theta = \mathrm{tr}\lambda - \mathrm{tr}\kappa$, $\theta' = \mathrm{tr}\lambda + \mathrm{tr}\kappa$. These are the expansions of the two families of null geodesics orthogonal to F. Let the area radius r be the square root of the area of F. The Hawking mass is defined by $m = -\frac{1}{2}r\nabla^\alpha r\nabla_\alpha r$. In this case $\nabla^A\eta_A = K = 0$ and equations (3.2) and (3.3) become:

$$\partial_u\theta = -H\theta - P, \tag{3.5}$$
$$\partial_u\theta' = H\theta' - P', \tag{3.6}$$

where the quantities P and P' are non-negative. It is also useful, following [17], to write these equations in the alternative form:

$$\begin{aligned}\partial_u(r\theta) &= -Q - \frac{1}{4r}(\theta^2r^2 + 4\theta Hr^2 + \theta r(\theta r - \theta' r)),\\ \partial_u(r\theta') &= -Q' - \frac{1}{4r}(\theta'^2r^2 - 4\theta' Hr^2 + \theta' r(\theta' r - \theta r))\end{aligned}, \tag{3.7}$$

where the quantities Q and Q' are non-negative. Consider now a symmetric Cauchy hypersurface S, i.e. one which is a union of surfaces of symmetry. Denote the maximum value attained by $r\theta$ and $r\theta'$ on this hypersurface by M_+ and the minimum by M_-. Let x_0 be a point where the maximum is attained and suppose without loss of generality that $\theta(x_0) \geq \theta'(x_0)$. Since x_0 is a critical point of $r\theta$, it follows from (3.7) that at that point either $r\theta \leq 0$ or

$$\theta^2r^2 + (4Hr)(\theta r) \leq 0. \tag{3.8}$$

It follows that $M_+ \leq 4|Hr|$. Similarly, $M_- \geq -4|Hr|$. These inequalities show that θ and θ' can be bounded in modulus by $4|H|$. The Hawking mass is related to the area radius and the expansions by $-2m/r = \frac{1}{4}r^2\theta\theta'$. Thus in a spacetime with local $U(1)\times U(1)$ symmetry which is foliated by compact CMC hypersurfaces with the mean curvature varying in a finite interval (t_1, t_2) and which satisfies the dominant energy condition, if r is bounded then $2m/r$ is bounded. These equations can also be used to prove a kind of positive mass theorem.

Proposition 3.1. *Let (M, g) be a spatially compact spacetime with local $U(1)\times U(1)$ symmetry which satisfies the dominant energy condition. Then the Hawking mass of each surface of symmetry is non-negative and if the Hawking mass of any surface of symmetry is zero the spacetime is flat.*

Proof. The proof is similar to that of Lemma 2.4 of [19]. If m vanishes on some surface F then θ or θ' is zero there. Suppose without loss of generality that it is θ. Then it can be concluded as in the proof of Lemma 2.4 of [19] that θ and P vanish on any symmetric compact Cauchy hypersurface containing F. When θ is zero the other expansion θ' is given by the rate of change of r along the compact Cauchy hypersurface. Hence θ' must vanish somewhere and, repeating the previous argument, θ' and P' vanish on the whole Cauchy hypersurface. Looking at the explicit forms of P and P' shows that $\rho = 0$, $\kappa_{AB} = 0$, $\lambda_{AB} = 0$ and $\eta_A = 0$ on the Cauchy hypersurface. The vanishing of λ_{AB} implies that $\tilde{g}_{AB}$ is independent of x. It follows that a linear transformation with constant coefficients of the coordinates y^A can be used to reduce the metric $\tilde{g}_{AB}$ on a given Cauchy hypersurface to the form δ_{AB}. This, together with the vanishing of κ_{AB} and η_A, shows that the initial data are plane symmetric. The subset of spacetime where $m = 0$ is closed. Because of the possibility of deforming spacelike hypersurfaces, it is also open and must be the whole spacetime. Hence the spacetime is plane symmetric and applying Lemma 2.4 of [19] shows that it is flat. If the spacetime is not flat then it follows that θ and θ' can never vanish. If they had opposite signs then this would mean that the gradient of r was everywhere spacelike and hence that the restriction of r to a Cauchy hypersurface was strictly monotonic. This is clearly impossible, since this restriction must have a critical point somewhere. Hence θ and θ' have opposite signs, the gradient of r is timelike and the Hawking mass is positive. □

The timelike vector $\nabla_a r$ is past-pointing. For otherwise θ would be negative and θ' positive. Integrating (3.5) from 0 to 2π on a hypersurface of constant time would then imply that H was positive somewhere, contrary to what has already been assumed. It follows that r is non-decreasing to the future along any timelike curve and that its value at any point with time coordinate t_1 is bounded by above by its value on the hypersurface $t = t_2$ if $t_1 < t_2$.

Let n^a denote the unit normal to the surfaces of symmetry in the hypersurfaces t =const. and define $K = k_{ab}n^a n^b$. Then, with respect to the coordinates introduced in Lemma 2.1, some of the field equations take the following explicit forms:

$$\begin{aligned}
\partial_x^2(A^{1/2}) &= -\tfrac{1}{8}A^{5/2}[\tfrac{3}{2}(K - \tfrac{1}{3}t)^2 - \tfrac{2}{3}t^2 \\
&\quad + 2\eta_A\eta^A + \tilde\kappa^{AB}\tilde\kappa_{AB} + \tilde\lambda^{AB}\tilde\lambda_{AB} + 16\pi\rho], && (3.9)\\
\partial_x^2\alpha + A^{-1}\partial_x A\partial_x\alpha &= \alpha A^2[\tfrac{3}{2}(K - \tfrac{1}{3}t)^2 + \quad \tfrac{1}{3}t^2 \\
&\quad + 2\eta_A\eta^A + \tilde\kappa_{AB}\tilde\kappa^{AB} + 4\pi(\rho + \mathrm{tr}S)] - A^2, && (3.10)\\
\partial_x K + 3A^{-1}\partial_x AK - A^{-1}\partial_x At - \tilde\kappa^{AB}\tilde\lambda_{AB} &= 8\pi JA, && (3.11)\\
\partial_x\beta^1 &= -a^{-1}\partial_t a + \tfrac{1}{2}\alpha(3K - t), && (3.12)\\
\partial_t a &= a[-\partial_x\beta^1 + \tfrac{1}{2}\alpha(3K - t)], && (3.13)\\
\partial_t A &= -\alpha KA + \partial_x(\beta^1 A), && (3.14)
\end{aligned}$$

These equations have been written in a form which makes as clear as possible how they differ from the form they take in the special case of plane symmetric spacetimes. The differences are not very great and in particular Eqs. (3.12)–(3.14) are identical to the corresponding Eqs. (2.6)–(2.8) in [19]. In terms of these variables the expansions are given by $\theta = 2A^{-2}\partial_x A - t + K$, $\theta' = 2A^{-2}\partial_x A + t - K$.

