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Examples of Newtonian limits of relativistic spacetimes

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Abstract. A frame theory encompassing general relativity and Newton–Cartan theory is reviewed. With its help, a definition is given for a one-parameter family of general relativistic spacetimes to have a Newton–Cartan or a Newtonian limit. Several examples of such limits are presented.

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1. Introduction

The relation between Newton's theory of space, time, dynamics and gravitation (NG) and Einstein's theory of general relativity (EG), which in essence accomodates all of classical physics, is by now rather well understood. As Cartan [1] and Friedrichs [2] have indicated and later authors have elaborated [3–7], Newton's theory can be re-expressed in generally covariant spacetime language. This reformulation naturally led to a frame theory [8, 9], the laws of which specialize to those of a slight generalization of Newton's theory, the so-called Newton–Cartan theory (NC), and those of Einstein's theory if a real parameter λ is restricted to be zero or positive, respectively. The λ dependence of the laws shows that the mathematical structure underlying NC is a degenerate limit of that on which EG is based. (In terms of principal bundles, the degeneracy EG $\to$ NC results from contracting the restricted Lorentz group to the restricted Galilei group, in the sense of Segal [10]; see Künzle [6].)

To understand a limit relation such as EG $\to$ NC, it is not sufficient to exhibit the *laws* of the former theory as specializations or limits of those of the latter one; the more difficult part consists in establishing relations between the *solutions* to these laws, as they represent testable *models of real situations*. In the present case, one such relation is provided by the frame theory which gives a rigorous meaning to the statement that 'a one-parameter family of EG spacetime models converges to an NC model' or, in particular, to an NG model [8].

Such a limit relation [11–13] may be used to throw new light on methods designed to construct 'nearly Newtonian' solutions of EG [14, 15] or to interpret Einsteinian solutions. The frame theory even led to the first existence theorem for stationary, axisymmetric EG solutions representing rigidly rotating fluid bodies surrounded by an asymptotically flat vacuum field [16].

The physically plausible heuristic, but mathematically obscure and incomplete, textbook arguments concerning the Newtonian limit of general relativity may therefore be replaced by the relations provided by the frame theory. Besides, the latter can be used as a counterexample to the claim that scientific 'revolutions' in the sense of Kuhn [17] are irrational leaps associated with incommensurability of concepts between the older theory and its successor (see e.g. [18, 19]).

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The modest purpose of this paper is to illustrate the limit relation by means of a few simple examples. In section 2, I summarize the frame theory and its specializations to NG, NC and EG. Then, in section 3, I define the limit relation alluded to above, and the remaining sections contain some examples.

It was Andrzej Trautman's elegant paper [5] which initiated my interest in these matters, and therefore I am happy to contribute the following remarks to this Festschrift, to thank and honour Andrzej.

2. The frame theory and its specializations

The theory deals with the following collection of fields on a 4-manifold M, *spacetime*:

- $t_{\alpha\beta}$, a nowhere vanishing symmetric, 2-covariant tensor field, the *temporal metric*;
- $s^{\alpha\beta}$, a nowhere vanishing, symmetric, 2-contravariant tensor field, the (inverse) *spatial metric*;
- $\Gamma^{\alpha}_{\beta\gamma}$, a symmetric, linear connection, the *gravitational field*;
- $T^{\alpha\beta}$, a symmetric, 2-contravariant tensor field, the mass–momentum–stress, or *matter tensor*.

The laws of the theory contain these fields and two real-valued parameters, λ and Λ. To formulate the laws concisely, it is convenient to define, for arbitrary tensors, index shift operations,

$$V_{\alpha}^{\bullet} := t_{\alpha\beta} V^{\beta}, \qquad \omega^{\alpha}_{\bullet} := s^{\alpha\beta}\omega_{\beta},$$

even when $t_{\alpha\beta}$ or $s^{\alpha\beta}$ are not invertible. To avoid confusion it is then necessary to indicate by dots whether the original object has upper or lower indices, as shown. (The indices may be considered as 'abstract' ones in Penrose's sense except when, in the examples, special coordinates are used.)

