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M-theory and Seiberg–Witten curves: orthogonal and symplectic groups

A. Brandhuber [a], J. Sonnenschein [a,1], S. Theisen [b,1], S. Yankielowicz [a,1]

[a] *School of Physics and Astronomy, Beverly and Raymond-Sackler Faculty of Exact Sciences, Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel*

[b] *Sektion Physik, Universität München, Theresienstraße 37, 80333 München, Germany*

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Abstract

We discuss $N = 2$ supersymmetric type IIA brane configurations within M-theory. This is a generalization of the work of Witten to all classical groups. © 1997 Published by Elsevier Science B.V.

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1. Introduction

Our understanding of supersymmetric field theories was greatly advanced by the seminal work of Seiberg and Witten [1]. Very early on it has been suggested that there should also be important ramifications for string theory which have then, in due course, been found and worked out [2–4]. The relation between string theory and supersymmetric field theory has become most transparent in the work of Hanany and Witten [5] who turned the brane technology into an efficient and easy tool to engineer supersymmetric field theories. Various dualities could be demonstrated this way; they are very natural in the brane picture [6–11]; see also Refs. [4,12,13] for an alternative brane picture of field theory dualities.

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One of the great surprises of the original work of Seiberg and Witten was that the low energy effective action of asymptotically free $N = 2$ supersymmetric field theories could be exactly computed. In this computation, which is possible due to the holomorphic structure of the Lagrangian, an auxiliary Riemann surface appears, whose period matrix is identified with the gauge couplings of the theory in its Coulomb phase. It was subsequently shown how this Riemann surface appears geometrically in string compactification on Calabi–Yau manifolds as a supersymmetric two-cycle around which the type IIA two-brane wraps, leaving a supersymmetric point-particle in uncompactified space-time [2,14,15]; for excellent recent reviews, see Refs. [16,17].

Recently, Witten [18] has shown how the Riemann surfaces naturally appear as supersymmetric cycles in the M-theory context, by reading the $N = 2$ supersymmetric brane configurations on the type IIA theory as one convoluted M-theory five-brane, whose internal part, which extends into the eleventh dimension, is a Riemann surface, holomorphically embedded into $\mathbb{R}^3 \times \mathbf{S}^1$. $\mathbb{R}^3$ are the three internal dimensions tangential to the configuration of Dirichlet four-branes (D4-branes) and NS five-branes, and $\mathbf{S}^1$ is the circle on which the eleventh dimension is compactified. Independently, Evans et al. [7] also noticed the connection between M-theory, type IIA brane configurations and Seiberg–Witten curves; their starting point was the generalization of the work of Ref. [6] to orthogonal and symplectic groups, with the orientifold plane playing a crucial role. Witten's analysis was restricted to the case of A_r gauge groups. We present here the extension of his results to the other classical groups.

Orthogonal and symplectic groups have previously been discussed in the brane context [7,10,12,13]. Here the appearance of an orientifold plane complicates the discussion. Our goal is to understand these theories in M-theory. One line of attack would be to argue for the brane configuration, which must of course respect the orientifold symmetries, and then write down the equation for the Riemann surface which should then agree with the hyperelliptic curves which have been constructed purely from field theory considerations. We have however chosen the reverse strategy, namely starting from the known curves, we infer the brane configurations. Here the curves which have been discussed in relation to Seiberg–Witten theory with integrable systems are the most appropriate. We find that the curve displays the orientifold plane only indirectly, namely via the symmetry of the brane configuration. Since in M-theory the brane configuration is smooth, there is no rationale for the jump in RR charge of the orientifold four-plane (O4 plane) as it crosses an NS five-brane. This was necessary in Ref. [7] to explain the symplectic flavor symmetry of the dual orthogonal gauge theory. In addition, in order to get a smooth transition from the electric to the magnetic theory, the authors of Ref. [10] had to assume that two of the D4-branes which extend between the two NS five-branes, must position themselves on the O4 plane. In the M-theory picture we find two infinite D4-branes which account for both of these phenomena.

An outline of this paper is as follows. In Section 2 we present our interpretation of the Seiberg–Witten curves in M- and type IIA theory. We also comment on the interpretation of the orientifold plane. In this section matter always comes, in the type II language, from D4-branes. D6-branes enter the stage in Section 3. We end with conclusions and an outlook.

While we were completing this manuscript, a preprint by Landsteiner et al. [26] appeared, which has a substantial overlap with our work.