On an interval where H is bounded the quantities θ and θ' and $K = t + \frac{1}{2}(\theta - \theta')$ are bounded. On the other hand, it follows from the lapse equation (3.10) that $\alpha \le 3/t^2$. Integrating Eq. (3.13) in space shows that on a finite time interval a and a^{-1} are bounded. Putting this back into the integrated equation shows that $\partial_t a$ is bounded. Equation (3.13) then implies that $\partial_x\beta^1$ is bounded. Equation (3.14) can be rewritten as

$$\partial_t(\log A) - \beta^1\partial_x(\log A) = -\alpha K + \partial_x\beta^1. \qquad (3.15)$$

Together with the bounds which have just been derived this implies that A and its inverse are bounded. Since $r = aA$ it follows immediately that r and its inverse are bounded. The inequalities just obtained serve as a replacement for the bound for m^{-1} obtained at the corresponding point in the argument in [19]. The argument of [19] would apparently not work in the present case because η_A makes a contribution to the equation for $\nabla_a m$ which has the wrong sign. The argument used here also has the advantage that it only requires the matter to satisfy the dominant and strong energy conditions and no assumption on the positivity of the pressure is needed. For this reason it applies to more general matter models and in particular to situations where an electromagnetic field is present.

The following theorem can now be proved:

Theorem 3.1. *Let a solution of the Einstein equations with local $U(1)\times U(1)$ symmetry be given and suppose that when coordinates are chosen which cast the metric into the form (2.5) with constant mean curvature time slices the time coordinate takes all values in the finite interval (t_1, t_2). Suppose further that:*
i) the dominant and strong energy conditions hold
ii) $t_2 < 0$
Then the following quantities are bounded on the interval (t_1, t_2):

$$\alpha, \partial_x\alpha, A, A^{-1}, \partial_x A, K, \beta^1, a, a^{-1}, \partial_t a. \tag{3.16}$$
$$\partial_t A, \partial_x\beta^1. \tag{3.17}$$

Proof. It has already been shown that α, A, A^{-1}, a, a^{-1}, K and $\partial_x\beta^1$ are bounded. The fact that θ and θ' are bounded implies that $\partial_x A$ is bounded. The boundedness of $\partial_x\beta^1$ and the fact that β^1 vanishes at one point show that β^1 is bounded. Equation (3.14) gives a bound for $\partial_t A$. Integrating equation (3.9) over the circle and using the bounds obtained already shows that $\int_0^{2\pi}\rho$ and $\int_0^{2\pi}(2\eta_A\eta^A+\tilde\kappa^{AB}\tilde\kappa_{AB}+\tilde\lambda^{AB}\tilde\lambda_{AB})$ are bounded. By the dominant energy condition it follows that $\int_0^{2\pi} j$ and $\int_0^{2\pi}\mathrm{tr}S$ are bounded. Finally, integrating (3.10) starting at a point where $\partial_x\alpha = 0$ gives a bound for $\partial_x\alpha$. □

4. Estimates for the Hyperbolic and Vlasov Equations

The field equations which are used to control W and V are hyperbolic. These quantities may be thought of as describing gravitational waves. The fact that these equations are coupled with the matter equations and are themselves nonlinear means intuitively that the waves interact with the matter and with each other. The equations will be written in terms of a 2+2 split of the metric. Here lower case Roman indices refer to objects which live on the quotient of spacetime by the symmetry group. Indices of objects of this kind are raised and lowered using the metric g_{ab} on the quotient space and its inverse. The equations are:

$$\begin{aligned}\nabla^a(r^2\nabla_a W) &= -2r^2\tanh V\nabla^a W\nabla_a V - r^2(\cosh V)^{-1}[e^{-W}T_{22} - e^W T_{33}\\ &\quad - \tfrac{1}{2}(e^{-W}(\eta_2)^2 - e^W(\eta_3)^2)],\end{aligned} \tag{4.1}$$
$$\begin{aligned}\nabla^a(r^2\nabla_a V) &= r^2\cosh V\sinh V\nabla^a W\nabla_a W - 2r^2(\cosh V)^{-1}\\ &\quad [(T_{23} - \tfrac{1}{2}\tilde h^{AB}T_{AB}\tilde g_{23}) - \tfrac{1}{2}(\eta_2\eta_3 - \tfrac{1}{2}(\tilde h^{AB}\eta_A\eta_B)\tilde g_{23})].\end{aligned} \tag{4.2}$$

The derivation of these equations is lengthy and it proved useful for this purpose to make use of the calculations of Kundu [15]. Let S_W and S_V denote the right hand sides of Eqs. (4.1) and (4.2) respectively. It will now be shown that the modulus of each of these quantities can be bounded by a constant multiple of the expression $\rho + \eta_A\eta^A + \tilde\kappa_{AB}\tilde\kappa^{AB} + \tilde\lambda_{AB}\tilde\lambda^{AB}$. It then follows from what was said in the proof of Theorem 3.1 that under the hypotheses of that theorem the L^1 norms of S_W and S_V in space are bounded by a constant which does not depend on time. For this purpose it is necessary to calculate $\tilde\lambda_{AB}$ and $\tilde\kappa_{AB}$ explicitly in terms of W and V,

$$\tilde\lambda_{AB}\tilde\lambda^{AB} = \tfrac{1}{2}A^{-2}(\cosh^2 V W_x^2 + V_x^2), \tag{4.3}$$
$$\tilde\kappa_{AB}\tilde\kappa^{AB} = \tfrac{1}{2}\alpha^{-2}[\cosh^2 V(W_t - \beta^1 W_x)^2 + (V_t - \beta^1 V_x)^2]. \tag{4.4}$$

