

THE SUGAWARA GENERATORS AT ARBITRARY LEVEL*

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Abstract: We construct an explicit representation of the Sugawara generators for arbitrary level in terms of the homogeneous Heisenberg subalgebra, which generalizes the well-known expression at level 1. This is achieved by employing a physical vertex operator realization of the affine algebra at arbitrary level, in contrast to the Frenkel–Kac–Segal construction which uses unphysical oscillators and is restricted to level 1. At higher level, the new operators are transcendental functions of DDF “oscillators” unlike the quadratic expressions for the level-1 generators. An essential new feature of our construction is the appearance, beyond level 1, of new types of poles in the operator product expansions in addition to the ones at coincident points, which entail (controllable) non-localities in our formulas. We demonstrate the utility of the new formalism by explicitly working out some higher-level examples. Our results have important implications for the problem of constructing explicit representations for higher-level root spaces of hyperbolic Kac–Moody algebras, and E_{10} in particular.

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1 Introduction

The Sugawara construction [18] and the GKO construction [8] have both come to play a prominent role in string theory and in the theory of Kac–Moody algebras (see e.g. [13, 15] for the general theory). As is well known, the Sugawara construction extends a representation of an affine Lie algebra \mathfrak{g} to that of its semidirect product with the Virasoro algebra $\text{Vir}^{\mathfrak{g}}$. The GKO construction, in turn, is based on the Sugawara construction: given an affine subalgebra $\mathfrak{p} \subset \mathfrak{g}$, there always exists another Virasoro algebra corresponding to the difference of Virasoro operators associated with \mathfrak{g} and \mathfrak{p} , respectively, such that the resulting coset Virasoro algebra $\text{Vir}^{\mathfrak{g}:\mathfrak{p}}$ commutes with the affine subalgebra \mathfrak{p} . It is for this reason that the GKO construction, which was originally developed for the explicit description of $c < 1$ Virasoro modules [8], has acquired great importance in the representation theory of affine algebras [14]. More specifically, every highest weight representation $L(\Lambda)$ of \mathfrak{g} can be decomposed w.r.t. the direct sum $\mathfrak{p} \oplus \text{Vir}^{\mathfrak{g}:\mathfrak{p}}$ as follows:

$$L(\Lambda) = \bigoplus_{\lambda \in \mathcal{P}^{\mathfrak{g}:\mathfrak{p}}} L^{\mathfrak{p}}(\lambda) \otimes U(\Lambda, \lambda). \quad (1.1)$$

The relevant specialization of this formula for tensor products of \mathfrak{g} modules is obtained by taking \mathfrak{p} to be the diagonal subalgebra of $\mathfrak{g} \otimes \mathbf{1} \oplus \mathbf{1} \otimes \mathfrak{g}$. In this case (1.1) becomes

$$L(\Lambda) \otimes L(\Lambda') = \bigoplus_j L(\Lambda_j) \otimes V(c_j, h_j), \quad (1.2)$$

where $V(c_j, h_j)$ is the Virasoro module with central charge c_j and highest weight h_j . Unfortunately, it is not so easy in general to compute the products (1.2) in practice. For simple examples, where $c_j < 1$ and unitarity restricts h_j to a finite set of allowed values, one can work out the product explicitly. However, at higher level the right-hand side of (1.2) will contain many terms; furthermore, these will in general correspond to Virasoro Verma modules with central charge $c_j > 1$ where the values of h_j are unrestricted (apart from $h_j \geq 0$). In order to better understand these representations it is thus desirable to find an explicit and manageable representation for the coset generators. A necessary prerequisite for this is the construction of an explicit representation for the higher-level Sugawara operators themselves (the explicit representation at level 1 is well known, see formula (3.1) below). This is the problem which we address and solve in the present paper.

Before going into the details we would like to briefly explain the new features of our construction. The famous FKS vertex operator realization [5, 16] of nontwisted affine Lie algebras corresponds to a spatially compactified bosonic string whose momentum lattice is taken to be the (Euclidean) root lattice of a finite-dimensional simple Lie algebra of ADE type. The Laurent coefficients (“modes”) of the tachyon vertex operators together with the string oscillators then constitute a basis of the affine algebra. This basis, however, is not physical in the sense of string theory since except for the zero mode these mode operators do not commute with the Virasoro constraints. Furthermore, the FKS construction has the drawback of being restricted to affine algebras at level 1. In [4, 9], it was noticed that if the momentum lattice of the string is enlarged by a two-dimensional Minkowski lattice then the zero mode operators by themselves already lead to a basis of the affine algebra which agrees with the FKS realization when the operator-valued expression $e^{in\delta \cdot \mathbf{X}(z)}$ is formally replaced by z^n . Our starting point was the observation that apart from being manifestly physical, this construction is applicable to affine Lie algebras at arbitrary level and thus more general than the FKS construction (this fact is apparently not widely known).

In the usual vertex operator formalism the computation of Lie algebra commutators is reduced to the evaluation of the singular terms in the expansion of certain operator products at coincident points. By contrast, we here find that, beyond level 1, additional poles appear at the non-coincident points $z = w_p := \zeta^p w$ in the expansion of the product of two conformal operators supported at z and w , where ζ is a primitive ℓ -th root of unity. A second unusual feature is that we are led to introduce a new “transversal coordinate” field, which has the form of the old Fubini-Veneziano field, but for which the usual string oscillators are replaced by the level- ℓ DDF operators, see Eq. (2.17) below. The final expression for the general level- ℓ Sugawara operators (cf. (3.15) below) involves an exponential dependence on this new field; since the DDF operators themselves are defined by exponentials of the string oscillators, our construction may thus be termed “doubly transcendental”. Moreover it is *non-local* in the sense that the integrand in (3.15) below depends both on w and w_p . This (controllable) non-locality accounts for a number of complications, such as the fact that the level- ℓ Sugawara operators contain products of the DDF operators of arbitrary order (depending on the state on which they act) whereas the level-1 Sugawara operators are always quadratic.

The present work is mainly motivated by and continues our previous investigation of hyperbolic Kac–Moody algebras corresponding to the canonical extensions of affine algebras by an over-extended root \mathbf{r}_{-1} [7, 6]. There

an attempt was made to understand the structure of such algebras, and in particular the maximally extended algebra E_{10} , via a novel realization in terms of DDF states which enabled us to give a simple and explicit representation for a non-trivial level-2 root space of E_{10} corresponding to a 75-fold multiple commutator of the Chevalley–Serre generators for the first time (meanwhile, further examples have been worked out). These results explicitly demonstrate the occurrence of *longitudinal* states for levels $|\ell| \geq 2$ and the simultaneous decoupling of certain transversal states, whereas the level ± 1 sectors can be simply realized as the set of purely transversal states [7, 4] (the level-0 sector is just the affine subalgebra). Let us recall that the higher-level elements of the algebra can be recursively defined as multiple commutators of level-1 elements. In principle one should thus be able to understand them from a representation theoretic point of view by analyzing multiple products of level-1 representations. However, a first difficulty here is that one must discard all those multiple commutators (and hence the corresponding affine representations) which contain the Serre relations somewhere inside. This difficulty is invisible in the string vertex algebra realization [1], which takes automatic care of the Serre relations (since there are no physical string states below the tachyon), but the tribute to this convenience is the phenomenon of “missing” (or “decoupled”) states, i.e., physical string states that can *not* be reached by multiple commutation of the Chevalley–Serre generators [7]. The second difficulty — already alluded to above — is that there is no general method for efficiently computing the relevant products of representations in practice. So we see again that we must gain a better understanding of the higher-level Sugawara generators. Although the challenge of finding explicit formulas for the coset Virasoro generators remains, we believe that the present results bring us one step closer towards the ambitious goal of finding a concrete realization of hyperbolic Kac–Moody algebras. With this future application in mind, we will give explicit examples of the new formula only for some special representations of E_9 which arise in the analysis of the hyperbolic algebra E_{10} .

