



On 2D quantum gravity coupled to a σ -model *

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1. INTRODUCTION

This contribution is a review of the method of isomonodromic quantization of dimensionally reduced gravity developed in [1-3]. The main idea of this approach is a complete separation of variables in the isomonodromic sector of the model and exploiting the related “two-time” Hamiltonian structure. This allows an exact quantization in the spirit of the scheme developed in the framework of integrable systems [4]. The possibilities to identify an appropriate quantum counterpart of the Kerr black hole are discussed. In addition, we briefly describe the relation of this model with Chern Simons theory.

2. THE MODEL

The Lagrangian of 2D gravity coupled to a $SL(2, \mathbf{R})/SO(2)$ σ -model is

$$\mathcal{L} = \rho \left(hR + \text{tr}(g_z g^{-1} g_{\bar{z}} g^{-1}) \right), \quad (1)$$

where the metric has been brought into conformal gauge

$$ds^2 = h(z, \bar{z}) dz d\bar{z}; \quad (2)$$

$R = (\log h)_{z\bar{z}}/h$ is the Gaussian curvature of the worldsheet, $g \in SL(2, \mathbf{R})/SO(2)$ and $\rho \in \mathbf{R}$ is the dilaton. The equation of motion for ρ derived from (1)

$$\rho_{z\bar{z}} = 0 \quad (3)$$

is solved by

$$\rho(z, \bar{z}) = \text{Im} \xi(z), \quad (4)$$

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where $\xi(z)$ is a (locally) holomorphic function. Now we can further specialize the gauge by identifying ξ with the worldsheet coordinate. Then the equation of motion for g coming from (1) is

$$\left((\xi - \bar{\xi}) g_{\xi} g^{-1} \right)_{\bar{\xi}} + \left((\xi - \bar{\xi}) g_{\bar{\xi}} g^{-1} \right)_{\xi} = 0. \quad (5)$$

Using the following parameterization of an arbitrary $SL(2, \mathbf{R})/SO(2)$ -valued matrix:

$$g = \frac{1}{\mathcal{E} + \bar{\mathcal{E}}} \begin{pmatrix} 2 & i(\mathcal{E} - \bar{\mathcal{E}}) \\ i(\mathcal{E} - \bar{\mathcal{E}}) & 2\mathcal{E}\bar{\mathcal{E}} \end{pmatrix} \quad (6)$$

in terms of the complex-valued function $\mathcal{E}(\xi, \bar{\xi})$, we can rewrite (5) in the familiar form of the Ernst equation [5]:

$$(\mathcal{E} + \bar{\mathcal{E}}) \left(\mathcal{E}_{\xi\bar{\xi}} - \frac{\mathcal{E}_{\xi} - \mathcal{E}_{\bar{\xi}}}{2(\xi - \bar{\xi})} \right) = 2\mathcal{E}_{\xi}\mathcal{E}_{\bar{\xi}} \quad (7)$$

To get from (1) the remaining equations of motion for the conformal factor h , we have to temporarily relax the conformal gauge and to vary (1) with respect to the off-diagonal elements of the metric. Restoring the conformal gauge then yields

$$(\log h)_{\xi} = \frac{\xi - \bar{\xi}}{4} \text{tr}(g_{\xi} g^{-1})^2 \quad \text{and c.c.} \quad (8)$$

It is well-known [6] that the same equations of motion arise in stationary axisymmetric reduction of the 4D Einstein equations. The quantum theory based on (1) may therefore be regarded as an example of the “midi-superspace” approximation to 4D quantum gravity.

3. DEFORMATION EQUATIONS AND τ -FUNCTION

Consider the following system of differential equations for 2×2 matrices $\{A_j(\xi, \bar{\xi})\}$ and the

functions $\{\gamma_j(\xi, \bar{\xi})\}$ with $j = 1, \dots, N$:

$$\frac{\partial A_j}{\partial \xi} = \frac{2}{\xi - \bar{\xi}} \sum_{k \neq j} \frac{[A_k, A_j]}{(1 - \gamma_k)(1 - \gamma_j)} \quad (9)$$

$$\frac{\partial A_j}{\partial \bar{\xi}} = \frac{2}{\bar{\xi} - \xi} \sum_{k \neq j} \frac{[A_k, A_j]}{(1 + \gamma_k)(1 + \gamma_j)}$$

$$\frac{\partial \gamma_j}{\partial \xi} = \frac{\gamma_j}{\xi - \bar{\xi}} \frac{1 + \gamma_j}{1 - \gamma_j} \quad (10)$$

$$\frac{\partial \gamma_j}{\partial \bar{\xi}} = \frac{\gamma_j}{\bar{\xi} - \xi} \frac{1 - \gamma_j}{1 + \gamma_j}$$

These (compatible) equations are solved by

$$\gamma_j = \frac{2}{\xi - \bar{\xi}} \times \left\{ w_j - \frac{\xi + \bar{\xi}}{2} \pm \sqrt{(w_j - \xi)(w_j - \bar{\xi})} \right\}, \quad (11)$$

where $w_j \in \mathbf{C}$ are constants of integration; in the sequel we shall assume γ_j to be defined by (11). One can easily check that the system (9) is always compatible if (10) holds.