This shows that the first term on the right-hand side of each of the Eqs. (4.1) and (4.2) can be bounded by $\tilde{\lambda}_{AB}\tilde{\lambda}^{AB} + \tilde{\kappa}_{AB}\tilde{\kappa}^{AB}$. To bound the other terms on the right hand side of (4.1) and (4.2), define an orthonormal frame on each orbit by:

$$e_2 = (Aa)^{-1}(e^{-W/2}\cosh(V/2)\partial/\partial y^2 - e^{W/2}\sinh(V/2)\partial/\partial y^3), \tag{4.5}$$

$$e_3 = (Aa)^{-1}(-e^{-W/2}\sinh(V/2)\partial/\partial y^2 + e^{W/2}\cosh(V/2)\partial/\partial y^3). \tag{4.6}$$

Then

$$e^{-W/2}\partial/\partial y^2 = Aa[\cosh(V/2)e_2 + \sinh(V/2)e_3], \tag{4.7}$$

$$e^{W/2}\partial/\partial y^3 = Aa[\sinh(V/2)e_2 + \cosh(V/2)e_3]. \tag{4.8}$$

The components of the covector η_A expressed in an orthormal frame can be bounded in terms of $\eta^A\eta_A$. Thus if the latter expression is bounded it follows that the components of η_A expressed with respect to the basis $(e^{-W/2}\partial/\partial y^2, e^{W/2}\partial/\partial y^3)$ can be bounded by a constant multiple of $\cosh(V/2)$ or, equivalently, by a constant multiple of $(\cosh V)^{1/2}$. This means that $e^{-W/2}\eta_2$ and $e^{W/2}\eta_3$ can be bounded by an expression of the form $C\eta^A\eta_A(\cosh V)^{1/2}$ for some constant C. This allows the expressions on the right-hand side of Eqs. (4.1) and (4.2) containing η_A to be bounded in modulus by a constant multiple of $\eta_A\eta^A$. The terms involving the energy-momentum tensor can be handled in a very similar way. The dominant energy condition implies that the components of the energy-momentum tensor in an orthonormal frame are bounded in modulus by ρ and using (4.7) and (4.8) allows this to be translated into a bound on the matter terms on the right-hand side of Eqs. (4.1) and (4.2) in terms of ρ.

Lemma 4.1. *Under the hypotheses of Theorem 3.1 the quantities W, V, η_A, β^A, $\partial_x\beta^A$ are bounded.*

Proof. Choose some t_3 in the interval (t_1, t_2) and let (t_4, x_4) be some point of the quotient manifold $\underline{M}$ parametrized by t and x with $t_4 < t_3$. (The case $t_4 > t_3$ is similar.) Equations (4.1) and (4.2) have the same characteristics. These are the null curves of the metric defined on $\underline{M}$. Let γ_1 and γ_2 be the two characteristics passing through (t_4, x_4) and let (t_3, x_5) and (t_3, x_6) be the coordinates of the points where they meet the hypersurface $t = t_3$. The left-hand side of Eq. (4.1) has the form of a divergence. Applying Stokes' theorem to the triangular region T bounded by γ_1, γ_2 and the curve $t = t_3$ gives the identity:

$$\int_T S_W\alpha A dtdx = \int_{t_4}^{t_3}(r^2DW/Dt)(t,\gamma_1(t))dt + \int_{t_4}^{t_3}(r^2DW/Dt)(t,\gamma_2(t))dt - \int_{x_5}^{x_6} r^2(W_t - \beta_1 W_x)(t_3,x)Adx ,$$

and hence, after integration by parts:

$$\begin{aligned}(r^2W)(t_4,x_4) &= \tfrac{1}{2}[(r^2W)(t_3,x_5) + (r^2W)(t_3,x_6)] \\ &\quad - \tfrac{1}{2}\int_{t_4}^{t_3}(2rWDr/Dt)(t,\gamma_1(t)) - \tfrac{1}{2}\int_{t_4}^{t_3}(2rWDr/Dt)(t,\gamma_2(t)) \\ &\quad - \tfrac{1}{2}\int_{x_5}^{x_6} r^2(W_t-\beta_1W_x)(t_3,x)Adx - \tfrac{1}{2}\int_T S_W\alpha A dtdx.\end{aligned} \tag{4.9}$$

Here D/Dt denotes a derivative in the direction of the characteristic along which the integration is carried out, with this characteristic being parametrized with respect to t. In other words, for any function f, $Df/Dt = d/dt(f(\gamma(t)))$. Most of the quantities in (4.9) are already known to be bounded. This is in particular true of Dr/Dt. Thus the following inequality holds:

$$\begin{aligned}&\|r^2W(t_3-t)\|_\infty\\ &\leq C\left[1+\int_0^t\left(\|r^2W(t_3-s)\|_\infty+\int_{\gamma_1(t_3-s)}^{\gamma_2(t_3-s)}|S_W(s,x)|dx\right)ds\right]\end{aligned} \tag{4.10}$$

Since $\int_0^{2\pi}|S_W(t,x)|dx$ is known to be bounded (as shown in the discussion preceding this lemma) and under the given hypotheses the number of times the characteristics can go around the circle between $t = t_1$ and $t = t_3$ is bounded, it follows from Gronwall's inequality that W is bounded. The same kind of argument shows that V is bounded. The remaining conclusions of the lemma are simple consequences of the boundedness of W and V, as will now be shown. The momentum constraint implies that:

$$\partial_x(A^2\eta_A) = 8\pi A^2 j_A, \tag{4.11}$$

which means that $(A^2\eta_A)(t, x_1) - (A^2\eta_A)(t, x_2)$ is bounded independently of t, x_1 and x_2. On the other hand the boundedness of $\int_0^{2\pi}\eta^A\eta_A(x)dx$ together with that of V and W shows that $\int_0^{2\pi}|A^2\eta_A(x)|dx$ is bounded. These two facts together show that η_A is bounded. The definition of the second fundamental form gives the equation:

$$\tilde{g}_{AB}\partial_x\beta^B = 2\alpha A^{-1}a^{-2}\eta_A. \tag{4.12}$$

This means that $\partial_x\beta^A$ is bounded and, remembering that by definition $\beta^A(0) = 0$, this implies that β^A is bounded. This completes the proof. □