Apart from the new structural insights afforded by the new formula, our results illustrate to what degree the higher-level sectors of hyperbolic Kac–Moody algebras are more complicated than the low-level sectors. This is acutely evident from the increasing “anisotropy” of the higher-level root spaces, which was already observed in [7], and which is now (partially) explained by the symmetry breaking of the full (affine) Weyl group down to a finite (and generically trivial) subgroup called “little Weyl group” in [7]. From a more technical perspective this phenomenon is due to the appearance of certain weighted sums of tensor products of roots (see (5.4)) which have not yet appeared in the literature to the best of our knowledge. We would like to emphasize that these and other special features of hyperbolic Kac–Moody algebras can *not* be explained by string compactification alone. In other words, such algebras reveal an enigma beyond the string vertex operator construction. By contrast, many of the generalized Kac–Moody (super)algebras which have recently received attention (see e.g. [2, 11, 12]) can presumably be realized as untruncated, hence *bona fide*, algebras of physical vertex operators of some compactified string. For instance, the so-called fake monster Lie algebra is just the algebra of *all* transversal physical states of the bosonic string compactified on $H_{25,1}$; its root spaces are therefore perfectly “isotropic” and the associated root multiplicities are simply given by the number of physical string states.

2 Preliminaries

We consider a nontwisted affine Lie algebra \mathfrak{g} with underlying simple finite-dimensional Lie algebra $\bar{\mathfrak{g}}$ of rank $d - 2$ ($d \geq 3$). The associated hyperbolic Kac–Moody algebra $\hat{\mathfrak{g}}$ of rank d is obtained by adjoining to the affine Dynkin diagram another node which is related to the over-extended simple root \mathbf{r}_{-1} . The extended (Minkowskian) affine root lattice is defined as $\hat{Q} := Q \oplus \mathbb{Z}\mathbf{\Lambda}_0$ (viewed as the even sublattice of the affine weight lattice), where Q denotes the affine root lattice and $\mathbf{\Lambda}_0 := \mathbf{r}_{-1} + \boldsymbol{\delta}$ is a null vector conjugate to the affine null root $\boldsymbol{\delta}$. Clearly, \hat{Q} is just the hyperbolic root lattice. For any element $\mathbf{\Lambda}$ of \hat{Q} , the level ℓ is defined by

$$\ell := -\mathbf{\Lambda} \cdot \boldsymbol{\delta}. \quad (2.1)$$

Now suppose that $\mathbf{\Lambda} \in \hat{Q}$ is a root of $\hat{\mathfrak{g}}$ of nonzero level. The DDF decomposition of $\mathbf{\Lambda}$ [7] is defined by

$$\mathbf{\Lambda} = \mathbf{a} - n\mathbf{k}_\ell, \quad (2.2)$$

where we have put

$$\mathbf{k}_\ell := -\frac{1}{\ell}\boldsymbol{\delta} \in Q_\mathbb{Q} := \mathbb{Q} \otimes_{\mathbb{Z}} Q, \quad (2.3)$$

and where the vector \mathbf{a} is uniquely determined by requiring $\mathbf{a}^2 = 2$, i.e., $n := 1 - \frac{1}{2}\mathbf{\Lambda}^2$. Note that n is always a non-negative integer since $\mathbf{\Lambda}^2 \leq 2$ for any root. We will refer to \mathbf{a} as the ‘tachyonic level- ℓ vector’ and to $|\mathbf{a}\rangle$

as the ‘tachyonic level- ℓ state’ associated to $\mathbf{\Lambda}$; beyond level 1, it need not be an element of the lattice \hat{Q} but only of its rational extension. Let us furthermore introduce the orthonormal polarization vectors $\xi_i \equiv \xi_i(\mathbf{a})$ satisfying $\xi_i \cdot \xi_j = \delta_{ij}$ and $\xi_i \cdot \delta = \xi_i \cdot \mathbf{\Lambda} = \xi_i \cdot \mathbf{a} = 0$. They constitute a basis of the complex vector space $\bar{\mathfrak{h}}^*$ dual to the Cartan subalgebra $\bar{\mathfrak{h}}$ of $\bar{\mathfrak{g}}$. Then we define the operators

$${}^{[\ell]}A_m^i(\mathbf{a}) := \oint \frac{dz}{2\pi i} \xi_i(\mathbf{a}) \cdot \mathbf{P}(z) e^{im\mathbf{k}_\ell \cdot \mathbf{X}(z)} \quad (2.4)$$

$${}^{[\ell]}E_m^{\mathbf{r}} := \oint \frac{dz}{2\pi i} :e^{i(\mathbf{r}+m\mathbf{k}_\ell) \cdot \mathbf{X}(z)}:, \quad (2.5)$$

for $m \in \mathbb{Z}$, $1 \leq i \leq d-2$, $\mathbf{r} \in \bar{\Delta}$ (roots of $\bar{\mathfrak{g}}$). Here we have used the well-known Fubini–Veneziano coordinate and momentum fields, respectively,

$$X^\mu(z) := q^\mu - ip^\mu \ln z + i \sum_{m \neq 0} \frac{1}{m} \alpha_m^\mu z^{-m}, \quad (2.6)$$

$$P^\mu(z) := i \frac{d}{dz} X^\mu(z) = \sum_{m \in \mathbb{Z}} \alpha_m^\mu z^{-m-1}, \quad (2.7)$$

expressed in terms of the string oscillators α_m^μ ($m \in \mathbb{Z}$, $1 \leq \mu \leq d$),

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu} \delta_{m+n,0}. \quad (2.8)$$

The shift of any polarization vector $\xi_i(\mathbf{a})$ along the δ direction leaves the associated DDF operator ${}^{[\ell]}A_m^i(\mathbf{a})$ unchanged for $m \neq 0$, because the residue of a total derivative always vanishes. Since the polarization vectors of two tachyonic level- ℓ states are related by

$$\xi_i(\mathbf{a}') = \xi_i(\mathbf{a}) + \frac{1}{\ell} (\xi_i(\mathbf{a}) \cdot \mathbf{a}') \delta, \quad (2.9)$$

we are thus effectively dealing with a single set of DDF operators for $m \neq 0$,

$${}^{[\ell]}A_m^i \equiv {}^{[\ell]}A_m^i(\mathbf{a}) = {}^{[\ell]}A_m^i(\mathbf{a}'); \quad (2.10)$$

so we can suppress the labels \mathbf{a} , \mathbf{a}' in the remainder, i.e. write ${}^{[\ell]}A_m^i \equiv {}^{[\ell]}A_m^i(\mathbf{a})$. Let us stress, however, that the zero mode operators do differ for different \mathbf{a} , viz.