Next define the τ -function $\tau(\xi, \bar{\xi})$ associated with (9) by

$$d \log \tau = \sum_{j < k} \text{tr}(A_j A_k) d \log(\gamma_j - \gamma_k), \quad (12)$$

where the differential is to be taken with respect to the variables $(\xi, \bar{\xi})$. Equivalently it can be computed with respect to the variables $\{\gamma_j\}$, which gives τ as a function of the parameters γ_j . Notice that the 1-form on the r.h.s. of (12) is always closed as a consequence of (9).

It is easy to check that $\text{tr} A_j$, $\text{tr} A_j^2$ and $\sum_{j=1}^N A_j$ are integrals of motion of the system (9).

Our purpose will be to exhibit the link between the system (9) with γ_j given by (11) and the equations of motion (5) and (8) of the previous section. A partial answer is given by

Theorem 1 *Let $\{A_j\}$ be an arbitrary solution of the system (9) with γ_j given by (11). Then*

1. *The system of equations*

$$g_\xi g^{-1} = \frac{2}{\xi - \bar{\xi}} \sum_j \frac{A_j}{1 - \gamma_j} \quad (13)$$

$$g_{\bar{\xi}} g^{-1} = \frac{2}{\bar{\xi} - \xi} \sum_j \frac{A_j}{1 + \gamma_j}$$

for the matrix-valued function $g(\xi, \bar{\xi}) \in GL(2, \mathbf{C})$ is always compatible.

2. *The solution $g(\xi, \bar{\xi})$ of this system satisfies equation (5).*

3. *The conformal factor h defined by (8) is related to the τ -function of the system (9) as follows:*

$$h = C(\xi - \bar{\xi})^{\frac{1}{2} \text{tr} A_\infty} \prod_j \left\{ \frac{\partial \gamma_j}{\partial w_j} \right\}^{\frac{1}{2} \text{tr} A_j^2} \tau \quad (14)$$

where $A_\infty \equiv \sum_{j=1}^N A_j$ and C is a constant.

To understand the precise correspondence between the solutions of (9) and the original model, one has to ensure the coset and reality conditions $g \in SL(2, \mathbf{R})/SO(2)$ and $h \in \mathbf{R}$. To this aim we define the rational function $A(\gamma)$ by

$$A(\gamma) = \sum_{j=1}^N \frac{A_j}{\gamma - \gamma_j}. \quad (15)$$

The proof of the following theorem may be found in [3].

Theorem 2 *Let $\{A_j\}$ be a solution of the system (9) satisfying the following additional conditions:*

1. *Reality:*

$$\overline{A(\gamma)} = -A(-\bar{\gamma}) \quad (16)$$

2. *Asymptotic regularity:*

$$A_\infty \equiv \sum_{j=1}^N A_j = 0 \quad (17)$$

3. *Invariance of the τ -function with respect to the involution $\gamma_j \rightarrow 1/\gamma_j$:*

$$\tau\left(\frac{1}{\gamma_1}, \dots, \frac{1}{\gamma_N}\right) = c_0 \tau(\gamma_1, \dots, \gamma_N) \quad (18)$$

with some constant $c_0 \neq 0$.

Then the constants of integration in (8) and (13) may be chosen in such a way that $h \in \mathbf{R}$ and $g \in SL(2, \mathbf{R})/SO(2)$.

At this point the relation between (9) and the original model may still appear obscure; it will be clarified in section 6. Let us just emphasize that the variables in the system (9) have been completely separated; thus we can treat the “left” (ξ) and “right” ($\bar{\xi}$) moving sectors as completely independent.

The link between the system (9) and the classical Schlesinger equations [7] for the variables A_j considered as functions of $\gamma_1, \dots, \gamma_N$

$$\begin{aligned} \frac{\partial A_j}{\partial \gamma_k} &= \frac{[A_j, A_k]}{\gamma_j - \gamma_k} \quad (k \neq j) \\ \frac{\partial A_j}{\partial \gamma_j} &= - \sum_{i \neq j} \frac{[A_j, A_i]}{\gamma_j - \gamma_i} \end{aligned} \quad (19)$$

is given by

Theorem 3 *Let $A_j(\{\gamma_j\})$ be a solution of the Schlesinger equations (19) satisfying the constraint (17). Then, assuming that all γ_j depend on $(\xi, \bar{\xi})$ according to (11), the functions $A_j(\xi, \bar{\xi})$ solve system (9).*