Everything which has been done up to now consists of obtaining bounds for parts of the geometry using nothing about the matter model except the dominant and strong energy conditions. Now the special case of the Vlasov equation will be considered. In this class of spacetimes the Vlasov equation for particles of unit mass takes the following form:

$$\partial f/\partial t + (\alpha A^{-1}(v^1/v^0) - \beta^1)\partial f/\partial x + F^i\partial f/dv^i = 0. \tag{4.13}$$

Here the quantities F^i are functions of t, the quantities listed in (3.16) and (3.17), β^A and their spatial derivatives, η_A and the first derivatives of W and V with respect to t and x. They depend linearly on the derivatives of W and V. The mass shell condition $v^0 = \sqrt{1 + \delta_{ij}v^iv^j}$ defines v^0 in terms of v^i. The characteristics of Eq. (4.13) satisfy the system:

$$\begin{aligned}dx^i/ds &= [\alpha A^{-1}(v^1/v^0) - \beta^1]\delta_1^i,\\ dv^i/ds &= F^i.\end{aligned} \tag{4.14}$$

The spacetimes considered here have two local Killing vectors. If k^α is a Killing vector in any spacetime and p^α the unit tangent vector to a timelike geodesic, then the quantity $p^\alpha k_\alpha$ is conserved along the geodesic. This allows two conserved quantities for the Eqs. (4.14) to be derived. They can be computed, using (4.7) and (4.8) to be:

$$\begin{aligned}&Aae^{W/2}[\cosh(V/2)v^2 + \sinh(V/2)v^3],\\ &Aae^{-W/2}[\sinh(V/2)v^2 + \cosh(V/2)v^3].\end{aligned} \tag{4.15}$$

It is easy to solve for v^2 and v^3 in terms of these two conserved quantities and the boundedness of W and V implies that v^2 and v^3 are bounded along a characteristic. Consider now a solution of the Einstein-Vlasov system where the initial datum for the distribution function has compact support. Let $P(t)$ be the supremum of $|v|$ over the support of $f(t)$. Since a, A, W and V have already been controlled pointwise the components T_{AB} of the energy-momentum tensor occurring on the right hand side of (4.1) and (4.2) can be estimated in terms of the corresponding frame components. Looking at the explicit expressions for these frame components and using the boundedness of v^2 and v^3 in the support of f shows that:

$$\|T_{AB}(t)\|_\infty \leq CP(t), \tag{4.16}$$

where C is a constant which only depends on the initial data. To make use of (4.16) an estimate for v^1 must be obtained. Define:

$$Q(t) = \|\partial_x W(t)\|_\infty + \|\partial_t W(t)\|_\infty + \|\partial_x V(t)\|_\infty + \|\partial_t V(t)\|_\infty. \tag{4.17}$$

Lemma 4.2. *If the hypotheses of Theorem 3.1 are satisfied by a solution of the Einstein-Vlasov system then the following inequality holds for $t_4 < t_3$:*

$$1 + P(t_4) \leq C\left(1 + P(t_3) + \int_0^{t_3-t_4} 1 + P(t_3 - t) + Q(t_3 - t)dt\right). \tag{4.18}$$

The analogous inequality holds for $t_4 > t_3$.

Proof. In a 3+1 decomposition of a general spacetime the Vlasov equation takes the following form when expressed in terms of frame components:

$$\begin{aligned}&\partial f/\partial t + (\alpha v^i/v^0 e_i^a - \beta^a)\partial f/\partial x^a \\ &\quad - [e_i(\alpha)v^0 + \alpha(-k_{ab}e_i^a e_j^b + \gamma^i_{0j})v^j + \alpha\gamma^i_{jk}v^j v^k/v^0]\partial f/\partial v^i = 0.\end{aligned} \tag{4.19}$$

Here γ^i_{0j} and γ^i_{jk} are Ricci rotation coefficients. Consider the terms appearing in F^1 in the case of the symmetry considered here which contain derivatives of V and W. No such terms arise from the terms in (4.19) involving the derivatives of α and the second fundamental form. To go further it is necessary to have more explicit expressions for the rotation coefficients. The four-dimensional rotation coefficients γ^i_{jk} are identical with corresponding three-dimensional ones while:

$$\gamma^i_{0j} = -\alpha^{-1}\gamma^i_{kj}\theta^k_a\beta^a + \tfrac{1}{2}\alpha^{-1}(e^a_j\nabla_a\beta^b\theta^i_b - \delta^{is}e^a_s\nabla_a\beta^b\theta^t_b\delta_{jt} + c^i_j - \delta^{is}c^t_s\delta_{jt}). \tag{4.20}$$

Here $c^i_j = e^a_j\partial_t\theta^i_a$. Each term in the expressions for the coefficients of the Vlasov equation is either independent of the derivatives of W and V, in which case it is bounded as a consequence of the estimates already proved, or it is linear in these derivatives. Consider now the equation for v^1 in the characteristic system. The estimate (4.18) is obtained by considering the dependence on v of those terms which are linear in the derivatives of W and V. The quantity v^i is bounded unless $i = 1$ while the quantity $v^j v^k/v^0$ is bounded unless $j = k = 1$. However, by the symmetry properties of the rotation coefficients, $\gamma^1_{j1} = 0$. Hence all terms on the right hand side of the equation for v^1 can be bounded by an expression of the form $C(1 + P + Q)$. □

Lemma 4.3. *If the hypotheses of Theorem 3.1 are satisfied by a solution of the Einstein-Vlasov system then the quantities P, $\partial_t W$, $\partial_t V$ $\partial_x W$, $\partial_x V$, ρ, α^{-1}, the derivative with respect to x of all the quantities in (3.16) and (3.17), $\partial_x\eta_A$ and $\partial_x^2\beta^A$ are bounded.*