2. Models with SO and Sp gauge groups

2.1. Generalities

We will use the same conventions as Ref. [18]. The classical brane configuration in the type IIA theory consists of infinite solitonic five-branes with the world-volume extending in the $x^0, x^1, x^2, x^3, x^4, x^5$ directions and Dirichlet four-branes with the world-volume along x^0, x^1, x^2, x^3, x^6. In Section 3 we will also introduce D6-branes with the world-volume along $(x^0, x^1, x^2, x^3, x^7, x^8, x^9)$. All brane configurations considered preserve $\frac{1}{4}$ of the 32 supercharges of the type IIA theory. As in Ref. [18] we have effectively a four-dimensional $N = 2$ supersymmetric theory on the world-volume of the D4-branes. We define

$$v = x^4 + ix^5, \tag{1}$$

$$s = (x^6 + ix^{10})/R, \tag{2}$$

where x^{10} is a periodic coordinate $x^{10} \sim x^{10} + 2\pi R$. It is convenient to make a transformation from the cylinder with coordinate s to the complex plane $t = \exp(-s)$.

In M-theory (compactified in the x^{10} direction on a circle of radius R) the configuration is described by a single five-brane with a complicated world-volume history,

$$\mathbb{R}^{3,1} \times \Sigma. \tag{3}$$

Here Σ is a (non-compact) Riemann surface holomorphically embedded in the complex two-plane parametrized by v and t. Σ is a supersymmetric cycle in the sense of Ref. [19]. Since in the type IIA theory we deal with infinitely extended branes, the Riemann surface which appears in M-theory is non-compact. Here we always refer to the compactified surface; see Ref. [18] for an explanation. We will now discuss the case of single B_r, C_r and D_r group factors, followed by the discussion of multiple group factors. The starting point will be the spectral curves associated with the various simple groups, as given in Refs. [20,21].

2.2. $SO(2r)$

For the curve for $N = 2$ gauge theories with gauge group $SO(2r)$ we take

$$F(t, v) = t^2v^2 + 2tP_r(v) + v^2 = 0 \tag{4}$$

with

$$P_r(v) = v^{2r} + c_2v^{2r-2} + c_4v^{2r-4} + \ldots + \tilde{c}_r^2. \tag{5}$$

c_{2n}, $n = 1, \ldots, r-1$ denote gauge invariant operators (Casimirs) of order $2n$ and $\tilde{c}_r$ is the exceptional Casimir of order r. Via the substitution $t \rightarrow (t - P_r)/v^2$ these curves assume the form of the $SO(2r)$ hyperelliptic curves of Ref. [22]. Substituting $t \rightarrow t/v^2$ we find the D_n curves of Refs. [20,21].

For fixed v the polynomial is of degree two in t and the two roots of Eq. (4) correspond to the fact that we have a type IIA configuration with two NS five-branes. In general, the degree of the polynomial in t equals the number of NS five-branes.

On the other hand $F(t, v)$ is of degree $2r$ in v, reflecting the presence of $2r$ D4-branes. Since the polynomial is even in v the configuration is symmetric under $v \rightarrow -v$. This hints towards the existence of an orientifold plane at $v = 0$, parallel to the D4-branes, in the classical type IIA picture. This O4 plane enforces a reflection-symmetric brane configuration. This incidentally, automatically removes the IR divergence in the five-brane kinetic energy, which, in the $U(r)$ case discussed in Ref. [18] led to a freezing out of the $U(1) \subset U(r)$ factor associated with the motion of the center of position of the D4-branes.

To get further information on the brane configuration, we now investigate the behavior of the curve in certain limits.

First we want to look at the limit where v is small. If $\tilde{c}_r = 0$ the polynomial factors into v^2 times a factor which is appropriate for the curve for gauge group $SU(2r-2)$ with all odd Casimir invariants set to zero. This corresponds to the situation where two infinite D4-branes coincide at $v = 0$ which, naively, would imply additional massless states from zero length strings between these two branes. But from a careful analysis in field theory [22] we know that the monodromy at this singularity is trivial. Therefore there are no additional particles becoming massless. This observation was first made in the context of type IIA brane configurations in Ref. [7].

In the case of non-zero $\tilde{c}_r$ the curve becomes, again in the limit of small v,

$$tv^2 + 2\tilde{c}_r^2 + v^2/t = 0. \tag{6}$$

Going to small t (i.e. $s \rightarrow \infty$) requires $t \sim v^2$, which means that two roots of $F(v)$ asymptotically approach $v = 0$. For large t ($s \rightarrow -\infty$) we find $t \sim 1/v^2$ which indicates two roots of $F(v)$ approaching $v = 0$. We interpret this as two infinite D4-branes which are deformed in the region of small x^6 but approach the position $v = 0$ as $x^6 \rightarrow \pm\infty$.