4. AN EXAMPLE: THE KERR-NUT SOLUTION

The general solution of the system (9) for arbitrary values of N and the parameters w_j is certainly not possible. However, one can try to understand which solutions of (9) correspond to known solutions of (7). For example, the Kerr-NUT solution of (7) corresponds to $N = 4$,

$$w_1 = w_3 = -\sigma \quad w_2 = w_4 = \sigma \quad \sigma \in \mathbf{R} \quad (20)$$

with $\gamma_3 = \gamma_1^{-1}$ and $\gamma_4 = \gamma_2^{-1}$. The integrals of motion $\text{tr} A_j^2$ should equal $1/2$ (since $\text{tr} A_j = 0$, this means that the eigenvalues of A_j are equal to $\pm 1/2$). It is not difficult to show that the solution $\{A_j\}$ of (9) satisfying these conditions and the constraints given by Thm.2 corresponds to the Ernst potential

$$\mathcal{E} = \frac{(\beta_2 - \beta_1)\mathcal{X} - (\beta_2 + \beta_1)\mathcal{Y} - 2}{(\beta_2 - \beta_1)\mathcal{X} - (\beta_2 + \beta_1)\mathcal{Y} + 2}, \quad (21)$$

where

$$\mathcal{X} = \frac{1}{2\sigma}\{S_1 + S_2\} \quad \mathcal{Y} = \frac{1}{2\sigma}\{S_1 - S_2\}, \quad (22)$$

with

$$\begin{aligned} S_1 &= \sqrt{(\xi + \sigma)(\bar{\xi} + \sigma)} \\ S_2 &= \sqrt{(\xi - \sigma)(\bar{\xi} - \sigma)} \end{aligned}$$

are prolate ellipsoidal coordinates; $\beta_{1,2}$ are complex constants satisfying $|\beta_1| = |\beta_2| = 1$. This is nothing but the Kerr-NUT solution; the Kerr solution itself corresponds to $\beta_2 = -\beta_1$.

5. TWO-TIME HAMILTONIAN STRUCTURE

We adopt here a “two-time” Hamiltonian formalism with the two “times” corresponding to the lightcone coordinates ξ and $\bar{\xi}$. One major advantage of this procedure is that the quantum theory is manifestly covariant under $2D$ coordinate transformations, a feature which is far from obvious (and possibly not even true) for the ADM formulation of canonical quantum gravity (see e.g. [8] for a recent discussion). Moreover, we must to treat the “times” ξ and $\bar{\xi}$ as phase space variables because they are really fields in a special gauge; then, according to the general canonical procedure, the related total Hamiltonians should weakly vanish, i.e. should be considered as first-class constraints.

The Hamiltonian structure which gives the complete set of equations of motion in terms of the variables $\{A_j\}$, $\xi, \bar{\xi}$, $(\log h)_\xi$ and $(\log h)_{\bar{\xi}}$ is described by the following

Theorem 4 *The system (8), (9) is a “two-time” Hamiltonian system with respect to the Poisson brackets*

$$\{A(\gamma) \otimes A(\mu)\} = [r, A(\gamma) \otimes \mathbf{1} + \mathbf{1} \otimes A(\mu)] \quad (23)$$

$$\{\xi, (\log h)_\xi\} = \{\bar{\xi}, (\log h)_{\bar{\xi}}\} = 1 \quad (24)$$

$$\{\bar{\xi}, (\log h)_\xi\} = \{\xi, (\log h)_{\bar{\xi}}\} = 0$$

$$\{A_j, (\log h)_\xi\} = \{A_j, (\log h)_{\bar{\xi}}\} = 0, \quad (25)$$

where $A(\gamma)$ is given by (15) and the classical rational R -matrix r is equal to

$$r(\gamma - \mu) = \frac{1}{\gamma - \mu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The mutually commuting Hamiltonian constraints in the ξ and $\bar{\xi}$ -directions are given by

$$\begin{aligned} \mathcal{C}^{(\xi)} &:= -(\log h)_\xi + \frac{1}{\xi - \bar{\xi}} \text{tr} A^2(1) \\ \mathcal{C}^{(\bar{\xi})} &:= -(\log h)_{\bar{\xi}} + \frac{1}{\bar{\xi} - \xi} \text{tr} A^2(-1) \end{aligned} \quad (26)$$

This theorem can be verified by direct calculation. The weak vanishing of $\mathcal{C}^{(\xi)}$ and $\mathcal{C}^{(\bar{\xi})}$ implies the equations (8) relating the gravitational and matter degrees of freedom. Commutativity of the Hamiltonian constraints may be obtained by use of the general relation

$$\left\{ \text{tr} A^2(\gamma), \text{tr} A^2(\mu) \right\} = 0, \quad (27)$$

which is valid for arbitrary γ and μ . The commutativity of the flows generated by $\mathcal{C}^{(\xi)}$ and $\mathcal{C}^{(\bar{\xi})}$ is equivalent to the decoupling of the classical equations of motion in (8) and (9) and may be viewed as a direct consequence of the compatibility of the system (8), (9). In terms of the standard “one time” canonical formalism with ρ as Euclidean time, the combination $\mathcal{C}^{(\rho)} = 1/2i(\mathcal{C}^{(\xi)} - \mathcal{C}^{(\bar{\xi})})$ corresponds to the Hamiltonian or Wheeler-DeWitt constraint while $1/2(\mathcal{C}^{(\xi)} + \mathcal{C}^{(\bar{\xi})})$ corresponds to the diffeomorphism constraint.