Proof. The first step is to obtain an estimate for the first derivatives of W and V. In order to do this, it is useful to write Eqs. (4.1) and (4.2) in a slightly different way:

$$\nabla^a\nabla_a W + 2\tanh V\nabla^a W\nabla_a V = -(2/r)\nabla^a r\nabla_a W - (\cosh V)^{-1}[e^{-W}T_{22} - e^W T_{33} - \tfrac{1}{2}(e^{-W}(\eta_2)^2 - e^W(\eta_3)^2)], \tag{4.21}$$

$$\nabla^a\nabla_a V - \sinh V\cosh V\nabla^a W\nabla_a W = -(2/r)\nabla^a r\nabla_a V - 2(\cosh V)^{-1} [(T_{23} - \tfrac{1}{2}\tilde{h}^{AB}T_{AB}\tilde{g}_{23}) - \tfrac{1}{2}(\eta_2\eta_3 - \tfrac{1}{2}(\tilde{h}^{AB}\eta_A\eta_B)\tilde{g}_{23})]. \tag{4.22}$$

The advantage of this is that if the right hand sides of (4.21) and (4.22) are replaced by zero the resulting equations are those for a wave map (hyperbolic harmonic map) with target space $\mathbf{R}^2$, endowed with the metric $\cosh^2 V dW^2 + dV^2$. This is a representation of the standard metric of the hyperbolic plane in a certain coordinate system. It is natural to try to generalize estimates which have been used in the study of wave maps to the present situation. Here this is done with an estimate of Gu [13], who used it to prove global existence of classical solutions in the Cauchy problem for wave maps defined on two-dimensional Minkowski space. Define two null vectors on the two-dimensional space coordinatized by t and r by

$$\begin{aligned} e_+ &= \alpha^{-1}(\partial/\partial t - \beta\partial/\partial x) + A^{-1}\partial/\partial x,\\ e_- &= \alpha^{-1}(\partial/\partial t - \beta\partial/\partial x) - A^{-1}\partial/\partial x. \end{aligned} \tag{4.23}$$

The (2-dimensional) covariant derivatives $\nabla_{e_-}e_+$ and $\nabla_{e_+}e_-$ are given by:

$$\begin{aligned} \nabla_{e_-}e_+ &= \alpha^{-1}(b_{++}e_+ + b_{+-}e_-),\\ \nabla_{e_+}e_- &= \alpha^{-1}(b_{-+}e_- + b_{--}e_-) \end{aligned} \tag{4.24}$$

for some bounded functions b_{++}, b_{+-}, b_{-+} and b_{--}. The normalization chosen for the vectors e_+ and e_- here is important, since otherwise the covariant derivatives could contain the time derivatives of α and β^1, quantities which have not yet been shown to be bounded. Let E_+ and E_- be the images of e_+ and e_- under the wave map, i.e.

$$\begin{aligned} E_+ &= e_+(W)\partial/\partial W + e_+(V)\partial/\partial V,\\ E_- &= e_-(W)\partial/\partial W + e_-(V)\partial/\partial V. \end{aligned} \tag{4.25}$$

Let γ_1 and γ_2 be integral curves of e_- and e_+ respectively and let $\hat{\gamma}_i = (W\circ\gamma_i, V\circ\gamma_i)$. The observation of Gu is that the equations obtained from (4.21) and (4.22) by replacing the right hand side by zero and the given metric by the flat metric just say that E_+ is parallelly transported along $\hat{\gamma}_1$ and that E_- is parallelly transported along $\hat{\gamma}_2$. A similar calculation can be done for Eqs. (4.21) and (4.22) and this gives rise to the following equation along $\hat{\gamma}_1$ (and an analogous equation along $\hat{\gamma}_2$):

$$\nabla_{\alpha E_-}E_+ = (b_{++} - r^{-1}\alpha e_-(r))E_+ + (b_{+-} - r^{-1}\alpha e_+(r))E_- + B_-, \tag{4.26}$$

where B_- satisfies an inequality of the form $|B_-| \le C(1 + \|T_{AB}\|_\infty)$. These equations allow the lengths of the vectors E_+ and E_- to be controlled. Multiplying (4.26) and the analogous equation for E_+ by α allows the following inequality to be derived:

$$Q(t_4) \le C[Q(t_3) + \int_0^{t_3-t_4} 1 + Q(t_3 - t) + \|T_{AB}(t_3 - t)\|_\infty dt]. \tag{4.27}$$

Putting together (4.16), (4.18) and (4.27) gives:

$$(1+P+Q)(t_4) \le (1+P+Q)(t_3) + C\int_0^{t_3-t_4} (1+P+Q)(t_3-t)dt. \qquad (4.28)$$

Hence by Gronwall's lemma P, $\partial_t W$, $\partial_t V$, $\partial_x W$ and $\partial_x V$ are bounded. It then follows immediately that ρ is bounded and (3.10) shows that α^{-1} is bounded. The equations (3.9)–(3.14) can be used directly to show that the first derivatives with respect to x of all the quantities in (3.16) and (3.17) are bounded. It follows from (4.11) that $\partial_x\eta_A$ is bounded and from (4.12) that $\partial_x^2\beta^A$ is bounded. □

Lemma 4.4. *If the hypotheses of Theorem 3.1 are satisfied by a solution of the Einstein-Vlasov system then the second derivatives of W and V and the first derivatives of f are bounded.*