$${}^{[\ell]}A_0^i(\mathbf{a}) = \xi_i(\mathbf{a}) \cdot \mathbf{p} = \xi_i(\mathbf{a}') \cdot \mathbf{p} - \frac{1}{\ell} (\xi_i(\mathbf{a}) \cdot \mathbf{a}') \delta \cdot \mathbf{p} = {}^{[\ell]}A_0^i(\mathbf{a}') - \frac{1}{\ell} (\xi_i(\mathbf{a}) \cdot \mathbf{a}') \delta \cdot \mathbf{p}. \quad (2.11)$$

For definiteness, we choose the polarization vectors to be $\xi_i(\mathbf{\Lambda}_0)$ throughout this paper, where $\mathbf{\Lambda}_0$ denotes the above fundamental dominant weight of level 1 with tachyonic level-1 vector $\mathbf{a}_0 = \mathbf{\Lambda}_0 - 2\delta \in \hat{Q}$. This vector \mathbf{a}_0 plays the role of the simple root \mathbf{r}_{-1} occurring in the canonical extension of \mathfrak{g} to the hyperbolic Kac–Moody algebra $\hat{\mathfrak{g}}$ with \hat{Q} as root lattice. Last not least we have the obvious relation

$${}^{[1]}A_m^i(\mathbf{a}) = {}^{[\ell]}A_{\ell m}^i(\mathbf{a}). \quad (2.12)$$

The above operators obey the commutation relations

$$[{}^{[\ell]}A_m^i, {}^{[\ell]}A_n^j] = m\delta^{ij} \delta_{m+n,0} {}^{[\ell]}K_0, \quad (2.13)$$

$$[{}^{[\ell]}A_m^i, {}^{[\ell]}E_n^{\mathbf{r}}] = (\xi_i \cdot \mathbf{r}) {}^{[\ell]}E_{m+n}^{\mathbf{r}}, \quad (2.14)$$

$$[{}^{[\ell]}E_m^{\mathbf{r}}, {}^{[\ell]}E_n^{\mathbf{s}}] = \begin{cases} 0 & \text{if } \mathbf{r} \cdot \mathbf{s} \geq 0, \\ \epsilon(\mathbf{r}, \mathbf{s}) {}^{[\ell]}E_{m+n}^{\mathbf{r}+\mathbf{s}} & \text{if } \mathbf{r} \cdot \mathbf{s} = -1, \\ -{}^{[\ell]}A_{m+n}^{\mathbf{r}} - m\delta_{m+n,0} {}^{[\ell]}K_0 & \text{if } \mathbf{r} \cdot \mathbf{s} = -2, \end{cases} \quad (2.15)$$

where ${}^{[\ell]}K_0 := \mathbf{k}_\ell \cdot \mathbf{p} = -\frac{1}{\ell} \delta \cdot \mathbf{p}$ denotes the operator realization of the central element of the affine algebra and

$${}^{[\ell]}A_m^{\mathbf{r}} := \oint \frac{dz}{2\pi i} \mathbf{r} \cdot \mathbf{P}(z) e^{im\mathbf{k}_\ell \cdot \mathbf{X}(z)} = \sum_i (\xi_i \cdot \mathbf{r}) {}^{[\ell]}A_m^i \quad \forall \mathbf{r} \in \bar{\Delta}.$$

As usual, we have to extend the Cartan subalgebra by an exterior derivative which we choose to be ${}^{[\ell]}d_0 := \ell(\mathbf{\Lambda}_0 + \boldsymbol{\delta}) \cdot \mathbf{p}$ for the basic (level 1) fundamental weight $\mathbf{\Lambda}_0$ (note that $(\mathbf{\Lambda}_0 + \boldsymbol{\delta})^2 = 0$).

The operators ${}^{[1]}K_0, {}^{[1]}d_0, {}^{[1]}A_m^i, {}^{[1]}E_m^{\mathbf{r}}$ ($1 \leq i \leq d-2, \mathbf{r} \in \bar{\Delta}, m \in \mathbb{Z}$) establish a realization of the affine Lie algebra \mathfrak{g} , the level being given by the eigenvalue of the operator ${}^{[1]}K_0$. Note that in contrast to the FKS construction this vertex operator realization works for *arbitrary level* and is *physical* in the sense of string theory, i.e.

$$[L_m, {}^{[1]}K_0] = [L_m, {}^{[1]}d_0] = [L_m, {}^{[1]}A_n^i] = [L_m, {}^{[1]}E_n^{\mathbf{r}}] = 0 \quad \forall m, n \in \mathbb{Z}, 1 \leq i \leq d-2, \mathbf{r} \in \bar{\Delta},$$

where the operators

$$L_m := \frac{1}{2} \sum_{n \in \mathbb{Z}} : \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_{m-n} :. \quad (2.16)$$

satisfy the standard Virasoro algebra with central charge $c = d$. There is yet another realization of the affine Lie algebra which is, however, restricted to level 1. Namely, on states with eigenvalue ℓ for ${}^{[1]}K_0$, the operators ${}^{[\ell]}K_0, {}^{[\ell]}d_0, {}^{[\ell]}A_m^i, {}^{[\ell]}E_m^{\mathbf{r}}$ ($1 \leq i \leq d-2, \mathbf{r} \in \bar{\Delta}, m \in \mathbb{Z}$) form a level-1 realization of \mathfrak{g} which is also physical.

Since we are working with the so-called homogeneous vertex operator construction we will refer to the algebra of operators ${}^{[\ell]}A_m^i$ as the **homogeneous Heisenberg subalgebra** of the affine algebra. The crucial observation for our analysis is that these operators not only occur as part of the affine algebra but also as part of the spectrum generating algebra for the physical string states. In this context, they are nothing but the well-known transversal DDF operators. A crucial new feature of our analysis is the appearance of the level- ℓ **transversal coordinate field**

$${}^{[\ell]}\mathcal{X}^i(z) := q^i - ip^i \ln z + i \sum_{m \neq 0} \frac{1}{m} {}^{[\ell]}A_m^i z^{-m}, \quad (2.17)$$

and the level- ℓ **transversal momentum field**

$${}^{[\ell]}\mathcal{P}^i(z) := iz \frac{d}{dz} {}^{[\ell]}\mathcal{X}^i(z) = \sum_{m \in \mathbb{Z}} {}^{[\ell]}A_m^i z^{-m}, \quad (2.18)$$

respectively, neither of which has appeared in the literature so far. Evidently, these fields are transcendental expressions in terms of the standard oscillator basis. The momentum field (2.18) is physical because it commutes with the Virasoro constraints term by term. This is not quite true of (2.17) due to the presence of the center of mass coordinate q^i in it; however, in all relevant expressions below we will be dealing with the fields

$${}^{[\ell]}\mathcal{Y}_p^i(z) := {}^{[\ell]}\mathcal{X}^i(z_p) - {}^{[\ell]}\mathcal{X}^i(z), \quad p = 1, \dots, \ell - 1, \quad (2.19)$$

where $z_p := \zeta^p z$ and ζ denotes a primitive ℓ -th root of unity. These fields are physical since the zero mode q^i drops out.