The “time evolutions” of any functional F are generated as usual by commutation with the total Hamiltonian constraints $\mathcal{C}^{(\xi)}$ and $\mathcal{C}^{(\bar{\xi})}$, i.e.

$$\frac{dF}{d\xi} = \{\mathcal{C}^{(\xi)}, F\} \quad \frac{dF}{d\bar{\xi}} = \{\mathcal{C}^{(\bar{\xi})}, F\}. \quad (28)$$

On the l.h.s. here we have the total derivatives with respect to ξ , $\bar{\xi}$; the first term of $\mathcal{C}^{(\xi)}$ or $\mathcal{C}^{(\bar{\xi})}$ generates the partial derivatives with respect to the coordinates and the second term takes care of the $(\xi, \bar{\xi})$ -dependence of A_j . Observe that we have $(\mathcal{C}^{(\xi)})^\dagger = \mathcal{C}^{(\bar{\xi})}$.

Defining

$$A_{j,\alpha\beta} := A_j^\alpha t_{\alpha\beta}^a, \quad (29)$$

where t^a are the generators of $SL(2, \mathbf{R})$, and inserting (15) into (23), we get

$$\{A_j^a, A_k^b\} = 2\delta_{jk} f_c^{ab} A_j^c,$$

where f_c^{ab} are the structure constants of $SL(2, \mathbf{R})$.

Observables in the sense of Dirac are by definition all those functionals \mathcal{O} on phase space which weakly commute with the constraints $\mathcal{C}^{(\xi)}$ and $\mathcal{C}^{(\bar{\xi})}$, but do not vanish on the constraint hypersurface $\mathcal{C}^{(\xi)} = \mathcal{C}^{(\bar{\xi})} = 0$, i.e.

$$\{\mathcal{C}^{(\xi)}, \mathcal{O}\} \approx 0, \quad \{\mathcal{C}^{(\bar{\xi})}, \mathcal{O}\} \approx 0. \quad (30)$$

By (28) the observables are independent of the coordinates and therefore highly non-local objects as one would expect on general grounds [9,10]. First of all, the parameters w_1, \dots, w_N trivially belong to this class since they commute with everything. Second, and more importantly, the monodromies M_1, \dots, M_N of the connection $A(\gamma)d\gamma$ defined by

$$M_j = \mathcal{P} \exp \oint_{l_j} A(\gamma)d\gamma, \quad (31)$$

where the contour l_j starts at $\gamma = \infty$ and encircles the point γ_j , are also observables for arbitrary N . For a discussion of this fact, see [3]. All observables can be generated from the set

$$Obs := \{w_1, \dots, w_N; M_1, \dots, M_N\} \quad (32)$$

by taking products and linear combinations. In this sense, Obs constitutes a complete set of classical (and quantum) observables for arbitrary N . These are the conserved “non-local charges” of dimensionally reduced gravity.

Notice also that the constraints mentioned in Thm.2 are in fact first class constraints with respect to our Poisson structure. In particular, the constraint $A_\infty = 0$ which closes into the $SL(2, \mathbf{R})$ algebra is nothing but the conserved charge which generates the Ehlers transformations $g \rightarrow Q^t g Q$ with a constant matrix $Q \in SL(2, \mathbf{R})$.

6. LINK TO CHERN SIMONS THEORY AND THE LINEAR SYSTEM

It is known that the Ernst equation can be obtained as the compatibility of a linear system [11,12]. The interpretation of the linear system as a zero curvature condition suggests a link with Chern Simons theory whose equations of motion also state the vanishing of some curvature. The new feature here is that the Chern Simons gauge connection lives on a space locally parameterized simultaneously by the spectral parameter and the true space time coordinate.

The relevant Chern Simons action (at level 1) reads

$$S = \frac{1}{4\pi i} \int \text{tr}(-A\partial_{\xi}A + 2A^{\xi}F)d\xi, \quad (33)$$

where ξ plays the role of time, $A = A^{\gamma}d\gamma + A^{\bar{\gamma}}d\bar{\gamma}$ is a time dependent connection 1-form on the Riemann surface locally coordinatized by $\gamma, \bar{\gamma}$, and $F \equiv F^{\gamma\bar{\gamma}}d\gamma d\bar{\gamma}$ is the curvature 2-form. The time component A^{ξ} appears as a Lagrangian multiplier for the first class constraints of vanishing curvature $F = 0$:

$$\{F^a(\gamma), F^b(\mu)\} = 2\pi i f^{abc} F^c(\gamma) \delta^{(2)}(\gamma - \mu). \quad (34)$$