Here are the (local) *laws*:

(i) For any timelike vector V^{α}, i.e. one which obeys $t_{\alpha\beta}V^{\alpha}V^{\beta} > 0$, the quadratic form $\omega_{\alpha} \to s^{\alpha\beta}\omega_{\alpha}\omega_{\beta}$ is positive definite on the subspace $\{\omega_{\alpha} | \omega_{\alpha}V^{\alpha} = 0\}$ of the covector space.
(ii) $t_{\alpha\beta}s^{\beta\gamma} = -\lambda\delta^{\gamma}_{\alpha}$.
(iii) $t_{\alpha\beta;\gamma} = 0$, $s^{\alpha\beta}{}_{;\gamma} = 0$.
(iv) $R^{\alpha\ \gamma}{}_{\beta\bullet\delta} = R^{\gamma\ \alpha}{}_{\delta\bullet\beta}$.
(v) $R_{\alpha\beta} = 8\pi\left(T^{\bullet\bullet}_{\alpha\beta} - \frac{1}{2}t_{\alpha\beta}T^{\gamma\bullet}_{\gamma}\right) - \Lambda t_{\alpha\beta}$.
(vi) $T^{\alpha\beta}{}_{;\beta} = 0$.

The covariant derivatives refer to the connection Γ, and $R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}$ denotes the Ricci tensor associated with the curvature tensor of Γ.

In addition to the general laws, *models of matter* have to be specified. Here I consider only the cases of vacuum, $T^{\alpha\beta} \equiv 0$, and *perfect fluid*,

(vii) $T^{\alpha\beta} = (\rho + \lambda p)U^{\alpha}U^{\beta} + ps^{\alpha\beta}$, where $t_{\alpha\beta}U^{\alpha}U^{\beta} = 1$, $\rho > 0$, $\rho + \lambda p > 0$.

$\rho = T^{\alpha\beta}U^{\bullet}_{\alpha}U^{\bullet}_{\beta}$ is the mass density, p is the pressure of the fluid. Note that the parameter λ appears in only *one* of the general axioms and in the perfect fluid matter tensor, and in the simplest possible manner.

If a set

$$(M, t_{\alpha\beta}, s^{\alpha\beta}, \Gamma^{\alpha}_{\beta\gamma}, T^{\alpha\beta}, \ \rho, p, U^{\alpha}, \lambda, \Lambda) \qquad (1)$$

satisfies the laws (i)–(vii), if $f\colon M \to \bar{M}$ is a diffeomorphism which maps $t_{\alpha\beta}$ into $\bar{t}_{\alpha\beta}$ etc, and if α, β are positive numbers, then the set

$$(\bar{M}, \alpha^{-2}t, \beta^2 s, \Gamma, \alpha^4 T, \alpha^2\rho, \alpha^4\beta^{-2}p, \alpha U, \alpha^{-2}\beta^2\lambda, \alpha^2\Lambda) \tag{2}$$

also satisfies those laws. This, of course, expresses the general covariance and the arbitrariness of the choice of units for duration and length. In order to dispense with physical dimensions and units (and in order to simplify the physical interpretation of the limit relation to be given in section 3), I adopt the view that all quantitative physical predictions are to be expressed in dimensionless form, e.g. as ratios of lengths. In accordance with this convention, a set (1) obeying the laws will be said to *represent a model* or *solution* of the frame theory, and representations of models related like (1) and (2) will be called *physically equivalent*; they may be interpreted as representing the same physical situation with respect to different units of time and length, keeping Newton's G equal to 1. With this terminology, the numerical values of λ and Λ have no physical significance, only their signs do.

The following facts can be inferred from [3–9].

If $\lambda > 0$, the frame theory reduces to general relativity, EG. In this case, one may choose $\lambda = 1$, $g_{\alpha\beta} \equiv -t_{\alpha\beta}, g^{\alpha\beta} \equiv s^{\alpha\beta}$, in which case the laws give those of EG with natural units ($c = 1, G = 1$) and spacelike signature $(-{+}{+}{+})$. (Some of the laws then become redundant.) For the purpose of this paper, however, it proves preferable *not* to fix the scaling and to work with $t_{\alpha\beta} = -\lambda g_{\alpha\beta}, s^{\alpha\beta} = g^{\alpha\beta}$ and arbitrary $\lambda > 0$.