The Sugawara generators built from the affine Cartan–Weyl basis (2.13)-(2.15) are

$$\mathcal{L}_m^{[\ell]} := \frac{1}{2(\ell + h^\vee)} \sum_{n \in \mathbb{Z}} \left(\sum_{i=1}^{d-2} \times {}^{[1]}A_n^i \times {}^{[1]}A_{m-n}^i \times + \sum_{\mathbf{r} \in \bar{\Delta}} \times {}^{[1]}E_n^{\mathbf{r}} \times {}^{[1]}E_{m-n}^{-\mathbf{r}} \times \right), \quad (2.20)$$

where h^\vee denotes the dual Coxeter number of $\bar{\mathfrak{g}}$. Normal-ordering is defined by

$$\times {}^{[1]}A_m^i \times \times {}^{[1]}A_n^j \times := \begin{cases} {}^{[1]}A_m^i {}^{[1]}A_n^j & \text{for } m \leq n, \\ {}^{[1]}A_n^j {}^{[1]}A_m^i & \text{for } m > n, \end{cases} \quad (2.21)$$

$$\times {}^{[1]}E_m^{\mathbf{r}} \times \times {}^{[1]}E_n^{\mathbf{s}} \times := \begin{cases} {}^{[1]}E_m^{\mathbf{r}} {}^{[1]}E_n^{\mathbf{s}} & \text{for } m \leq n, \\ {}^{[1]}E_n^{\mathbf{s}} {}^{[1]}E_m^{\mathbf{r}} & \text{for } m > n. \end{cases} \quad (2.22)$$

It is well known that the operators $\mathcal{L}_m^{[\ell]}$, $m \in \mathbb{Z}$, form a Virasoro algebra (see e.g. [10] and references therein),

$$[\mathcal{L}_m^{[\ell]}, \mathcal{L}_n^{[\ell]}] = (m - n) \mathcal{L}_{m+n}^{[\ell]} + \frac{c(\ell)}{12} (m^3 - m) \delta_{m+n,0} {}^{[\ell]}K_0, \quad (2.23)$$

with central charge

$$c(\ell) := \frac{\ell \dim \bar{\mathfrak{g}}}{\ell + h^\vee}. \quad (2.24)$$

These operators act as outer derivations on the affine algebra so that one obtains a semidirect product $\text{Vir}_{\mathcal{L}^{[\ell]}} \ltimes \mathfrak{g}$:

$$[\mathcal{L}_m^{[\ell]}, {}^{[1]}A_n^i] = -n {}^{[1]}A_{m+n}^i, \quad [\mathcal{L}_m^{[\ell]}, {}^{[1]}E_n^{\mathbf{r}}] = -n {}^{[1]}E_{m+n}^{\mathbf{r}}. \quad (2.25)$$

By construction, the Sugawara generators are physical, viz.

$$[\mathcal{L}_m^{[\ell]}, L_n] = 0 \quad \forall m, n \in \mathbb{Z}. \quad (2.26)$$

Thus the above semidirect product is a symmetry of the physical string spectrum, whereas in the FKS approach only the full Fock space carries a (level-1) representation of the affine algebra. It should be mentioned that in addition to the operators $\mathcal{L}_m^{[\ell]}$, there is another infinity of ‘‘physical’’ Virasoro algebras (but with uniform central charge $c = 26 - d$) generated by the longitudinal DDF operators ${}^{[\ell]}A_m^-(\mathbf{a})$, all of which commute with the Sugawara generators (2.20). However, we will not elaborate on this point here; for further information, the interested reader may consult [7].

3 The main formula

Our aim is to rewrite the Sugawara generators $\mathcal{L}_m^{[\ell]}$ in terms of the homogeneous Heisenberg subalgebra spanned by the ${}^{[\ell]}A_m^i$'s. This will be the generalization of the well-known result

$$\mathcal{L}_m^{[1]} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{i=1}^{d-2} \times {}^{[1]}A_n^i {}^{[1]}A_{m-n}^i \times, \quad (3.1)$$

which is referred to in the literature as ‘the equivalence of the Sugawara and the Virasoro construction at level 1’.¹ For this purpose, we wish to evaluate the operator products occurring in the second part of the Sugawara generators. We start from the well-known formulas

$$\begin{aligned} :e^{i(\mathbf{r}+n\boldsymbol{\delta})\cdot\mathbf{X}(z)}: :e^{-i(\mathbf{r}+(m+n)\boldsymbol{\delta})\cdot\mathbf{X}(w)}: &= (z-w)^{-2} :e^{i\mathbf{r}\cdot[\mathbf{X}(z)-\mathbf{X}(w)]}: e^{-im\boldsymbol{\delta}\cdot\mathbf{X}(w)} e^{in\boldsymbol{\delta}\cdot[\mathbf{X}(z)-\mathbf{X}(w)]} \\ :e^{-i(\mathbf{r}+(m-n)\boldsymbol{\delta})\cdot\mathbf{X}(w)}: :e^{i(\mathbf{r}-n\boldsymbol{\delta})\cdot\mathbf{X}(z)}: &= (z-w)^{-2} :e^{i\mathbf{r}\cdot[\mathbf{X}(z)-\mathbf{X}(w)]}: e^{-im\boldsymbol{\delta}\cdot\mathbf{X}(w)} e^{-in\boldsymbol{\delta}\cdot[\mathbf{X}(z)-\mathbf{X}(w)]}, \end{aligned}$$

where the exponentials involving $\boldsymbol{\delta}$ need not be normal ordered since $\boldsymbol{\delta}^2 = \boldsymbol{\delta}\cdot\mathbf{r} = 0$. Invoking the algebraic identity

$$\sum_{n>0} q^{-n} = (1-q^{-1})^{-1} - 1 = -(1-q)^{-1} = -\sum_{n\geq 0} q^n,$$

we get

$$\begin{aligned} &\sum_{\mathbf{r} \in \bar{\Delta}} \sum_{n \in \mathbb{Z}} \times {}^{[1]}E_n^{\mathbf{r}} {}^{[1]}E_{m-n}^{-\mathbf{r}} \times \\ &= \sum_{\mathbf{r} \in \bar{\Delta}} \left[\oint_0 \frac{dw}{2\pi i} \oint_{|z|>|w|} \frac{dz}{2\pi i} \sum_{n\geq 0} :e^{i(\mathbf{r}+n\boldsymbol{\delta})\cdot\mathbf{X}(z)}: :e^{-i(\mathbf{r}+(m+n)\boldsymbol{\delta})\cdot\mathbf{X}(w)}: \right. \\ &\quad \left. + \oint_0 \frac{dw}{2\pi i} \oint_{|z|<|w|} \frac{dz}{2\pi i} \sum_{n>0} :e^{-i(\mathbf{r}+(m-n)\boldsymbol{\delta})\cdot\mathbf{X}(w)}: :e^{i(\mathbf{r}-n\boldsymbol{\delta})\cdot\mathbf{X}(z)}: \right] \\ &= \sum_{\mathbf{r} \in \bar{\Delta}} \oint_0 \frac{dw}{2\pi i} \sum_{w_p \in \{\text{poles}\}} \oint_{z=w_p} \frac{dz}{2\pi i} (z-w)^{-2} :e^{i\mathbf{r}\cdot[\mathbf{X}(z)-\mathbf{X}(w)]}: e^{-im\boldsymbol{\delta}\cdot\mathbf{X}(w)} \sum_{n\geq 0} e^{in\boldsymbol{\delta}\cdot[\mathbf{X}(z)-\mathbf{X}(w)]}, \quad (3.2) \end{aligned}$$

where the second sum runs over all poles of the integrand in the region

$$C_w := \lim_{\epsilon \rightarrow 0} \{z \mid |w| - \epsilon \leq |z| \leq |w| + \epsilon\} = \{z \mid |z| = |w|\},$$

i.e., on a circle of radius $|w|$ in the z plane. A crucial observation for our construction is that besides the the obvious pole at $z = w$, there will be extra poles for level $|\ell| \geq 2$ in the operator-valued function

$$Y(z, w) := \sum_{n\geq 0} e^{in\boldsymbol{\delta}\cdot[\mathbf{X}(z)-\mathbf{X}(w)]}.$$

¹Although, strictly speaking, this statement has only been proven in the framework of the FKS construction.