Proof. If f were zero (the vacuum case) then it would be rather simple to prove this theorem, since the equations obtained by differentiating the equations for W and V with respect to x are linear in the highest order derivatives in that case. With the coupling to f things are less straightforward. When the Vlasov equation is differentiated with respect to x terms come up which involve second derivatives of W and V multiplied by first derivatives of f. In other words, there are terms which are quadratic in the quantities to be estimated and this precludes a direct application of Gronwall's inequality. This problem can be solved using a device of Glassey and Strauss [12], which can be seen in a particularly simple form, adequate for the present application, in the paper [10] of Glassey and Schaeffer (see also [11]). The equation for W can be written in the following form:

$$l^a\nabla_a(n^b\nabla_b W) = (Y_1(W,V)l^a + Y_2(W,V)n^a)\nabla_a W + Z(W,V), \qquad (4.29)$$

where $Z(W,V)$ contains no derivatives of W or V and $Y_1(W,V)$ and $Y_2(W,V)$ contain them at most linearly. Here, for ease of notation, $l = e_+$ and $n = e_-$. The equation for V can of course be written in a similar form. There are also alternative forms of both of these equations where the roles of l and n are interchanged. Differentiating Eq. (4.29) with respect to x gives an equation of the form

$$l^a\nabla_a(\partial_x(n^b\nabla_b W)) = (Y_1(W,V)l^a + Y_2(W,V)n^a)\partial_x(\nabla^a W) + \tilde{Z}(W,V), \qquad (4.30)$$

where the expression $\tilde{Z}(W,V)$ does not depend on second derivatives of W and V. Suppose now that we integrate Eq. (4.30) along the characteristic which is an integral curve of l^a. The only terms which cannot be bounded straightforwardly (even before integration) are those which contain derivatives of the energy momentum tensor with respect to x. It will now be shown how a typical term of this type can be handled. The others which occur can be taken care of in a strictly analogous way.

The term which is to be bounded is:

$$\int_0^{t_3-t_4} [(e^W \cosh V)^{-1}\partial_x T_{22}](t_3-t)dt. \qquad (4.31)$$

In fact the coordinate components of the energy-momentum tensor may be replaced by frame components at this stage since their spatial derivatives only differ by terms which are bounded. Substituting the definition of the frame component $T(e_2,e_2)$ into the expression of interest gives

$$\int_0^{t_3-t_4}\int [(e^W \cosh V)^{-1}(v_2)^2(1+|v|^2)^{-1}]\partial_x f dv dt. \tag{4.32}$$

The idea of [12] is to express ∂_x as a linear combination of l and the vector

$$m = \partial/\partial t + (\alpha A^{-1}(v^1/v^0) - \beta^1)\partial/\partial x. \tag{4.33}$$

The result is:

$$\partial/\partial x = \alpha^{-1}A(1 - v^1/v^0)^{-1}(l - m). \tag{4.34}$$

This allows the integral in (4.32) to be rewritten as a sum of two terms, one containing l and the other containing m. Now it is possible to substitute for mf using the Vlasov equation and the result contains only derivatives of f with respect to the velocity variables. These derivatives can be eliminated by an integration by parts in v and the result is a bounded quantity. The other term is equal to

$$\begin{aligned}&\int\int_0^{t_3-t_4}[\alpha^{-1}A(e^W \cosh V)^{-1}(v_2)^2(1+|v|^2)^{-1}l^a\nabla_a f](\gamma_1(t_3-t))dt(1-v^1/v^0)^{-1}dv\\ &= \int [(\alpha^{-1}A(e^W \cosh V)^{-1})(t_4,x_4)f(t_4,x_4,v)\\ &- (\alpha^{-1}A(e^W \cosh V)^{-1})(t_3,x_5)f(t_3,x_5,v)](v_2)^2(1+|v|^2)^{-1}(1-v^1/v^0)^{-1}dv + \ldots.\end{aligned} \tag{4.35}$$

This is obtained by integrating by parts in t along the characteristic. Only the boundary terms are written explicitly. The other terms are integrals in t of bounded quantities. Thus the term involving lf coming from (4.32) is also bounded. The same trick can be applied when W is replaced by V and when l and n are interchanged. (In the last case $\partial/\partial x$ must be replaced by a combination of m and n.) The result of all this is that if

$$Q_1 = \|\partial_x(l^a\nabla_a W)\|_\infty + \|\partial_x(n^a\nabla_a W)\|_\infty + \|(l^a\nabla_a V)\|_\infty + \|(n^a\nabla_a V)\|_\infty \tag{4.36}$$

then an estimate of the form:

$$1 + Q_1(t_4) \le 1 + Q_1(t_3) + C\int_0^{t_3-t_4}(1 + Q_1(t_3 - t))dt \tag{4.37}$$

is obtained. It follows from Gronwall's inequality that Q_1 is bounded. Hence the derivatives W_{xx}, W_{tx}, V_{xx} and V_{tx} are bounded. Using this information in the equations obtained by differentiating the Vlasov equation with respect to x or v shows that the first derivatives of f with respect to these variables are bounded. □

5. The Main Result

In this section the estimates collected in Sect. 4 are applied to prove the main result. First one last auxiliary lemma is required.

Lemma 5.1. *Consider a CMC initial data set for the Einstein-Vlasov system with local $U(1)\times U(1)$ symmetry. Then there exists a local Cauchy evolution of this data which has local $U(1)\times U(1)$ symmetry, so that the hypotheses of Lemma 2.1 are satisfied. Consider next a family of initial data sets of this type on the same manifold such that:*
(i) the data in the family are uniformly bounded in the C^∞ topology,
(ii) the metrics are uniformly positive definite,

(iii) the supports of the distribution functions are contained in a common compact set,
(iv) the mean curvatures are uniformly bounded away from zero.
Then the time interval in the conclusion of Lemma 2.1 can be chosen uniformly for the Cauchy evolutions of all data in the family.