If $\lambda = 0$, the frame theory reduces to what is called the Newton–Cartan theory, NC. In this case, the temporal metric can be expressed in terms of a scalar *absolute time* t, as $t_{\alpha\beta} = t_{,\alpha}t_{,\beta}$, and it is possible to choose (local) coordinates (t, x^a) such that

$$t_{\alpha\beta} = \mathrm{diag}(1, 0, 0, 0), \tag{3a}$$

$$s^{\alpha\beta} = \mathrm{diag}(0, 1, 1, 1), \tag{3b}$$

$$\Gamma^\alpha_{\beta\gamma} = 0 \quad \text{except} \quad \Gamma^a_{tt} = -g^a \quad \text{and} \quad \Gamma^b_{ta} = s^{bc}\varepsilon_{acd}\omega^d. \tag{3c}$$

With respect to such a coordinate system, the field equations show that the spatial vector fields $\vec{g}(t, \vec{x})$ and $\vec{\omega}(t, \vec{x})$ play the part of a gravitational acceleration and Coriolis angular velocity, respectively.

To reduce this theory to Newton's, one has to add a condition which ensures that $\vec{\omega}$ depends on time only; then the coordinates can be specialized further to 'non-rotating' ones, with $\vec{\omega} = 0$, and then

$$\Gamma^\alpha_{\beta\gamma} = t_{,\beta}t_{,\gamma}s^{\alpha\delta}U_{,\delta}, \tag{4}$$

where the Newtonian potential U appears in the role of a connection potential. The simplest such condition, due to Trautman [3], reads

$$R^{\alpha\beta}{}_{\bullet\gamma\delta} = 0 \tag{5a}$$

or, equivalently,

$$t_{,[\alpha}R^{\beta}{}_{\gamma]\delta\varepsilon} = 0. \tag{5b}$$

Equation (5b) says that parallel transport of spacelike vectors V^α, which are characterized by $t_{,\alpha}V^\alpha = 0$, is integrable, i.e. path independent, a well known property of Newton's theory.

Equation (5a) cannot, of course, be added to the general laws of the frame theory since for $\lambda \neq 0$ it implies flatness. If one restricts the solutions of the frame theory to spatially asymptotically flat ones, i.e. considers isolated systems only, then in the case when $\lambda = 0$ equations (5) follow [8]. In this case, EC = NC, and the connection can be split uniquely into a flat connection and a tensorial gravitational field.

3. Definition of Newtonian limits of Einsteinian spacetimes

The facts reviewed in the preceding section suggest the following.

Definition. Let $\mathcal{M}(\lambda), 0 \leqslant \lambda < a$, be a family of (representatives of) models (1) parametrized by λ (i.e. $t_{\alpha\beta}(\lambda), s^{\alpha\beta}(\lambda), \ldots$), all defined on open submanifolds $M(\lambda)$ of a fixed manifold M. Then, by definition,

$$\lim_{\lambda\to 0} \mathcal{M}(\lambda) = \mathcal{M}(0) \tag{6}$$

means that the members of $\mathcal{M}(\lambda)$ as well as $R^{\alpha}{}_{\beta\gamma\delta}(\lambda)$ converge pointwise to those of $\mathcal{M}(0)$ and $R^{\alpha}{}_{\beta\gamma\delta}(0)$, respectively. (One could generalize Geroch's definition of limits of spacetimes [20] to the present situation to formulate a more general limit definition, but this will not be pursued here.)

One would like to have a criterion which ensures that a family $\mathcal{M}(\lambda)$ of EG models, $0 < \lambda < a$, *does have* an NC (or, in particular, an NG) limit. One such condition is easily verified:

Lemma. A family $\mathcal{M}(\lambda), 0 < \lambda < a$, converges to an NC model $\mathcal{M}(0)$ if

(i) the members of $\mathcal{M}(\lambda)$ and their first covariant derivatives with respect to some arbitrary, fixed symmetric connection converge, locally uniformly on M, to fields $t_{\alpha\beta}(0), \ldots, T^{\alpha\beta}(0)$;
(ii) $s^{\alpha\beta}(0)$ has rank 3;
(iii) $\lim_{\lambda\to 0}(\lambda^{-1} \det s^{\alpha\beta}(\lambda))$ exists and is negative.