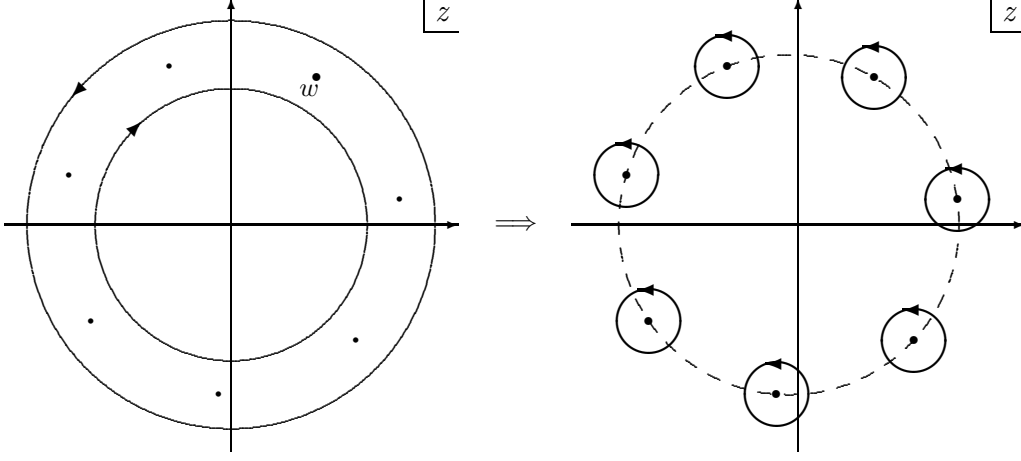


Figure 1: Location of poles for level $\ell = 7$

These are due to the replacement of $(w/z)^n$ by $(w/z)^{\ell n}$ in the momentum mode contributions when the infinite sum defining $Y(z, w)$ acts on a level- ℓ state $|\mathbf{a}\rangle$. More specifically, we shall see that, when acting on such states, this operator gives rise to poles of arbitrary order located at (see Fig. 3 below)

$$z = w_p := \zeta^p w, \quad 1 \leq p \leq \ell, \quad (3.3)$$

where $\zeta := e^{2\pi i/\ell}$ and ℓ denotes the eigenvalue of ${}^{[1]}K_0$. These extra poles will lead to non-local (in the sense of quantum field theory) integrands in our final expressions.

Let us first analyze the pole that $Y(z, w)$ gives rise to at $z = w \equiv w_\ell$; expansion around $z = w$ yields

$$Y(z, w) = -\frac{1}{(z-w)f_\ell(z, w)}, \quad (3.4)$$

where the function $f_\ell(z, w)$ does not vanish at $z = w$; explicitly,

$$\begin{aligned} f_\ell(z, w) &= \sum_{k \geq 1} \frac{1}{k!} (z-w)^{k-1} \left(\frac{\partial^k}{\partial z^k} e^{i\delta \cdot [\mathbf{X}(z) - \mathbf{X}(w)]} \right) \Big|_{z=w} \\ &= \delta \cdot \mathbf{P}(w) + \frac{1}{2} (z-w) [\delta \cdot \mathbf{P}'(w) + (\delta \cdot \mathbf{P}(w))^2] \\ &\quad + \frac{1}{6} (z-w)^2 [\delta \cdot \mathbf{P}''(w) + 3\delta \cdot \mathbf{P}(w) \delta \cdot \mathbf{P}'(w) + (\delta \cdot \mathbf{P}(w))^3] + \dots, \end{aligned} \quad (3.5)$$

with the momentum field $\mathbf{P}(z)$ already defined in Eq. (2.7). When we insert the expansion of $Y(z, w)$ back into Eq. (3.2), we observe that the integrand has a pole of third order at $z = w_\ell \equiv w$. Application of Cauchy's theorem therefore yields

$$\begin{aligned} &\sum_{\mathbf{r} \in \Delta} \sum_{n \in \mathbb{Z}} \times E_n^{\mathbf{r}} E_{m-n}^{-\mathbf{r}} \times \\ &= \sum_{\mathbf{r} \in \Delta} \oint \frac{dw}{2\pi i} \left\{ -\frac{1}{2} \frac{\partial^2}{\partial z^2} \left(\frac{:e^{i\mathbf{r} \cdot [\mathbf{X}(z) - \mathbf{X}(w)]}:}{f_\ell(z, w)} \right) \Big|_{z=w} \right. \\ &\quad \left. + \sum_{p=1}^{\ell-1} \oint \frac{dz}{2\pi i} \left[\frac{:e^{i\mathbf{r} \cdot [\mathbf{X}(z) - \mathbf{X}(w)]}:}{(z-w)^2} Y(z, w) \right] \right\} e^{-im\delta \cdot \mathbf{X}(w)}. \end{aligned} \quad (3.6)$$

The first term may be further simplified by noting that the sum over both the positive and negative roots of $\bar{\mathbf{g}}$ cancels expressions linear in \mathbf{r} . Hence

$$\sum_{\mathbf{r} \in \Delta} \frac{\partial^2}{\partial z^2} \left(\frac{:e^{i\mathbf{r} \cdot [\mathbf{X}(z) - \mathbf{X}(w)]}:}{f_\ell(z, w)} \right) \Big|_{z=w} = \sum_{\mathbf{r} \in \Delta} \left[\frac{:(\mathbf{r} \cdot \mathbf{P}(w))^2:}{\delta \cdot \mathbf{P}(w)} + \frac{\delta \cdot \mathbf{P}(w)}{6} - \frac{\delta \cdot \mathbf{P}''(w)}{3(\delta \cdot \mathbf{P}(w))^2} + \frac{(\delta \cdot \mathbf{P}'(w))^2}{2(\delta \cdot \mathbf{P}(w))^3} \right]. \quad (3.7)$$

Next recall that the physical states of a subcritical bosonic string are finite linear combinations of states of the form

$${}^{[\ell]}A_{-m_1}^{i_1} \cdots {}^{[\ell]}A_{-m_M}^{i_M} {}^{[\ell]}A_{-n_1}^- \cdots {}^{[\ell]}A_{-n_N}^- |\mathbf{a}\rangle, \quad (3.8)$$

where $|\mathbf{a}\rangle$ is any tachyonic state with ${}^{[1]}K_0$ -eigenvalue ℓ and the operators ${}^{[\ell]}A_m^i$ and ${}^{[\ell]}A_m^-$ denote the transversal and the longitudinal DDF operators, respectively [3]. In order to know the action of the Sugawara operators on arbitrary physical states, it is therefore sufficient to work out explicitly the action of the $\mathcal{L}_m^{[\ell]}$'s on a tachyonic ground state and then to determine their commutation relations with the DDF operators.