In the usual treatment A^{ξ} is gauged to zero which leads to static components A^{γ} and $A^{\bar{\gamma}}$. In particular, the singularities of this connection are then time-independent and treated by inserting static Wilson-lines in the action (33). Alternatively, we consider the gauge

$$A^{\xi}(\gamma) = \frac{2A^{\gamma}(1) - \gamma(1 + \gamma)A^{\bar{\gamma}}(\gamma)}{(\xi - \bar{\xi})(1 - \gamma)}. \quad (35)$$

The residual gauge freedom corresponding to (34) is fixed by demanding

$$A^{\bar{\gamma}} = 0 \quad (36)$$

on the whole surface except for some set of zero measure. Because of (34) and $F = 0$ the remaining component A^{γ} then becomes holomorphic up to poles. To allow such singularities in A^{γ} as in the previous section, it is clear that (36) cannot be imposed everywhere because the singularities arising via the relation $\partial_{\bar{\gamma}} \frac{1}{\gamma} = 2\pi i \delta^{(2)}(\gamma)$ would spoil the constraint (34). Instead one should

think of $A^{\bar{\gamma}}$ as being localized on some string with endpoints at the singularities of A^{γ} .

The remaining equation of motion is

$$\frac{\partial A^{\xi}}{\partial \gamma} - \frac{\partial A^{\gamma}}{\partial \xi} + [A^{\xi}, A^{\gamma}] = 0 \quad (37)$$

The constraints can now be treated by introducing Dirac brackets. The original Poisson-bracket that comes from the action (33):

$$\{A^{\gamma a}(\gamma), A^{\bar{\gamma} b}(\mu)\} = 2\pi i \delta^{ab} \delta^{(2)}(\gamma - \mu) \quad (38)$$

is thereby changed to a bracket between the remaining meromorphic components $A^a(\gamma) \equiv A^{\gamma a}(\gamma)$ [14,15]:

$$\{A^a(\gamma), A^b(\mu)\} = -f^{abc} \frac{A^c(\gamma) - A^c(\mu)}{\gamma - \mu} \quad (39)$$

This may be translated into a bracket structure on the coefficients of the poles of A^{γ} , which — together with the positions of the poles — now parameterize the phase space:

$$\{A_i^a, A_j^b\} = 2\delta_{ij} f^{abc} A_j \quad (40)$$

for

$$A(\gamma) = \sum_j \frac{A_j}{\gamma - \gamma_j}$$

It coincides with the Poisson structure introduced in Thm. 4 of the previous section. The equations of motion (37) give rise to equations (9) and (10).

Among the surviving first class constraints is the total sum of the residues:

$$\int F(\gamma) = A_{\infty} \approx 0$$

as well as the Chern Simons Hamiltonian

$$C(\xi) = \frac{1}{2\pi i} \int \text{tr} A^{\xi} F, \quad (41)$$

which generates the equations of motion for the holomorphic component of the connection:

$$\partial_{\xi} A(\gamma) = \partial_{\gamma} A^{\xi}(\gamma) + [A^{\xi}(\gamma), A(\gamma)]$$

Splitting the Hamiltonian (41) it is now possible to identify its parts with the expressions obtained in the previous section. A short calculation reveals

$$\frac{1}{2\pi i} \text{tr} \int A^{\xi} \partial_{\bar{\gamma}} A^{\gamma} d\gamma d\bar{\gamma} = \frac{1}{\xi - \bar{\xi}} \text{tr} A(1)A(1),$$

such that defining

$$(\log h)_\xi \equiv \frac{1}{2\pi i} \text{tr} \int A^\xi \left(\partial_\gamma A^\gamma + [A^\gamma, A^\gamma] \right) d\gamma d\bar{\gamma},$$

we have

$$\{(\log h)_\xi, A^\gamma\} = \frac{\partial A^\xi}{\partial \gamma} \quad (42)$$

in agreement with (25) if A^γ is given by (15). All equations of motion are now generated by

$$\mathcal{C}^{(\xi)} = -(\log h)_\xi + \frac{1}{\xi - \bar{\xi}} \text{tr} A^2(1) \quad (43)$$

In this way the Poisson structure as well as the Hamiltonian and the constraints have a natural explanation in the context of Chern Simons theory. Similar considerations lead to the analogous results for the $\bar{\xi}$ -sector. However, further work is required to embed this two-time treatment in one unified canonical approach.