The limit whose existence is asserted in the lemma will, in general, only be a Newton–Cartan model. In view of the last remark in the foregoing section one may expect the limit to be strictly Newtonian if the members of $\mathcal{M}(\lambda)$ are uniformly spatially asymptotically flat in a suitable sense, still to be made explicit. Examples confirm this expectation.

The limit definition in terms of 'dimensionless' models which may be rescaled without change of their physical meanings has the advantage that it avoids taking the physically meaningless limit $c \to \infty$ of a dimensional constant. Also, it ensures that if $\mathcal{M}(0)$ is the limit of $\{\mathcal{M}(\lambda)\}$ and if the convergence is uniform on a compact domain D of M, then the values of corresponding observables of the physical models represented by the $\mathcal{M}(\lambda)$ will converge to those of $\mathcal{M}(0)$, provided those observables, e.g. proper times of segments of world lines representing particles or angles between light rays connecting such particles, refer to 'figures' and fields in D, and provided the identification of events in the different models $\mathcal{M}(\lambda)$ via points of their common manifold M (point identification gauge) has been chosen such that figures interpreted as 'the same objects' in the $\mathcal{M}(\lambda)$ are represented by the same sets of M. (This, of course, restricts the freedom of applying diffeomorphisms $f(\lambda)$ to the $\mathcal{M}(\lambda)$, just as in physical applications of perturbation theory, but it does not restrict rescalings). Physical interpretations of the limit relations have been discussed in [9, 14].

4. Limits of the Schwarzschild and Kerr black holes

A dimensionless, parametrized representation of the metric of the *Schwarzschild black hole* is given by†

$$\mathrm{d}s^2 = -\lambda^{-1}\left(1 - \frac{2\lambda}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - 2\lambda/r} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) \tag{7}$$

† For convenience I use the old-fashioned notation $\mathrm{d}s^2$ for $g_{\alpha\beta}\,\mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta$.

where $0 < 2\lambda < r$ and t, ϑ, φ have their usual ranges. The family $\mathcal{M}(\lambda) = \{-\lambda g_{\alpha\beta}(\lambda), g^{\alpha\beta}(\lambda), \Gamma^{\alpha}_{\beta\gamma}(\lambda)\}$ with $\Lambda = 0$ converges to the field of a mass point at the spatial origin, the horizon at 2λ shrinks to a point in the limit $\lambda = 0$.

The reason for displaying this trivial example is only to show how the description in terms of the frame theory provides, simply and rigorously, the Newtonian spacetime metric $(t_{\alpha\beta}, s^{\alpha\beta})$ of equations (3*a, b*) along with the gravitational field

$$\Gamma^{r}_{tt}(0) = \frac{1}{r^2}, \qquad \Gamma^{\alpha}_{\beta\gamma} = 0 \quad \text{otherwise}$$

of a Newtonian mass point (cf equation (4)), as limits of the relativistic fields.

Enriching the relativistic models by test particles, the world lines of which also have Newtonian limits, one can compute observables such as ratios of proper times or frequencies or angles associated with these particles, and find out how relativistic relations between such observables tend to those of the Newtonian limit model, provided the test objects keep some distance from the horizon. Replacing such (analytic) relations by low-order Taylor polynomials in λ then gives post-Newtonian approximations.

(The time-dependent, interior part of the Kruskal extension, given by $0 < r < 2\lambda$, does not admit a Newtonian limit in this parametrization, and probably also not in any other parametrization.)

The following considerations extend straightforwardly if one adds a cosmological term Λ to the metric (7).