So let us consider a state $|\mathbf{a}\rangle$ satisfying $\mathbf{a}^2 = 2$ and ${}^{[1]}K_0 |\mathbf{a}\rangle = -(\boldsymbol{\delta} \cdot \mathbf{a}) |\mathbf{a}\rangle = \ell |\mathbf{a}\rangle$ for some $\ell \in \mathbb{N}$. Evidently, $|\mathbf{a}\rangle$ is a highest weight state for $\text{Vir}_{\mathcal{L}^{[\ell]}}$,

$$\mathcal{L}_m^{[\ell]} |\mathbf{a}\rangle = 0 \quad \forall m > 0,$$

because $(\mathbf{a} - m\boldsymbol{\delta})^2 = 2(1 + \ell m) > 2$ for $m > 0$, but there is no physical string state below the tachyon. For $m \geq 0$, we first have to evaluate $Y(z, w) |\mathbf{a}\rangle$. We find that

$$\begin{aligned} Y(z, w) |\mathbf{a}\rangle &= \sum_{n \geq 0} e^{in\boldsymbol{\delta} \cdot [\mathbf{X}(z) - \mathbf{X}(w)]} |\mathbf{a}\rangle \\ &= \sum_{k \geq 0} \sum_{n \geq 0} \frac{n^k}{k!} \left(\frac{w}{z}\right)^{\ell n} [i\boldsymbol{\delta} \cdot \mathbf{X}_<(z) - i\boldsymbol{\delta} \cdot \mathbf{X}_<(w)]^k |\mathbf{a}\rangle \\ &= \sum_{k \geq 0} \frac{z^{\ell(k+1)} p_k \left(\frac{w}{z}\right)^\ell}{k! [z^\ell - w^\ell]^{k+1}} [i\boldsymbol{\delta} \cdot \mathbf{X}_<(z) - i\boldsymbol{\delta} \cdot \mathbf{X}_<(w)]^k |\mathbf{a}\rangle, \end{aligned} \quad (3.9)$$

where

$$i\boldsymbol{\delta} \cdot \mathbf{X}_<(z) := \sum_{n > 0} \frac{1}{n} (\boldsymbol{\delta} \cdot \boldsymbol{\alpha}_{-n}) z^n,$$

and

$$p_0 \equiv 1, \quad p_{k+1}(x) := x[(1-x)p'_k(x) + (k+1)p_k(x)] \quad \forall k \geq 0. \quad (3.10)$$

The latter recursion relation follows from the formula

$$\sum_{n \geq 0} n^k x^n = \left(x \frac{d}{dx}\right)^k \left(\frac{1}{1-x}\right) = \frac{p_k(x)}{(1-x)^{k+1}} \quad (|x| < 1). \quad (3.11)$$

The polynomials $p_k(x)$ only have positive coefficients. Indeed, the above recursion relations translate into

$$p_{k+1, i} = ip_{k, i} + (k - i + 2)p_{k, i-1} \quad \forall k > 0, \quad 0 \leq i \leq k + 1,$$

where²

$$p_k(x) = \sum_{i=0}^k p_{k, i} x^i.$$

Hence, in particular, the polynomials cannot vanish at $x = 1$ which proves that the term

$$\frac{z^{\ell(k+1)} p_k \left(\frac{w}{z}\right)^\ell}{(z^\ell - w^\ell)^{k+1}}$$

contains ℓ poles at $z = w_p \equiv \zeta^p w$, each of order $k + 1$.

On the other hand, the expression $[i\boldsymbol{\delta} \cdot \mathbf{X}_<(z) - i\boldsymbol{\delta} \cdot \mathbf{X}_<(w)]^k$ is a sum of terms of the form

$$(z^{n_1} - w^{n_1}) \cdots (z^{n_k} - w^{n_k}) (\boldsymbol{\delta} \cdot \boldsymbol{\alpha}_{-n_1}) \cdots (\boldsymbol{\delta} \cdot \boldsymbol{\alpha}_{-n_k}), \quad n_i > 0 \quad \forall i,$$

²By induction, it is easy to show that the coefficients are symmetric in the sense that $p_{k, i} = p_{k, k-i+1} \quad \forall k > 0, \quad 0 \leq i \leq k$. In particular, $p_{k, k} = p_{k, 1} = 1 \quad \forall k > 0$.

each of them having a zero at $z = w$ of order k . In total, $Y(z, w)|\mathbf{a}\rangle$ always has a simple pole at $z = w \equiv w_\ell$, which was already evaluated in (3.7), but exhibits a much more complicated pattern at the other poles. For example, if $(n_i, \ell) = m > 0$ (highest common divisor) then the poles at $z = e^{2\pi ik/m}$, $1 \leq k \leq m$, in $(z^\ell - w^\ell)^{-1}$ cancel against the zeros in $z^{n_i} - w^{n_i}$. Up to oscillator number two, for instance, one has the explicit formula

$$Y(z, w)|\mathbf{a}\rangle = \left\{ \frac{z^\ell}{z^\ell - w^\ell} + \frac{z^\ell w^\ell}{(z^\ell - w^\ell)^2} [(z - w)(\boldsymbol{\delta} \cdot \boldsymbol{\alpha}_{-1}) + \frac{1}{2}(z^2 - w^2)(\boldsymbol{\delta} \cdot \boldsymbol{\alpha}_{-2}) + \dots] \right. \\ \left. + \frac{z^\ell w^\ell (z^\ell + w^\ell)}{2(z^\ell - w^\ell)^3} [(z - w)^2 (\boldsymbol{\delta} \cdot \boldsymbol{\alpha}_{-1})^2 + \dots] + \dots \right\} |\mathbf{a}\rangle. \quad (3.12)$$

It is obvious from this result that a direct evaluation of $\mathcal{L}_{-m}^{[\ell]}|\mathbf{a}\rangle$ quickly becomes unfeasible with increasing m . There is, however, an elegant argument which allows us to shortcut this calculation and to read off the result directly from the expression (3.6). We recall that the leading oscillator contribution of a DDF operator is

$${}^{[\ell]}A_{-m}^i \sim \boldsymbol{\xi}_i \cdot \boldsymbol{\alpha}_{-m} + \dots, \quad {}^{[\ell]}A_{-m}^- \sim \mathbf{a} \cdot \boldsymbol{\alpha}_{-m} + \dots$$