The question of how to identify segments of the curve with branes of type IIA can be addressed as follows. Eq. (4) defines a multivalued map from the v plane to the t plane with $4r$ branch points. We want to identify the objects that extend to $x^6 \rightarrow \infty$ with v small. Examining the map mentioned above, we find that circling $v = 0$ once maps to circling $t = 0$ *twice*. But a closed contour around the origin of the t plane means going around the $\mathbf{S^1}$. A D4-brane is distinguishable from an NS brane in that the former wraps $\mathbf{S^1}$ and the latter does not. We thus identify the objects that stretch to $x_6 = \pm\infty$ as *two* D4-branes. Another way to see that we are dealing with two semi-infinite D4-branes is that for the codimension-one subspace of the moduli space defined by $\tilde{c}_r = 0$ we have two branes precisely at $v = 0$, which means that they have to wrap around the $\mathbf{S^1}$. By continuity this also holds for generic values of $\tilde{c}_r$.

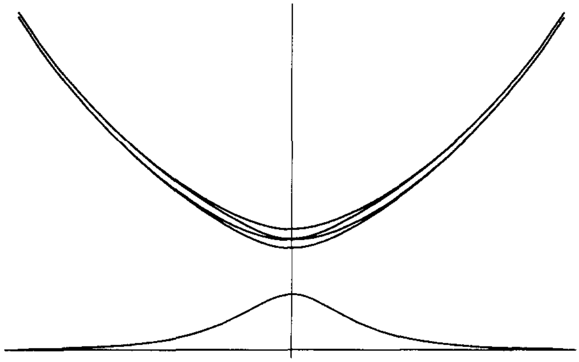

Fig. 1. The M5-brane for the gauge group $SO(10)$.

Let us now consider the situation where v is large. For t large ($s \to -\infty$), the roots for v are approximately at

$$t \equiv v^{2r-2}, \tag{7}$$

and for t very small ($s \to \infty$) approximately at

$$t \equiv v^{-(2r-2)}. \tag{8}$$

This describes the bending of an NS five-brane when a net number of $2r-2$ D4-branes end on it from the right and the left, respectively. Again, note that the D4-branes are located symmetrically with respect to the $v = 0$ plane. $2(r-1)$ of the D4-branes have the same asymptotics which diverges exponentially. It is to be identified with the position of the NS five-branes at large v. In addition, there are two infinite D4-branes which asymptotically approach $v = 0$, the position of the IIA orientifold plane which is not directly visible in the M-theory picture; it is encoded in the curve only through the symmetry and the presence of the infinite D4-branes.

As an example we have drawn in Fig. 1 the M5-brane for the gauge group $SO(10)$ with generic values of the moduli. Each line corresponds to two D4-branes due to the $v \to -v$ symmetry. More specifically, we have chosen a specific slice through Σ with $x^{10} = 0$. Therefore, each of the lines is actually a tube (times $\mathbb{R}^{3,1}$). The horizontal axis corresponds to x^6 and the vertical axis to the absolute value of v.

The generalization to the cases with $N_f = N_1 + N_2$ matter multiplets is

$$v^2 t^2 \prod_{j=1}^{N_1} (v^2 - m_j^2) + 2tP_r(v) + v^2 \prod_{i=N_1+1}^{N_f} (v^2 - m_i^2) = 0, \tag{9}$$

which corresponds to adding N_1 semi-infinite mirror pairs of four-branes to the left of all five-branes and N_2 to the right.

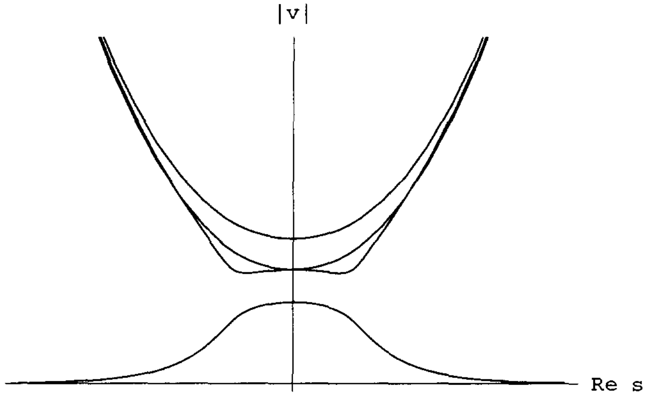

Fig. 2. $SO(5)$.