A parametrization of the *Kerr metric* analogous to (7) is

$$\begin{aligned} \mathrm{d}s^2 = {} & -\lambda^{-1}\left(1 - \frac{2\lambda r}{\Sigma}\right)\mathrm{d}t^2 + \frac{\Sigma}{r^2 - 2\lambda r + k^2}\mathrm{d}r^2 \\ & + \Sigma\,\mathrm{d}\vartheta^2 + \sin^2\vartheta\left(r^2 + k^2 + \frac{2\lambda r}{\Sigma}k^2\sin^2\vartheta\,\mathrm{d}\varphi^2\right) + \frac{4k\sqrt{\lambda}}{\Sigma}\sin^2\vartheta\,\mathrm{d}t\,\mathrm{d}\varphi, \end{aligned} \tag{8}$$

where $\Sigma = r^2 + k^2\cos^2\vartheta$, $0 < r < \infty$, t, ϑ, φ as usual.

The λ-family of spacetime models determined by (8) has an NC limit if and only if $\lim_{\lambda\to 0} k(\lambda) = k_0$ exists [14]. If $k_0 \neq 0$, the angular momentum of the model diverges if $\lambda \to 0$, for sufficiently small λ the spacetime contains closed timelike lines, and the limit spacetime (which was also constructed in a different way by Keres [21]) does not correspond to a physically acceptable, non-negative mass distribution. If all members of the family are assumed to be black holes—i.e. regular up to their horizons—then $k_0 = 0$, and the limit is the same as that of the Schwarzschild black hole. These results are compatible with the still unproven conjecture that the Kerr field does not admit a physically acceptable material source.

5. Newtonian limits of fluid balls

Let us again take the Schwarzschild metrics of equation (7), but restrict their domain to the region $r > 1$. Within the ball $r \leqslant 1$, we take Schwarzschild's interior metric

$$\mathrm{d}s^2 = \frac{-1}{4\lambda}(3\sqrt{1-2\lambda} - \sqrt{1-2\lambda r^2})^2\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2\lambda r^2} + r^2\,\mathrm{d}\omega^2 \tag{9}$$

with its perfect fluid matter tensor as in (vii), where

$$\rho = \frac{3}{4\pi}, \qquad \frac{\lambda p}{\rho} = \frac{\sqrt{1-2\lambda r^2} - \sqrt{1-2\lambda}}{3\sqrt{1-2\lambda} - \sqrt{1-2\lambda r^2}} \tag{10}$$

and where $d\omega^2$ denotes the standard metric of S^2, the angular part of (7). (The 4-velocity is the normalized tangent of the t-lines.) In this way we get a one-parameter family of 'star' models surrounded by their vacuum fields. The parameter 2λ equals the ratio of the Schwarzschild radius/geometrical radius of the star, its range has to be restricted to $0 < 2\lambda < \frac{8}{9}$ in order for the pressure to be finite. This family has as its limit a Newtonian fluid ball of constant density with its field. Rendall and Schmidt have shown that corresponding limits exist for static, spherical fluid balls for a large range of equations of state [22].

6. The limit of a plane gravitational wave

While the two preceding examples give what one expects, it is perhaps a little surprising that even a plane gravitational wave admits a Newtonian limit [8]. The metric of such a wave may be written in Kerr–Schild form, as

$$ds^2 = -\lambda^{-1}\,dt^2 + dx^2 + dy^2 + dz^2 + 2U(dt - \sqrt{\lambda}\,dz)^2, \tag{11}$$

where

$$U = \tfrac{1}{4}(x^2 - y^2)A(t - z\sqrt{\lambda}) + \tfrac{1}{2}xyB(t - z\sqrt{\lambda}) \tag{12}$$

depends on two amplitudes A, B. This λ-family has, in fact, a Newtonian limit the connection of which corresponds, via (4), to the time-dependent quadrupolar potential obtained by putting $\lambda = 0$ in (12). (Note that, in contrast to all other examples given in this paper, the temporal metric $-\lambda g_{\alpha\beta}(\lambda)$ is smooth in $\sqrt{\lambda}$, not in λ.) The Newtonian test particle motions indeed imitate the action of a 'slow' gravitational wave on the particles.