Since these oscillators are linearly independent we can immediately rewrite a given physical state in terms of DDF states simply by identifying the leading oscillators. An important assumption here, without which this argument would be invalid, is that there must not be any null physical state present, because their appearance would spoil the nice oscillator structure. Now a glance at Eq. (3.6) shows that *longitudinal oscillators are absent altogether*. This means that the Sugawara generators when applied to any physical state neither produce null physical states nor additional longitudinal excitations (apart from those already contained in the initial state (3.8)). We conclude that the Sugawara generators can be rewritten in terms of the transversal DDF operators alone and that the result can be obtained by isolating those terms which do not contain $\boldsymbol{\delta} \cdot \boldsymbol{\alpha}_{-n}$ oscillators. For the second term in the Sugawara generators we find in this way that

$$\sum_{\mathbf{r} \in \bar{\Delta}} \sum_{n \in \mathbb{Z}} \times \times {}^{[1]}E_n^{\mathbf{r}} {}^{[1]}E_{m-n}^{-\mathbf{r}} \times \times \\ = \sum_{\mathbf{r} \in \bar{\Delta}} \oint_0 \frac{dw}{2\pi i} \left\{ \frac{w}{2\ell} \cdot (\mathbf{r} \cdot \mathbf{P}(w))^2 \cdot + \frac{\ell^2 - 1}{12\ell w} + \sum_{p=1}^{\ell-1} \oint_{z=w_p} \frac{dz}{2\pi i} \left[\frac{e^{i\mathbf{r} \cdot [\mathbf{X}(z) - \mathbf{X}(w)]} : z^\ell}{(z - w)^2 (z^\ell - w^\ell)} \right] \right\} w^{\ell m} \\ + \text{terms containing } \boldsymbol{\delta} \cdot \boldsymbol{\alpha}_{-n} \text{'s}. \quad (3.13)$$

Note that the integrals around $z = w_p$ for the displayed terms can be immediately performed since the integrands have only simple poles. The above reasoning ensures that all terms involving $\boldsymbol{\delta}$'s in (3.13) must combine with the other terms precisely in such a way that the ordinary string oscillators are replaced by DDF oscillators. After this ‘‘leap of faith’’ we arrive at

$$\sum_{\mathbf{r} \in \bar{\Delta}} \sum_{n \in \mathbb{Z}} \times \times {}^{[1]}E_n^{\mathbf{r}} {}^{[1]}E_{m-n}^{-\mathbf{r}} \times \times \\ = \sum_{\mathbf{r} \in \bar{\Delta}} \oint_0 \frac{dw}{2\pi i} \left\{ \frac{1}{2\ell w} \times \times ({}^{[\ell]}P^{\mathbf{r}}(w))^2 \times \times + \frac{\ell^2 - 1}{12\ell w} + \sum_{p=1}^{\ell-1} \frac{w_p}{\ell(w_p - w)^2} \times \times e^{i\mathbf{r} \cdot [{}^{[\ell]}X(w_p) - {}^{[\ell]}X(w)]} \times \times \right\} w^{\ell m} \quad (3.14)$$

Our main result is thus the following new realization of the Sugawara generators at arbitrary level:

Theorem 1 *The operators*

$$\mathcal{L}_m^{[\ell]} = \frac{1}{2(\ell + h^\vee)} \sum_{n \in \mathbb{Z}} \sum_{i=1}^{d-2} \times \times {}^{[1]}A_n^i {}^{[1]}A_{m-n}^i \times \times + \frac{h^\vee}{2\ell(\ell + h^\vee)} \sum_{n \in \mathbb{Z}} \sum_{i=1}^{d-2} \times \times {}^{[\ell]}A_n^i {}^{[\ell]}A_{m-n}^i \times \times + \frac{(\ell^2 - 1)(d - 2)h^\vee}{24\ell(\ell + h^\vee)} \delta_{m,0} \\ - \frac{1}{2\ell(\ell + h^\vee)} \sum_{\mathbf{r} \in \bar{\Delta}} \sum_{p=1}^{\ell-1} \frac{1}{|\zeta^p - 1|^2} \oint_0 \frac{dw}{2\pi i} \left\{ w^{\ell m - 1} \times \times e^{i\mathbf{r} \cdot [{}^{[\ell]}Y_p(w)]} \times \times \right\}, \quad (3.15)$$

form a Virasoro algebra with central charge

$$c(\ell) := \frac{\ell \dim \bar{\mathfrak{g}}}{\ell + h^\vee}.$$

In deriving this result we have made use of the identity

$$\sum_{\mathbf{r} \in \bar{\Delta}} \times_{\times}^{[\ell]} \mathcal{P}^{\mathbf{r}}(z) \times_{\times}^{[\ell]} \mathcal{P}^{\mathbf{r}}(z) = 2h^{\vee} \sum_{i=1}^{d-2} \times_{\times}^{[\ell]} \mathcal{P}^i(z) \times_{\times}^{[\ell]} \mathcal{P}^i(z).$$

Observe that (3.15) is “doubly transcendental” as a function of the ordinary string oscillators because the new coordinate field (2.17), which itself is already a transcendental function of the string oscillators, appears in the exponential. Moreover, this expression is manifestly physical as it depends only on the difference of the coordinate field. (3.15) contains the well known formula (3.1) as a special case for $\ell = 1$.

With the above formula, the level- ℓ energy-momentum tensor

$$\mathcal{L}^{[\ell]}(z) := \sum_{m \in \mathbb{Z}} \mathcal{L}_m^{[\ell]} z^{-\ell m}, \quad (3.16)$$

becomes nonlocal, to wit

$$\begin{aligned} \mathcal{L}^{[\ell]}(z) &= \frac{1}{2(\ell + h^{\vee})} \sum_{i=1}^{d-2} \times_{\times}^{[1]} \mathcal{P}^i(z^{\ell}) \times_{\times}^{[1]} \mathcal{P}^i(z^{\ell}) + \frac{h^{\vee}}{2\ell(\ell + h^{\vee})} \sum_{i=1}^{d-2} \times_{\times}^{[\ell]} \mathcal{P}^i(z) \times_{\times}^{[\ell]} \mathcal{P}^i(z) + \frac{(\ell^2 - 1)(d - 2)h^{\vee}}{24\ell(\ell + h^{\vee})} \\ &\quad - \frac{1}{2\ell(\ell + h^{\vee})} \sum_{\mathbf{r} \in \bar{\Delta}} \sum_{p=1}^{\ell-1} \frac{1}{|\zeta^p - 1|^2} \times_{\times} e^{i\mathbf{r} \cdot [\ell \mathcal{X}(z_p) - \ell \mathcal{X}(z)]}. \end{aligned} \quad (3.17)$$

4 General Properties and Proof of Theorem

Before we prove that the expressions (3.15) for the level- ℓ Sugawara operators indeed satisfy the Virasoro algebra with the correct central charge, we would like to discuss some general features of our new formula. In the next section we will work out some explicit examples.