2.3. $SO(2r+1)$

For the gauge groups $SO(2r+1)$ we need a brane setup with $2r+1$ D4-branes between the two NS five-branes. This is achieved by curves of the form

$$F(t,v) = t^2v^2 + 2tvP_r(v^2) + v^2 = 0,$$
$$P_r(v) = v^{2r} + c_2v^{2r-2} + c_4v^{2r-4} + \ldots + c_{2r}, \tag{10}$$

which are related, via the substitution $t \to t/v^2$ to the B_r curves of Refs. [20,21], whereas $t \to (t-P)/v$ reproduces those of Ref. [23]. Important is the overall factor v, which is needed to get the correct number of D4-branes. Note that the same argument that was used in Section 2.2 to demonstrate that there are indeed two semi-infinite D4-branes stretching to $x^6 \to \pm\infty$, now tells us that there is just one of these branes. In addition, there is still the infinite D4-brane at $v = 0$ which necessarily wraps around the $\mathbf{S}^1$.

Fig. 2 shows an example with gauge group $SO(5)$. Every branch corresponds now to one D4-brane in contrast to the $SO(2r)$ case because of a different reflection symmetry; see below. Furthermore, the horizontal line corresponds to the additional infinite D4-brane at $v = 0$.

We now have again two D4-branes in the region to the left and to the right of the NS branes. As for $SO(2r)$ they approach $v = 0$ as $x^6 \to \pm\infty$. $F(t,v) = 0$ is now invariant under $(t,v) \to (-t,-v)$. Note that $t \to -t$ implies $x^{10} \to x^{10} + \pi R$. This means that the orientifolding also involves a non-trivial transformation in the x^{10} direction.

In the $SO(2r+1)$ case the singularity at $c_{2r} = 0$ has a non-trivial monodromy, in contrast to the $\tilde{c}_r = 0$ singularity of the $SO(2r)$ case, and a dyon becomes massless, which is related to a short root of the Lie algebra of $SO(2r+1)$ [22]. In the type IIA picture there is an additional D4-brane on top of the orientifold plane such that there are additional states from strings between this special D4-brane and the other D4-branes. These states are necessary to lift the $SO(2r)$ multiplet to an $SO(2r+1)$ multiplet. In our M-theory configuration we have taken into account this additional D4-brane at $v = 0$

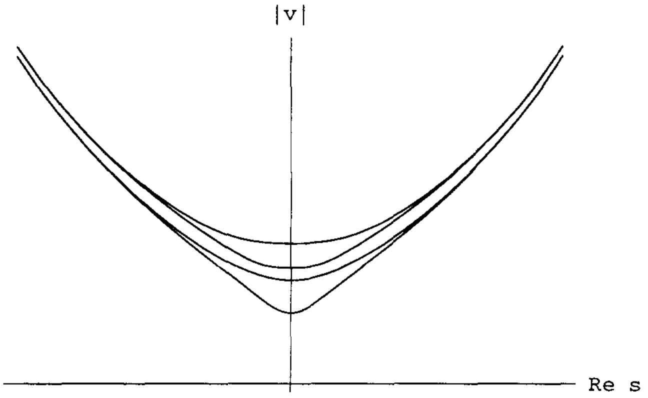

Fig. 3. $Sp(6)$.

by introducing an additional factor of v in the curve. As $c_{2r} \to 0$ the curve develops another infinite D4-brane on top of that which gives rise to an additional massless state, as expected.

Adding matter in the fundamental representation is straightforward: we attach N_1 semi-infinite D4-branes (and their mirror images) from the left to the left NS five-brane and N_2 semi-infinite D4-branes (and their mirror images) from the right to the right NS five-brane, with $N_1 + N_2 = N_F$ being the total number of fundamental flavors. The curve then takes the form

$$v^2t^2\prod_{j=1}^{N_1}(v^2 - m_j^2) + 2tvP_r(v) + v^2\prod_{i=N_1+1}^{N_f}(v^2 - m_i^2) = 0. \tag{11}$$

2.4. $Sp(2r)$

We take the holomorphic curve which is to represent the brane configuration in the pure gauge case to be

$$t^2 + 2tv^2P_r(v) + 1 = 0, \qquad P_r(v) = v^{2r} + c_2v^{2r-2} + \ldots + c_{2r}. \tag{12}$$

Note the symmetry under $v \to -v$. Via the substitution $t \to tv^2$ this curve is seen to be equivalent to the C_r curve in Ref. [21]. The relation with the curves of Ref. [24] is however not so obvious.[2]

Fig. 3 shows the generic form of the curve for the gauge group $Sp(6)$. As in the $SO(2r)$ case each branch corresponds to two D4-branes. From the form of the curve we read off that there are two additional D4-branes between the two NS five-branes. These two additional D4-branes have the property that they touch the orientifold at $v = 0$, $t_{1,2} = \pm i$ for all values of the moduli. This is not visible in Fig. 3 since we took a slice where t is real.