It is quite instructive to see how the well-known auxiliary linear system [11,12] arises in this framework. The analogous treatment of Chern Simons model in $(\gamma, \bar{\gamma}, \bar{\xi})$ space with the gauge choice

$$A^{\bar{\xi}}(\gamma) = \frac{2A^\gamma(-1) + \gamma(1 - \gamma)A^\gamma(\gamma)}{(\xi - \bar{\xi})(1 + \gamma)} \quad (44)$$

gives the equation of motion supplementing (37):

$$\frac{\partial A^{\bar{\xi}}}{\partial \gamma} - \frac{\partial A^\gamma}{\partial \bar{\xi}} + [A^{\bar{\xi}}, A^\gamma] = 0 \quad (45)$$

The vanishing of the curvatures (37) and (45) implies the existence of a gauge transformation $\Psi(\gamma; \xi, \bar{\xi})$ such that

$$\frac{\partial \Psi}{\partial \gamma} = A^\gamma \Psi \quad \frac{\partial \Psi}{\partial \xi} = A^\xi \Psi \quad \frac{\partial \Psi}{\partial \bar{\xi}} = A^{\bar{\xi}} \Psi \quad (46)$$

Substituting (35) and (44) into the last two equations and using (13) we get just the linear system of [12] with γ playing the role of the spectral parameter. The solutions of (5) for which A^γ can be represented as in (15) are called isomonodromic; in particular, they contain all known solutions such as multisoliton solutions [12] and the algebro-geometrical solutions of [13], as well as many others. Of course in assuming (15) we truncate the total phase space of the original model. We

would expect that there exists a topology on the space of solutions for which the isomonodromic solutions constitute a dense subset of the phase space of “all solutions” (notice that the Poisson structure given by (23), (24) and (42) is independent of the ansatz (15)).

7. QUANTIZATION

To quantize the model, we replace the Poisson brackets (23) by commutators in the usual fashion:

$$[A(\gamma) \otimes A(\mu)] = i\hbar \left[r, A(\gamma) \otimes \mathbf{1} + \mathbf{1} \otimes A(\mu) \right] \quad (47)$$

$$[\xi, (\log h)_\xi] = [\bar{\xi}, (\log h)_{\bar{\xi}}] = i\hbar \quad (48)$$

$$[\bar{\xi}, (\log h)_\xi] = [\xi, (\log h)_{\bar{\xi}}] = 0$$

Suppose now that all γ_j are imaginary (i.e. $w_j \in \mathbf{R}$); then by Thm.2 we should require all elements of A_j to be real at the classical level. Quantum mechanically we get

$$A_j \equiv \frac{i\hbar}{2} \begin{pmatrix} h_j & 2e_j \\ 2f_j & -h_j \end{pmatrix}, \quad (49)$$

where e_j, f_j, h_j are the anti-hermitian Chevalley generators of $SL(2, \mathbf{R})$ obeying the standard commutation relations

$$[h_j, e_j] = 2e_j \quad (50)$$

$$[h_j, f_j] = -2f_j$$

$$[e_j, f_j] = h_j$$

Unitary representations of (50) with Casimir operator

$$\frac{-4}{\hbar^2} \text{tr} A_j^2 = \frac{1}{2} h_j^2 + e_j f_j + f_j e_j$$

equal to $s_j(s_j - 2)$ are given by

$$e_j = \zeta_j^2 \frac{d}{d\zeta_j} + s_j \zeta_j \quad (51)$$

$$f_j = -\frac{d}{d\zeta_j}$$

$$h_j = 2\zeta_j \frac{d}{d\zeta_j} + s_j,$$

where $\{\zeta_j\}$ are the arguments of the functions spanning the representation space \mathcal{H}_j , which may

belong to the principal, supplementary or discrete series of $SL(2, \mathbf{R})$.

According to (48) one can choose

$$(\log h)_\xi = -i\hbar \frac{\partial}{\partial \xi} \quad (\log h)_{\bar{\xi}} = -i\hbar \frac{\partial}{\partial \bar{\xi}} \quad (52)$$

Thus the wave function Φ of a given isomonodromic sector with $w_j \in \mathbf{R}$ should depend on $(\xi, \bar{\xi})$ and live in the direct product

$$\mathcal{H}^{(N)} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$$

of N unitary representation spaces of $SL(2, \mathbf{R})$. This means that Φ can be realized as a function

$$\Phi \equiv \Phi(\xi, \bar{\xi}; \zeta_1, \dots, \zeta_N).$$

8. WHEELER-DEWITT EQUATIONS AND KNIZHNIK-ZAMOŁODCHIKOV SYSTEM FOR $SL(2, \mathbf{R})$

The Wheeler-DeWitt equations now take the form

$$\frac{d\Phi}{d\xi} = \frac{d\Phi}{d\bar{\xi}} = 0 \quad (53)$$

or, equivalently,

$$C^{(\xi)}\Phi = C^{(\bar{\xi})}\Phi = 0 \quad (54)$$

which can be written out by use of the explicit form of the constraints $C^{(\xi)}$ and $C^{(\bar{\xi})}$ given in (26) (49),(51) and (52):

$$\begin{aligned} \frac{\partial \Phi}{\partial \xi} &= -i\hbar \sum_{k \neq j} \frac{\Omega_{jk}}{(1-\gamma_j)(1-\gamma_k)} \Phi \\ \frac{\partial \Phi}{\partial \bar{\xi}} &= -i\hbar \sum_{k \neq j} \frac{\Omega_{jk}}{(1+\gamma_j)(1+\gamma_k)} \Phi \end{aligned} \quad (55)$$