Proof. The first statement of the proof is essentially a direct consequence of the standard local existence theorem for the Einstein-Vlasov system and for CMC hypersurfaces and the fact that the resulting spacetimes inherit any symmetry which is present. When there is only local symmetry the inheritance of symmetry argument should be applied to the universal cover (cf. [19]). The second part of the lemma, concerning families is a consequence of the stability of various operations. Firstly, the statement is used that the time of existence of a solution of the Einstein-Vlasov system, measured with respect to an appropriate time coordinate, depends only on the size of the initial data and that on a fixed closed time interval the solution depends continuously on the initial data. Secondly, the fact is used that the interval on which a CMC foliation exists in a neighbourhood of a given CMC hypersurface depends only on the size of the metric coefficients in an appropriate coordinate system and a positive lower bound for the lapse function, provided the mean curvature of the starting hypersurface remains bounded away from zero. □

Theorem 5.1. *Let (M, g, f) be a C^∞ solution of the Einstein-Vlasov system with local $U(1) \times U(1)$ symmetry which is the maximal globally hyperbolic development of data on a symmetric hypersurface of constant mean curvature $H_0 < 0$. Then the part of the spacetime to the past of the initial hypersurface can be covered by a foliation of CMC hypersurfaces with the mean curvature taking all values in the interval $(-\infty, H_0]$. Moreover, the CMC foliation can be extended to the future of the initial hypersurface in such a way that the mean curvature attains all negative real values.*

Proof. Let T be the largest number (possibly infinite) such that the local foliation by CMC hypersurfaces which exists near the initial hypersurface can be extended so that the mean curvature takes on all values in the interval $(-T, H_0)$. Suppose that T is finite. Then Theorem 3.1 and the results of Sect. 4 imply the boundedness of many quantities on the interval $(-T, H_0]$. It will now be shown by induction that the spatial derivatives of all orders of all quantities of interest are bounded on the given interval. The inductive hypothesis is that the following quantities are bounded:

$$\begin{aligned} &D^n f, D^{n+1}W, D^n(\partial_t W), D^{n+1}V, D^n(\partial_t V), D^{n+1}\alpha, \\ &D^{n+1}\beta^1, D^{n+1}A, D^n(\partial_t A), D^n K, D^n \eta_A, D^{n+1}\beta^A. \end{aligned} \tag{5.1}$$

It follows from the results of Sect. 4, and in particular Lemma 4.4, that the inductive hypothesis is satisfied for $n = 1$. Suppose now that it is satisfied for a given value of n. Then it follows immediately from the field Eqs. (3.9)–(3.14) and (4.11)–(4.12) that all quantities which are required to be bounded by the inductive hypothesis at the next step are bounded, except possibly for the relevant derivatives of f, W and V. Consider the equation obtained by differentiating the Vlasov equation $n + 1$ times with respect to x. There results a linear equation for $D^{n+1}f$ with coefficients which are known to be bounded, except for terms involving derivatives of W and V in the inhomogeneous term. If $F_n(t) = \|D^n f\|_\infty$ and

$$Q_n(t) = \|D^{n+1}W\|_\infty + \|D^{n+1}V\|_\infty + \|D^n(\partial_t W)\|_\infty + \|D^n(\partial_t V)\|_\infty,$$

then this equation implies an inequality of the form

$$F_{n+1}(t) \leq F_{n+1}(t_0) + C\int_0^{t_0-t} F_{n+1}(t_0-s) + Q_{n+1}(t_0-s)ds. \qquad (5.2)$$

Similarly, differentiating Eqs. (4.29) and (4.30) n times with respect to x gives a linear system of equations for derivatives of V and W with coefficients which are known to be bounded, except for terms involving derivatives of order $n+1$ of matter quantities in the inhomogeneous term. Hence:

$$G_{n+1}(t) \leq G_{n+1}(0) + C\int_0^{t_0-t} F_{n+1}(t_0-s) + G_{n+1}(t_0-s)ds. \qquad (5.3)$$

Putting together (5.2) and (5.3) and applying Gronwall's inequality proves that F_{n+1} and G_{n+1} are bounded and completes the inductive step. Thus the quantities in (5.1) are bounded for all n. Consider now the data obtained by restricting the given solution to the hypersurfaces t =const. By what has just been proved, this family of data satisfies the conditions of Lemma 5.1. Hence there exists some $\epsilon > 0$ such that each of these initial data has a corresponding solution on a time interval of length ϵ. Hence the original solution extends to the interval $(-T-\epsilon, H_0)$, contradicting the maximality of T. It follows that in fact $T = \infty$, as desired. This means in particular that the spacetime has a crushing singularity in the past, and hence that the CMC foliation covers the entire past of the initial hypersurface.

Now let T' be the largest number such that the CMC foliation can be extended to the interval $(-\infty, T')$. Since the spacetime contains no compact maximal hypersurface $T' \leq 0$. If T' were strictly less than zero it could be argued as in the first part of the proof that the CMC foliation could be extended further, which would contradict the definition of T'. Hence in fact $T' = 0$. □

This argument does not prove that the entire future of the initial hypersurface is covered by the CMC foliation. In connection with this it is interesting to note that if instead of assuming, as is done in this paper, that the cosmological constant Λ vanishes, it is assumed that $\Lambda < 0$, then the same types of arguments apply to give a stronger theorem. (The choice of sign convention for the cosmological constant used here is such that $\Lambda < 0$ corresponds to anti-de Sitter space.) With $\Lambda < 0$ the result is that the whole spacetime can be covered by a CMC foliation with the mean curvature taking on all real values. The reason for this difference can be traced to the estimate for α following from the lapse equation, which in general reads $\alpha \leq (\frac{1}{3}t^2 - \Lambda)^{-1}$.

6. The Case of Wave Maps

In this section we consider what happens when the collisionless matter described by the Vlasov equation is replaced by a wave map as source in the Einstein equations. This is quite natural, given that, as was seen in Sect. 4, a wave map comes up automatically in the case of vacuum spacetimes. Let (N, h) be a complete Riemannian manifold. If (M, g) is a Lorentz manifold a wave map ϕ from M to N is a map which satisfies the equation whose expression in local coordinates x^α on M and y^I on N is:

$$\nabla_\alpha \nabla^\alpha \phi^I + \Gamma^I_{JK} \nabla_\alpha \phi^J \nabla^\alpha \phi^K = 0. \qquad (6.1)$$

(Wave maps are also known as (hyperbolic) harmonic maps or nonlinear sigma models.) The global Cauchy problem for wave maps on two-dimensional Minkowski space was solved by Gu [13] and for wave maps on three-dimensional Minkowski space which are invariant or equivariant under rotations by Christodoulou, Shatah and Tahvildar-Zadeh [3, 4, 21]. The results of [4] were applied to the Einstein-Maxwell equations in [2]. Associated to a wave map ϕ is the energy-momentum tensor:

$$T_{\alpha\beta} = [\nabla_\alpha\phi^I\nabla_\beta\phi^J - \tfrac{1}{2}(\nabla_\gamma\phi^I\nabla^\gamma\phi^J)g_{\alpha\beta}]h_{IJ}, \tag{6.2}$$

and this can be used to couple the wave map to the Einstein equations. In harmonic coordinates the coupled equations form a system of nonlinear wave equations and so a local existence and uniqueness theorem can be proved by the usual methods. The energy-momentum tensor of a wave map satisfies both the dominant and strong energy conditions. This can be seen by noting that both these conditions are purely algebraic in nature and can be checked using normal coordinates based at a given point of N. Then the energy-momentum is reduced at a point to a sum of terms, each of which is the energy-momentum tensor of a massless scalar field.