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[2] We acknowledge the instructive correspondence about the various forms of the curves with A. Mironov and A. Morozov. Note that there is also a disagreement between the C_r curves of Refs. [20] and [21].

Again, adding matter in the fundamental $2r$-dimensional representation of $Sp(2r)$ is straightforward: we simply add N_1 and N_2 semi-infinite D4-branes (and their mirror images) to the left and to the right, respectively, with $N_1 + N_2 = N_F$.

We have thus seen that there is a natural correspondence between the A_r, B_r, C_r, D_r Seiberg–Witten curves as spectral curves of appropriate integrable systems and the type IIA brane configuration which leads to $SU(r)$, $SO(2r)$, $SO(2r+1)$ and $Sp(2r)$ gauge theories on the D4-branes. In the M-theory context these curves simply describe the internal part of the five-brane.

2.5. The orientifold plane

We would now like to examine the question how the structure of an orientifold emerges from the M-theory curves that correspond to the orthogonal and symplectic groups. In string theory we have, as a consequence of dividing by the world-sheet parity inversion times a space-time symmetry, an orientifold plane which also carries an RR charge, namely if p is the dimension of the orientifold plane, $\pm 2^{p-5}$ units of charge of a physical Dp-brane, i.e. the Dp-brane and its mirror image. If we normalize the RR charge of a physical D4-brane to be $+2$, the charge of an O4 plane is ± 1. In the geometric brane arrangement of the type IIA theory, the orientifold plane makes its appearance by enforcing symmetry under reflection on the orientifold plane, but, in the discussion of Seiberg dualities with SO and Sp gauge groups, also via its RR charge, in particular via its charge induced on the NS branes. Central in the discussion of Refs. [7,10] was the fact that the charge of the orientifold plane switches sign on traversing an NS five-brane, so that in the simplest arrangement of two NS five-branes its charge is -1 between the five-branes and $+1$ outside. This is for orthogonal gauge groups and the sign of the charge is reversed for symplectic groups.

Let us now see how the orientifold plane and these charge assignments might be understood from the M-theoretic point of view. We saw that with the $SO(2r)$ projection we get additional semi-infinite D4-branes, two on each side of the NS five-brane arrangement, while with the $SO(2r+1)$ projection we get one infinite D4- and one additional semi-infinite D4-brane on each side. The Sp projection leads to two additional D4-branes between the two NS five-branes. If we now agree to assign the RR charge -1 to an infinite four-dimensional four-plane along the (x^1, x^2, x^3, x^6) direction, with $x^4 = x^5 = 0$ and x^7, x^8, x^9 fixed by the NS five-branes, we find that the $SO(2r)$ projection effectively leads to a charge assignment $(+1, -1, +1)$ for the three regions to the left, between, and to the right of the two NS five-branes. For the Sp projection we get $(-1, +1, -1)$ and for the $SO(2r+1)$ projection $(+1, 0, +1)$ instead.

2.6. Product gauge groups

We now consider more general models with chains of NS five-branes connected by D4-branes. The $n+1$ five-branes are labeled from 0 to n and the $(\alpha-1)$th five-brane is connected to the αth five-brane by k_α D4-branes.

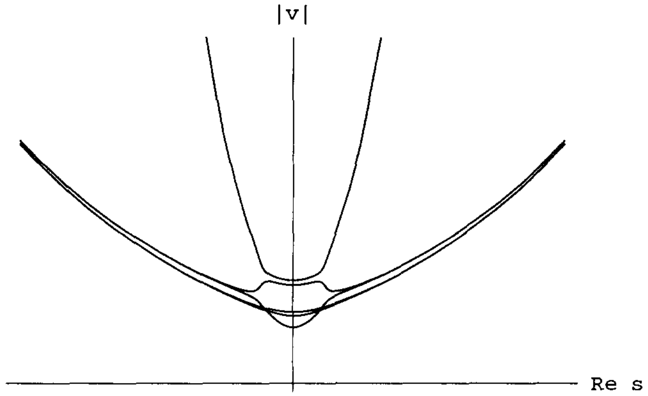

Fig. 4. $Sp(6) \times SO(10) \times Sp(6)$.