7. Limits of Friedmann–Lemaître cosmological models

A one-parameter family of Friedmann–Lemaître models for $\lambda > 0$ is determined by three functions $R(t,\lambda), \rho(t,\lambda), p(t,\lambda)$ and a constant E. The functions have to satisfy the equations

$$\dot{R}^2 - \frac{8\pi}{3}\rho R^2 = E, \tag{13}$$

$$(\rho R^3)^{\boldsymbol{\cdot}} + \lambda p (R^3)^{\boldsymbol{\cdot}} = 0. \tag{14}$$

(If desired, one could add an equation of state, $p = f(\rho, \lambda)$.) The metric can be written as

$$ds^2 = -\lambda^{-1}\,dt^2 + \frac{\delta_{ab}(dX^a - (\dot{R}/R)X^a\,dt)(dX^b - (\dot{R}/R)X^b\,dt)}{(1 - \frac{1}{4}\lambda E R^{-2}\vec{X}^2)^2}, \tag{15a}$$

or, more simply, as

$$ds^2 = -\lambda^{-1}\,dt^2 + R^2\frac{d\vec{\xi}^2}{(1 - \frac{1}{4}\lambda E\vec{\xi}^2)^2}, \tag{15b}$$

where

$$\xi^a = \frac{X^a}{R(t,\lambda)} \tag{16}$$

is a λ-dependent coordinate transformation. The fluid has density ρ, pressure p and world lines given by $\xi^a =$ constant. Again, this family has a Newtonian limit. If one employs the Eulerian coordinates X^a, the Newtonian connection of the limit model has the form (4) with the well known potential $U = \frac{2\pi}{3}\rho\vec{X}^2$. If, on the other hand, one works with Lagrangian coordinates ξ^a, the connection is given by

$$\Gamma^a_{t\ b} = \frac{\dot{R}}{R}\delta^a_b, \qquad \Gamma^\gamma_{\beta\gamma} = 0 \quad \text{otherwise} \tag{17}$$

which does *not* have the form (4) and does not come from a 'scalar' potential in the standard way, although, of course, it describes the same gravitational field as U above. In the (t, ξ^a) description, all 'Newtonian' fields $(t_{\alpha\beta}, s^{\alpha\beta}, \Gamma^\alpha_{\beta\gamma}, \rho, p, U^\alpha)$ of the limit model are independent of the Lagrangian coordinates ξ^a, and therefore one may consider the 3-space of the model to be a torus $\mathbb{T}^3$ rather than $\mathbb{R}^3$, which is not at all obvious in the 'Eulerian' description. These Newtonian toroidal models are useful especially for studying perturbations (see e.g. [23]).

The spatial Gaussian curvatures of the relativistic models (13)–(16) are given by $K = -\lambda E/R^2$. A spherical relativistic model $(E < 0)$, for example, can be approximated by its spatially flat Newtonian limit in any comoving region which is small compared to the global radius, in which the convergence is uniform—as one would expect.

An elegant account of Newton–Cartan cosmology including perturbations has recently been given in [23].

8. A limit of the Gödel model

The metric of the stationary, homogenous, rotating, dust-filled, cosmological model of Gödel can be parametrized in the form

$$\mathrm{d}s^2 = -\lambda^{-1}\left(\mathrm{d}t - \frac{2}{\omega}\sinh^2\left(\tfrac{1}{2}\omega r\sqrt{2\lambda}\right)\mathrm{d}\varphi\right)^2 + \mathrm{d}r^2 + \mathrm{d}z^2 + \frac{1}{2\lambda\omega^2}\sinh^2(\omega r\sqrt{2\lambda})\,\mathrm{d}\varphi^2, \tag{18}$$

where (except for dimensions) ω is the angular velocity, $\rho = \omega^2/4\pi$ the density, $\Lambda = -\rho$ the cosmological constant and the coordinates r, z, φ are comoving. This metric and the fields $t_{\alpha\beta} = -\lambda g_{\alpha\beta}$, $s^{\alpha\beta} = g^{\alpha\beta}$, etc, determined by it depend analytically on λ at all values of λ, and one easily verifies that one obtains, for $\lambda = 0$, a (strictly!) Newtonian model with properties analogous to its relativistic progenitor. The metric rotates rigidly relative to local inertial frames everywhere, and relative to non-rotating coordinates the Newtonian connection has the standard form (4), and its potential obeys the Poisson equation with a Λ term.