Consider now the case of a solution of the Einstein equations with local $U(1)\times U(1)$ symmetry coupled to an invariant wave map. To say that the wave map is invariant means that each surface of symmetry is mapped to a single point of N. Since the relevant energy conditions hold, it follows that the analogues of the results obtained in Sects. 2 and 3 for the Einstein-Vlasov system are also valid for the Einstein-wave map system. Given that in proving Theorem 5.1 a wave map was already estimated, albeit for a special target manifold (N, h), it appears straightforward to generalize that theorem to the case of the Einstein-wave map system. In fact the equation of motion for the wave map does not involve W, V or η_A while the combinations of matter terms occurring in (4.1) and (4.2) vanish identically for the energy-momentum tensor of an invariant wave map. Thus there is no direct coupling between the wave map describing the matter and the wave-map-like equation satisfied by W and V. The one difficulty which occurs is that, in contrast to the special case of the hyperbolic plane, there is no global coordinate system on N in the general case. The equation for an invariant wave map can be written in the form:

$$\nabla_a(r^2\nabla^a\phi^I) + r^2\Gamma^I_{JK}\nabla_a\phi^J\nabla^a\phi^K = 0. \tag{6.3}$$

This bears a strong resemblance to Eqs. (4.1)–(4.2), with the difference that there are no terms involving η or matter quantities in (6.3). This makes the analogue of the calculation (4.9) for the wave map superfluous. This is just as well, since it seems difficult to formulate an analogue of (4.9) in the case that there is no global coordinate system on N. What can be done instead is to go directly to the analogue of (4.26) for the wave map. Define:

$$\begin{aligned} \tilde{E}_+ &= e_+(\phi^I)\partial/\partial\phi^I, \\ \tilde{E}_- &= e_-(\phi^I)\partial/\partial\phi^I. \end{aligned} \tag{6.4}$$

Then $\tilde{E}_+$ and $\tilde{E}_-$ satisfy propagation equations like (4.26) along $\hat{\gamma}_1$ and $\hat{\gamma}_2$ respectively. There is no term corresponding to B_- in this case. It follows that under the hypotheses of Theorem 3.1 the length of the vectors $\tilde{E}_+$ and $\tilde{E}_-$ is bounded on the given time interval. This implies a bound on the distance of any point of the image under ϕ of this time interval from the image of the initial hypersurface. In particular, the image of this time interval under ϕ is contained in a compact subset of N. This compact set can be covered by a finite number of charts, each of which can be chosen to be defined on a domain with compact closure in a larger chart domain. In each of these charts the quantities ϕ^I, $\partial_t\phi^I$ and $\partial_x\phi^I$ are bounded. Moreover, in any of these charts the following analogue of (4.30) holds:

$$l^a\nabla_a(\partial_x(n^b\nabla_b\phi^I)) = (Y_1(\phi^I)l^a + Y_2(\phi^I)n^a)\partial_x(\nabla^a\phi^I) + \tilde{Z}(\phi^I). \tag{6.5}$$

This equation and the equations obtained by differentiating it repeatedly with respect to x can be used to inductively bound all spatial derivatives of ϕ^I. This proceeds essentially as in the proof of Theorem 5.1; it is merely necessary to be careful about the different charts which occur. In the case of a wave map define $F_n(t)$ to be the maximum over the finite set of charts chosen of $\|D^{n+1}\phi^I\|_\infty$ and $\|D^n(\partial_t\phi^I)\|_\infty$. When the derivatives of lower orders are known to be bounded, this is equivalent to choosing for each point one chart which contains its image and only taking the supremum over those values. When the quantity $F_n(t)$ is bounded the derivatives of order n of the frame components of the energy-momentum tensor are bounded. In order to get an inequality which can be used to control $F_n(t)$, we would like to integrate a derivative of (6.5) along a characteristic (integral curve of l or n). The image of this characteristic under ϕ need not be contained in a single chart. Consider such a characteristic γ, parametrized by t from $t = 0$ to $t = T$. For each $t \in [0, T]$ there exists an interval I, open in $[0, T]$, whose image under $\phi \circ \gamma$ is contained in one of the chosen charts on N. By compactness of $[0, T]$, finitely many of these intervals cover it. It follows that there is a finite sequence of times $\{0 = t_1, t_2, \ldots, t_k\} = T$ such that $\phi\circ\gamma([t_i, t_{i+1}])$ is contained in one of the chosen charts for all i between 1 and $k - 1$. It follows from (6.5) that

$$F_n(t_k) \le F_n(t_{k-1})e^{C(t_k-t_{k-1})}. \tag{6.6}$$

This is enough to allow F_n to be bounded for all $t \in [0, T]$. Thus the following analogue of Theorem 5.1 is obtained:

Theorem 6.1. *Let (N, h) be a complete Riemannian manifold and let (M, g, ϕ) be a C^∞ solution of the Einstein equations with local $U(1) \times U(1)$ symmetry coupled to an invariant wave map with target space (N, h) which is the maximal globally hyperbolic development of data on a symmetric hypersurface of constant mean curvature $H_0 < 0$. Then the part of the spacetime to the past of the initial hypersurface can be covered by a foliation of CMC hypersurfaces with the mean curvature taking all values in the interval $(-\infty, H_0]$. Moreover, the CMC foliation can be extended to the future of the initial hypersurface in such a way that the mean curvature attains all negative real values.*

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