As in the $N = 1$ supersymmetric case [11] it is not possible to create models with gauge groups SO–SO–... or Sp–Sp–..., only alternating chains of the form SO–Sp–SO–... or Sp–SO–Sp–... are possible. For example, the second possibility is realized by the following curve:

$$F(t,v) = t^n + t^{n-1}v^2 P_1(v) + t^{n-2}P_2(v) + t^{n-3}v^2 P_3(v) + t^{n-4}P_4 + \ldots = 0 \quad (13)$$

with

$$P_\alpha(v) = v^{2r_\alpha} + c_2^{(\alpha)} v^{2r_\alpha - 2} + c_4^{(\alpha)} v^{2r_\alpha - 4} + \ldots + c_{2r_\alpha}^{(\alpha)}. \quad (14)$$

From the previous sections it is clear that this must correspond to a gauge theory with gauge group

$$G = Sp(2r_1) \otimes SO(2r_2) \otimes Sp(2r_3) \otimes \ldots \quad (15)$$

and matter content

$$\bigoplus_{\alpha=1}^{n-1} (2r_\alpha, 2r_{\alpha+1}). \quad (16)$$

In the case that the chain starts with an SO gauge factor, the first terms of the curve are

$$t^n v^2 + t^{n-1} P_1(v) + t^{n-2} v^2 P_2(v) + \ldots = 0. \quad (17)$$

In Fig. 4 we have drawn a simple example with three gauge group factors $G = Sp(6) \times SO(10) \times Sp(6)$. The branches going to infinity for large $\mathrm{Re}\, s$ correspond to the four NS five-branes. They are all bent differently according to the number of D4-branes ending on them from the left and the right. Each branch corresponds to two D4-branes in a type IIA configuration. Chains without matter and $SO(2r+1)$ factors are not possible as follows immediately from our discussion of the curves and the brane configurations derived from it.

We can add matter by putting semi-infinite D4-branes in the usual way. For example, we can multiply the highest power of t by $\prod_{i=1}^{N_f}(v^2 - m_i^2)$ which will lead to N_f

matter hypermultiplets in the fundamental representation of the first gauge group factor, $SO(2r_1)$ or $Sp(2r_1)$.

If we compactify these chains on a circle $\mathbf{S}^1$ in the x^6 direction we find the generalizations of the elliptic models of Ref. [18]. This means that the semi-infinite D4-branes to the left and to the right are connected and produce an additional gauge group factor. This is only consistent if we have an even number of NS branes. In this case the gauge group is

$$G = SO(2r_1) \times Sp(2r_2) \times SO(2r_3) \times Sp(2r_4) \times \ldots \times Sp(2r_{2n}), \tag{18}$$

and matter comes in the usual mixed representations with the exception that there is an additional hypermultiplet in the $(2r_{2n}, 2r_1)$ representation. If the number of D6-branes $b_l = 0$ (see below) one can make the beta functions vanish for all gauge group factors,

$$r_{2i+1} = r_{2j} + 1 = n + 1 \qquad \text{for } 1 \leqslant i, j \leqslant n. \tag{19}$$

In this case the gauge group is $SO(2n+2) \times Sp(2n) \times \ldots$

3. Configurations with D6-branes

We now incorporate D6-branes in our configurations, following closely Ref. [18]. We now place d_α D6-branes between the $(\alpha-1)$th and αth NS five-brane. Each D6-brane is located at definite values of x^4, x^5 and x^6. Due to the symmetry $v \to -v$ we always have to place D6-branes in pairs at locations v and $-v$.

The interpretation of the resulting world-volume theory is clear with the gauge group and matter from D4-branes as before. In addition, we have d_α hypermultiplets in the fundamental representation of the corresponding orthogonal or symplectic gauge group. The v positions of the D6-branes give the bare masses. Their x^6 positions decouple from the low energy four-dimensional physics; they become relevant in the discussion of Higgs or mixed branches only [18].

In this section we will discard the semi-infinite D4-branes which gave rise to (massive) hypermultiplets; we can generate an arbitrary number of hypermultiplets using D6-branes only.