9. A Newton–Cartan limit of the NUT spacetime

The metric of the NUT spacetime can be written as

$$\mathrm{d}s^2 = -\lambda^{-1}V\left(\mathrm{d}t + 4\lambda a\sin^2\frac{\vartheta}{2}\mathrm{d}\varphi\right)^2 + V^{-1}\,\mathrm{d}r^2 + (r^2 + \lambda a^2)\,\mathrm{d}\omega^2 \tag{19}$$

with $V = 1 - 2\lambda(mr + a^2)/(r^2 + \lambda a^2)$.

m and a are positive constants, $0 < r$, and $\mathrm{d}\omega^2$ again denotes the standard metric on S^2 in terms of (ϑ, φ). This family of vacuum spacetimes also depends analytically on λ. As in the other examples, the limit model has a flat Galilean metric with t as absolute time and r, ϑ, φ as Euclidean polar coordinates. With respect to these coordinates and in the notation of (3*c*), however, one finds

$$U = -\frac{m}{r} - \frac{a^2}{r^2}, \qquad \vec{g} = -\vec{\nabla}U = \left(-\frac{m}{r^2} - \frac{2a^2}{r^3}\right)\frac{\partial}{\partial r}, \qquad \vec{\omega} = -\frac{a}{r^2}\frac{\partial}{\partial r}.$$

Here, at last, we have a *truly Newton–Cartan limit* whose Coriolis field $\vec{\omega}$ is not spatially constant. In fact, near the point singularity, for $r \to 0$, the Coriolis potential dominates. It is the only case known to me where an EG family has a genuinely NC limit.

References

[1] Cartan E 1923 *Ann. Ecole Norm.* **40** 325
Cartan E 1924 *Ann. Ecole Norm.* **41** 1
Cartan E 1924 1955 *Œuvres Complétes* vol III/1, (Paris: Gauthier-Villars) pp 659, 799

[2] Friedrichs K 1927 *Math. Ann.* **98** 566

[3] Trautman A 1963 *C. R. Acad. Sci. Paris* **357** 617

[4] Havas P 1964 *Rev. Mod. Phys. Polon.* **65** 637

[5] Trautman A 1966 *Perspectives in Geometry and Relativity* ed B Hoffmann (Bloomington, IN: Indiana University Press) pp 413–25

[6] Künzle H P 1972 *Ann. Inst. H. Poincaré* **17** 337

[7] Künzle H P 1983 *Geometrodynamics Proc. 37 (Bologna)*

[8] Ehlers J 1981 *Grundlagenprobleme der modernen Physik* ed J Nitsch *et al* (Mannheim: Bibliographisches Institut)

[9] Ehlers J 1991 The Newtonian limit of general relativity *Classical Mechanics and Relativity: Relationship and Consistency* ed G Ferrarese (Naples)

[10] Segal J E 1951 *Duke Math. J.* **18** 221

[11] Dautcourt G 1964 *Acta Phys. Polon.* **65** 637

[12] Künzle H P 1976 *Gen. Rel. Grav.* **7** 445

[13] Dautcourt G 1990 *Acta Phys. Polon.* **BZI** 755

[14] Lottermoser M 1988 *Dissertation* Universität München (unpublished)

[15] Rendall A 1992 *Proc. R. Soc.* A **438** 341

[16] Heilig U 1993 *Dissertation* Universität Tübingen
Heilig U 1995 *Commun. Math. Phys.* **166** 457

[17] Kuhn Th 1962 *The Structure of Scientific Revolutions*

[18] Ehlers J 1986 *Logic, Methodology and Philosophy of Science VII* ed B Marcus *et al* (Amsterdam: Elsevier) pp 387–403

[19] Ehlers J 1995 *Physik, Philosophie und die Einheit der Wissenschaften* ed L Krüger and B Falkenburg (Heidelberg: Spektrum) pp 165–76

[20] Geroch R P 1969 *Commun. Math. Phys.* **13** 180

[21] Keres H 1967 *Sov. Phys.–JETP* **25** 504

[22] Rendall A and Schmidt B G 1991 *Class. Quantum Grav.* **8** 985

[23] Buchert Th and Ehlers J 1993 *Mon. Not. R. Astron. Soc.* **264** 375

[24] Rüede Ch and Straumann N *Preprint* ZU-TH-12/96 (ETH Zürich)