where

$$\begin{aligned} \Omega_{jk} &\equiv \frac{1}{2} h_j h_k + e_j f_k + e_k f_j \\ &= -(\zeta_j - \zeta_k)^2 \frac{\partial^2}{\partial \zeta_j \partial \zeta_k} \\ &\quad + (\zeta_k - \zeta_j) \left(s_j \frac{\partial}{\partial \zeta_k} - s_k \frac{\partial}{\partial \zeta_j} \right) + \frac{s_j s_k}{2} \end{aligned} \quad (56)$$

According to Thm.2, the wave functionals satisfying the coset constraints should be symmetric

with respect to the involution $\gamma_j \rightarrow 1/\gamma_j$ and satisfy the constraint

$$\sum_{j=1}^N A_j \Phi = 0. \quad (57)$$

The general solution of the system (55) is not known. However, these equations turn out to be intimately related to the Knizhnik-Zamolodchikov system for $SL(2, \mathbf{R})$ [16,17]:

$$\frac{\partial \Phi_{KZ}}{\partial \gamma_j} = -i\hbar \sum_{k \neq j} \frac{\Omega_{jk}}{\gamma_j - \gamma_k} \Phi_{KZ} \quad (58)$$

with an $\mathcal{H}^{(N)}$ -valued function $\Phi_{KZ}(\xi, \bar{\xi})$.

Theorem 5 *If Φ_{KZ} is annihilated by the “total spin”*

$$\sum_{j=1}^N A_j \Phi_{KZ} = 0$$

and the γ_j depend on $(\xi, \bar{\xi})$ according to (11), then

$$\Phi = \prod_{j=1}^N \left(\frac{\partial \gamma_j}{\partial w_j} \right)^{-\frac{1}{4} \hbar^2 s_j (s_j - 2)} \Phi_{KZ} \quad (59)$$

solves the constraint (Wheeler DeWitt) equations (55).

Thus, the task of solving (55) reduces to the solution of (58).

The full set of quantum observables is related to the algebra of monodromies for the KZ equations (58) which is well understood only for $SU(2)$ where it gives rise to certain quantum groups [18].

The only solutions of KZ equations for the non-compact group $SL(2, \mathbf{R})$ known so far are solutions corresponding to the unitary discrete series representations (either positive for all j or negative for all j) all of which possess a ground (lowest weight) state in $\mathcal{H}^{(N)}$. However, it is possible to show [3] that solutions of this kind cannot satisfy the constraint (57). Moreover, a simple analysis of the sign of the Casimir operator shows that in order to construct wave functions corresponding to physically interesting classical solutions (such as Kerr-NUT) one would have to consider representations of the continuous series.

Namely, for all known classical solutions (including Kerr-NUT) we have $\text{tr}A_j^2 > 0$. However, in the quantum case,

$$\text{tr}A_j^2 = -\frac{\hbar^2}{2}s(s-2)$$

For the discrete series, when s is real and integer, this is always negative. For the continuous series we have $s = 1 + iq$ with $q \in \mathbf{R}$, and the eigenvalue of $\text{tr}A_j^2$ is positive.

In the next section we shall briefly discuss how one might go about constructing a quantum state whose semiclassical limit would reduce to the Kerr-NUT solution.

9. TOWARDS A QUANTUM KERR SOLUTION

According to the previous section, the desired solution of (58) for $N = 4$ is a function of the four positions of the poles γ_i , defined by (20) and of the four auxiliary variables ζ_i , on which the algebra $sl(2, \mathbf{R})$ is represented.

The constraints (57) have to annihilate this function which hence is $SL(2, \mathbf{R})$ invariant. Therefore it essentially depends only on the ratio

$$\frac{(\zeta_1 - \zeta_2)(\zeta_3 - \zeta_4)}{(\zeta_1 - \zeta_4)(\zeta_3 - \zeta_2)}$$

just as in conformal field theory where the conformal Ward identities reduce the correlation functions to a function of a single variable [20]. Moreover, the validity of the KZ equations implies an analogous reduction of the γ_i -dependence.