Our basic guideline throughout this paper is to interpret type IIA brane configurations in M-theory. So we have to identify the type IIA six-brane in M-theory which was first done in Ref. [25]. We consider M-theory on $\mathbb{R}^{10} \times \mathbf{S}^1$ which is equivalent to type IIA on $\mathbb{R}^{10}$. The RR $U(1)$ gauge field of type IIA is associated in M-theory with shifts along the $\mathbf{S}^1$. Momentum states in the $\mathbf{S}^1$ direction are electrically charged with respect to this $U(1)$ and are interpreted in type IIA as D0-branes. The monopoles of this $U(1)$ correspond to D6-branes in type IIA.

The object that is magnetically charged under this $U(1)$ is the Kaluza–Klein monopole $\mathbb{R}^6 \times \widetilde{Q}$, where $\widetilde{Q}$ is a Taub-NUT space. It can be described as

$$\widetilde{Q} = \left\{ (v, y, z) \in \mathbb{C}^3 \middle| yz = \prod_{a=1}^{d} (v^2 - e_a^2) \equiv Q(v^2) \right\}. \tag{20}$$

Here we have incorporated the fact that the D6-branes come in pairs located at $\pm e_a$. As explained in Ref. [18], asymptotically y and z can be identified with t and t^{-1}. $\widetilde{Q}$ is smooth as long as $e_a \neq e_b$ $\forall a \neq b$. Otherwise the singularity has to be resolved.

We start with models with a single gauge group and matter hypermultiplets. To incorporate D6-branes we have to replace $Q = \mathbb{R}^3 \times \mathbf{S}^1$ by $\widetilde{Q}$. As before, the type IIA configuration of D4- and NS five-branes is described by a complex curve Σ, now embedded in $\widetilde{Q}$. Σ will again be given by an polynomial equation $F(y, v) = 0$; any dependence on z has been eliminated using $z = Q(v)/y$. The C_n and D_n configurations are symmetric under $(v, y, z) \to (-v, y, z)$, the B_n configurations are symmetric under $(v, y, z) \to (-v, -y, -z)$ due to the asymptotic relation between t, t^{-1}, and y, z.

With two NS five-branes, $F(y, v)$ is quadratic in y. Furthermore, we assume that there are no semi-infinite D4-branes to the left or to the right, except those needed for the orthogonal gauge groups. In this example we will choose $G = SO(2r)$ but the discussion also applies to gauge groups of the B_r and C_r series. Thus F has the form

$$A(v)v^2y^2 + B(v)y + C(v)v^2 = 0, \tag{21}$$

where A, B, and C are relatively prime polynomials, which as all polynomials appearing here and below, depend on v through v^2 only. The condition that there be no semi-infinite D4-branes implies that A is constant; we set it to 1. Expressing (21) in terms of $z = Q(v^2)/y$, we obtain

$$C(v)v^2z^2 + B(v)Q(v)z + Q(v)^2v^2 = 0. \tag{22}$$

The absence of semi-infinite D4-branes implies that C divides BQ and Q^2. In particular this means that Q^2 is divisible by C. So any zero of C must be a zero of Q and may appear at most quadratically in C. This means we can split Q into three factors: Q_0, Q_1, and Q_2, whose roots are roots of C of order 0, 1, and 2, respectively. We will denote the number of zeroes of the Q_i by q_i. Thus we have

$$C(v) = fQ_2(v)^2Q_1(v), \tag{23}$$

with f being a non-zero constant. In addition, $BQ = BQ_0Q_1Q_2$ has to be divisible by C, leading to

$$B(v) = \widetilde{B}(v)Q_2(v) \tag{24}$$

for some polynomial $\widetilde{B}(v)$. Now F assumes the form

$$v^2y^2 + \widetilde{B}(v)Q_2(v)y + v^2fQ_2(v)^2Q_1(v) = 0. \tag{25}$$

In terms of the coordinate $\tilde{y} = y/Q_2(v)$ this becomes

$$v^2\tilde{y}^2 + \widetilde{B}(v)\tilde{y} + v^2fQ_1(v) = 0. \tag{26}$$

If $\widetilde{B}(v)$ is a polynomial of degree $2N$ in v, this presents a curve for $SO(2N)$ with q_1 flavors in the fundamental representation. The zeroes of Q_1 correspond to D6-branes between the two NS five-branes, the zeroes of Q_0 and Q_2 correspond to D6-branes to the right and to the left of all NS five-branes, respectively. As long as $q_1 \neq 2N-2$ we can set $f=1$ by rescaling v and $\tilde{y}$.

Finally, we want to include D6-branes in the models with chains of NS five-branes. The curve Σ will now be defined by the zero locus of a polynomial $F(y,v)$ of the form

$$\begin{aligned} &y^{n+1} + P_1(v)v^2y^n + P_2(v)y^{n-1} + P_3(v)v^2y^{n-2} + P_4(v)y^{n-3} + \dots \\ &+P_{n+1}(v) = 0. \end{aligned} \tag{27}$$

The substitution $y = Q(v)/z$ leads to

$$P_{n+1}z^{n+1} + v^2QP_nz^n + Q^2P_{n-1}z^{n-1} + \dots + Q^{n+1} = 0. \tag{28}$$

The absence of semi-infinite D4-branes implies that Q^mP_{n-m+1} is divisible by P_{n+1}. Hence all zeroes of P_{n+1} are zeroes of Q and the multiplicities of the zeroes of P_{n+1} must lie between 0 and $n+1$. Define polynomials

$$Q_m = \prod_{a=i_m+1}^{i_{m+1}} (v^2 - e_a^2), \qquad Q = \prod_{m=0}^{n+1} Q_m, \qquad 0 \leqslant i_0 \leqslant i_1 \leqslant \dots \leqslant i_n \tag{29}$$

such that

$$P_{n+1} = f\prod_{m=0}^{n+1} Q_m^{n+1-m} \tag{30}$$

and for $1 \leqslant m \leqslant n$

$$P_m = B_m(v)\prod_{j=0}^{m-1} Q_j^{m-j}. \tag{31}$$

Via the transformation $y \to yQ_0$ the curve (27) takes the following form:

$$\begin{aligned} &y^{n+1} + v^2B_1(v)y^n + B_2(v)Q_1(v)y^{n-1} + v^2B_3(v)Q_1(v)^2Q_2(v) + \dots \\ &+f(\delta_{-1,(-1)^n} + v^2\delta_{1,(-1)^n})\prod_{j=1}^{n} Q_j^{n+1-j} = 0. \end{aligned} \tag{32}$$

For $n=1$ this reduces to the case of two NS five-branes, discussed earlier in this section. For general n we conclude from the matter content that $d_s = i_{s+1} - i_s$ is the number of D6-branes between the sth and the $(s+1)$th NS five-brane. If the degree of B_s is $2r_s$, the gauge group is

$$G = SO(2r_1) \times Sp(2r_2) \times SO(2r_3) \times \dots \tag{33}$$

The hypermultiplets are in the representations $(2r_s, 2r_{s+1})$ plus d_s hypermultiplets in the fundamental representation of $SO(2r_s)$ $(Sp(2r_s))$ for even (odd) s.

4. Conclusions and outlook

Using the ideas of Ref. [18] we have generalized the interpretation of the type IIA brane configuration giving rise to $N = 2$ supersymmetric field theories with and without matter, to all classical groups. To a large part this is a straightforward extension of Ref. [18]. The interesting new aspect is how the orientifold plane, which is present in the type IIA picture, manifests itself from the M-theory point of view. Here we have taken the point of view that the orientifold plane is not directly visible in the Riemann surface which describes the internal, one-complex-dimensional part of the M5-brane. We only detect it via the presence of additional semi-infinite D4-branes, which do not add to the spectrum and via the symmetry of the brane configuration. We do not use the knowledge from type IIA theory about the charge assignment of the orientifold plane but try to understand it from the M-theoretic brane configuration. We determined the RR charge of D4-branes, for instance those that stretch to $x_6 = \pm\infty$ in the $SO(2r)$ case, by observing that they have a non-trivial monodromy around the origin of the t plane, namely, they are wrapped around the circle in the x^{10} direction. In fact, this may be a useful tool in general to determine the RR charge. A five-brane of M-theory, when it wraps the circle N times, implies a configuration of N D4-branes in the type IIA description. A similar situation applies to the NS charge of the strings in type IIA that are obtained from wrapping the M-theory membrane. Our starting point was the known curves for the various gauge groups. We find a satisfactory picture, which can be connected to the type IIA picture for B_r, C_r and D_r gauge groups.

The real challenge, in our mind, is the generalization to $N = 1$ theories. The type IIA picture has been fully developed for all classical groups. The Riemann surface in this case will be embedded in $\mathbb{R}^5 \times \mathbf{S}^1$. Presumably it encodes information about the Coulomb branch of the $N = 1$ theories, but many new insights into $N = 1$ theories might be gained by a better understanding in the framework of M-theory.

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