The quantum state then reduces to the following form:

$$\Phi = \prod_{j=1}^4 \left(\frac{\partial \gamma_j}{\partial w_j} \right)^{-\frac{1}{2}\hbar^2 s(s-2)} F(\zeta_i, \gamma_i), \quad (60)$$

with

$$\begin{aligned} s &= s_1 = s_2 = s_3 = s_4 \\ F(\zeta_i, \gamma_i) &= (\gamma_1 - \gamma_4)^{-\Delta} (\gamma_2 - \gamma_3)^{-\Delta} \times \\ &\quad (\zeta_1 - \zeta_4)^{-s} (\zeta_2 - \zeta_3)^{-s} G(x, y), \end{aligned}$$

$$x = \frac{(\zeta_1 - \zeta_2)(\zeta_3 - \zeta_4)}{(\zeta_1 - \zeta_4)(\zeta_3 - \zeta_2)}$$

$$\begin{aligned} y &= \frac{(\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4)}{(\gamma_1 - \gamma_4)(\gamma_3 - \gamma_2)} \\ \Delta &= \frac{i\hbar}{2}s(s-2) \end{aligned}$$

The remaining KZ equation for the function G can be obtained by a lengthy but straightforward calculation which gives

$$\partial_y G(x, y) = i\hbar \left(\frac{D(x)}{y} - \frac{D(1-x)}{1-y} \right) G(x, y) \quad (61)$$

with

$$\begin{aligned} D(x) &= x^2(1-x)\partial_x^2 + 2sx(1-x)\partial_x \\ &\quad + \frac{1}{2}s^2(1-2x) \end{aligned}$$

An equivalent form of this equation appeared in the study of four-point correlation-functions in Liouville theory [19].

Equation (61) is very similar to the standard hypergeometric equation, where $D(x)$ and $D(1-x)$ are just two 2×2 matrices. The singular points $y = 0, 1, \infty$ have a very definite physical meaning from the point of view of the classical Kerr solution. Namely, we can express the variable y in terms of prolate ellipsoidal coordinates (22) as follows:

$$y = \frac{1 - \mathcal{Y}^2}{1 - \mathcal{X}^2} = -\frac{\rho^2}{\sigma^2(\mathcal{X}^2 - 1)^2}$$

This shows that $y = 0$ corresponds classically to the spatial infinity and the part of the symmetry axis outside of the event horizon. The value $y = 1$ corresponds to the poles of the event horizon, and $y = \infty$ corresponds to the surface of the event horizon.

The analysis of equation (61) should give asymptotical expansions of the wave functional at these singular points and allow us to relate them. This would then enable us to understand the behavior of physically interesting expectation values at these points and to clarify the meaning and the fate of the classical singularities in the quantum theory.

The classical limit leading to the Kerr-NUT solution should look like

$$\frac{\hbar^2}{4}(1+q^2) \rightarrow 1$$

If this limit is equal to an integer k , the related classical solution should be the k th member of the Tomimatsu-Sato hierarchy.

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REFERENCES

1. D. Korotkin and H. Nicolai, Phys. Rev. Lett. **74** (1995) 1272.
2. D. Korotkin and H. Nicolai, Phys. Lett. **356B** (1995) 211.
3. D. Korotkin and H. Nicolai, *Isomonodromic quantization of dimensionally reduced gravity*, to appear
4. L. Faddeev, in *Les Houches, Session XXXIX*, eds. J.-B. Zuber and R. Stora, Elsevier Science Publishers B.V., 1984.
5. F. Ernst, Phys. Rev. **167** (1968) 1175.
6. D. Kramer, H. Stephani, E. Herlt and M. MacCallum, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, 1980.
7. L. Schlesinger, J.Reine u. Angew. Math. **141** (1912) 76.
8. N. P. Landsman, *Against the Wheeler-DeWitt equation*, preprint gr-qc/9510033;
H.J. Matschull, *Causal structure and diffeomorphisms in Ashtekar's gravity*, preprint KCL-TH-95-10, gr-qc/9511034.
9. C. Isham, in *Recent Aspects of Quantum Fields*, eds. H. Mitter and H. Gausterer, Springer Verlag, 1991.
10. A. Ashtekar, in *Gravitation and Quantizations*, eds. B. Julia and J. Zinn Justin, North Holland Publishing Company, 1995.
11. D. Maison, Phys. Rev. Lett. **41** (1978) 521.
12. V. Belinskii and V. Zakharov, Sov. Phys. JETP **48** (1978) 985.
13. D. Korotkin, Theor. Math. Phys, **77** (1989) 1018;
D. Korotkin and V. Matveev, St.Petersburg Math. J. **1** (1990) 379.
14. V. Fock and A. Rosly, *Poisson structure on the moduli space of flat connections on Riemann surfaces and r-matrix*, preprint ITEP 72-92, June 1992, Moscow.
15. D. Korotkin and H. Samtleben, *On the quantization of isomonodromic deformations on the torus*, preprint DESY 95-202, hep-th/9511087.
16. V. Knizhnik and A. Zamolodchikov, Nucl. Phys. **B247** (1984) 83.
17. A. M. Polyakov, Mod. Phys. Lett. A **2** No.11 (1987) 893.
18. V.G. Drinfeld in: *Problems of modern quantum field theory*, Proceedings Alushta 1989, Research reports in physics, 1989, Springer, Heidelberg;
C.Kassel, *Quantum groups*, 1995, Springer, New York.
19. A.B. Zamolodchikov and V.A. Fateev, Yad. Fiz. **43** (1986) 1031.
20. A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333.