Since the operators (3.15) are purely transversal in terms of DDF oscillators, we immediately see that they commute with the longitudinal DDF operators,

$$[{}^{[\ell]} A_n^-, \mathcal{L}_m^{[\ell]}] = 0 \quad \forall m, n \in \mathbb{Z}. \quad (4.1)$$

The commutation relations with the transversal DDF operators can be verified as follows. A straightforward calculation yields

$$[{}^{[\ell]} A_n^j, \times_{\times} e^{i\mathbf{r} \cdot [\ell] \mathcal{Y}_p(w)} \times_{\times}] = (\mathbf{r} \cdot \boldsymbol{\xi}_j) (\zeta^{pn} - 1) w^{n \times} \times_{\times} e^{i\mathbf{r} \cdot [\ell] \mathcal{Y}_p(w)} \times_{\times}. \quad (4.2)$$

Similarly, one finds that

$$\sum_{k \in \mathbb{Z}} [{}^{[\ell]} A_n^j, \times_{\times} [{}^{[\ell]} A_k^i, \times_{\times} [{}^{[\ell]} A_{m-k}^i, \times_{\times}]] = 2n \delta^{ij} [{}^{[\ell]} A_{m+n}^j].$$

Inserting these results into formula (3.15) we obtain

$$\begin{aligned} [{}^{[\ell]} A_n^j, \mathcal{L}_m^{[\ell]}] &= \frac{n}{(\ell + h^{\vee})} \sum_{k \in \mathbb{Z}} \delta_{\ell k + n, 0} [{}^{[1]} A_{m-k}^j] + \frac{nh^{\vee}}{\ell(\ell + h^{\vee})} [{}^{[\ell]} A_{\ell m + n}^j] \\ &\quad - \frac{1}{2\ell(\ell + h^{\vee})} \sum_{\mathbf{r} \in \bar{\Delta}} (\mathbf{r} \cdot \boldsymbol{\xi}_j) \sum_{p=1}^{\ell-1} \frac{\zeta^{pn} - 1}{|\zeta^p - 1|^2} \oint_0 \frac{dw}{2\pi i} \left\{ w^{\ell m + n - 1} \times_{\times} e^{i\mathbf{r} \cdot [\ell] \mathcal{Y}_p(w)} \times_{\times} \right\}, \end{aligned} \quad (4.3)$$

and therefore recover the formula $[{}^{[1]} A_n^j, \mathcal{L}_m^{[\ell]}] = n [{}^{[1]} A_{m+n}^j]$ in agreement with Eq. (2.25).

It is instructive to have a closer look at the new expression for $\mathcal{L}_0^{[\ell]}$, which reads

$$\mathcal{L}_0^{[\ell]} | \mathbf{a} \rangle = \left[\frac{1}{2\ell} \bar{\mathbf{a}}^2 + \frac{(\ell^2 - 1)(d - 2)h^{\vee}}{24\ell(\ell + h^{\vee})} - \frac{1}{2\ell(\ell + h^{\vee})} \sum_{\mathbf{r} \in \bar{\Delta}} \sum_{p=1}^{\ell-1} \frac{\zeta^{p\mathbf{r} \cdot \mathbf{a}}}{|\zeta^p - 1|^2} \right] | \mathbf{a} \rangle, \quad (4.4)$$

where $\bar{\mathbf{a}} \equiv \bar{\Lambda}$ denotes the projection of \mathbf{a} (resp. Λ) onto $\bar{\mathfrak{h}}^*$. Let us focus on the last term. We have the following

Lemma 1 *Let ζ be a primitive ℓ -th root of unity and let $k = 0, \dots, \ell - 1$. Then*

$$\sum_{p=1}^{\ell-1} \frac{\zeta^{pk}}{|\zeta^p - 1|^2} = \frac{1}{12}(\ell^2 - 1) - \frac{1}{2}k(\ell - k). \quad (4.5)$$

*Proof:*³ With the elementary algebraic identity (for $p \not\equiv 0 \pmod{\ell}$)

$$\frac{1}{1 - \zeta^p} = -\frac{1}{\ell} \sum_{j=1}^{\ell-1} j \zeta^{pj},$$

we immediately obtain

$$\sum_{p=1}^{\ell-1} \frac{\zeta^{pk}}{|\zeta^p - 1|^2} = \frac{1}{\ell^2} \sum_{i,j,p=1}^{\ell-1} ij \zeta^{p(k+i-j)} = \frac{1}{\ell^2} \sum_{i,j=1}^{\ell-1} ij \sum_{p=0}^{\ell-1} \zeta^{p(k+i-j)} - \frac{1}{4}(\ell - 1)^2.$$

Invoking the following well-known property of sums of roots of unity,

$$\sum_{p=0}^{\ell-1} \zeta^{pn} = \begin{cases} \ell & \text{if } n \equiv 0 \pmod{\ell}, \\ 0 & \text{else,} \end{cases} \quad (4.6)$$

we find that

$$\begin{aligned} \sum_{i,j=1}^{\ell-1} ij \sum_{p=0}^{\ell-1} \zeta^{p(k+i-j)} &= \ell \sum_{i=1}^{\ell-1-k} i(i+k) + \ell \sum_{i=\ell-k}^{\ell-1} i(i+k-\ell) \\ &= \ell \sum_{i=1}^{\ell-1} i(i+k-\ell) + \ell^2 \sum_{i=1}^{\ell-1-k} i \\ &= \ell \left[\frac{1}{6}(\ell-1)\ell(2\ell-1) + \frac{1}{2}(k-\ell)\ell(\ell-1) + \frac{1}{2}\ell(\ell-1-k)(\ell-k) \right], \end{aligned}$$

and thus

$$\begin{aligned} \sum_{p=1}^{\ell-1} \frac{\zeta^{pk}}{|\zeta^p - 1|^2} &= \frac{1}{6}(\ell-1)(2\ell-1) - \frac{1}{2}k(\ell-k) - \frac{1}{4}(\ell-1)^2 \\ &= \frac{1}{12}(\ell^2 - 1) - \frac{1}{2}k(\ell - k) \quad \text{q.e.d.} \end{aligned}$$

By use of this result and some well-known facts about the finite root system, we may rewrite the last term of the above formula for $\mathcal{L}_0^{[\ell]}$ as

$$\begin{aligned} \sum_{\mathbf{r} \in \bar{\Delta}} \sum_{p=1}^{\ell-1} \frac{\zeta^{p\mathbf{r} \cdot \mathbf{a}}}{|\zeta^p - 1|^2} &= \sum_{\mathbf{r} \in \bar{\Delta}_+} \left[\frac{1}{6}(\ell^2 - 1) - (\mathbf{r} \cdot \mathbf{a})(\ell - (\mathbf{r} \cdot \mathbf{a})) \right] \\ &= \frac{(\ell^2 - 1)(d - 2)h^\vee}{12} - 2\ell \bar{\rho} \cdot \bar{\mathbf{a}} + h^\vee \bar{\mathbf{a}}^2, \end{aligned}$$

where $\bar{\rho}$ denotes the Weyl vector for the finite subalgebra.⁴ If we insert this into (4.4) we arrive at the formula

$$\mathcal{L}_0^{[\ell]}|\mathbf{a}\rangle = \frac{(\bar{\mathbf{a}} + 2\bar{\rho}) \cdot \bar{\mathbf{a}}}{2(\ell + h^\vee)}|\mathbf{a}\rangle, \quad (4.7)$$

in agreement with [13, Lemma12.8.b)]. Note that we have not employed any properties of the affine Casimir operator in our calculation.

³We would like to thank H. Samtleben for the crucial idea.

⁴Note that the term linear in $k = \mathbf{r} \cdot \mathbf{a}$ does not drop out upon summation over the roots but rather reproduces the Weyl vector. This is due to the fact that the Lemma is valid only for $0 \leq k \leq \ell - 1$ and thus different values of $\mathbf{r} \cdot \mathbf{a}$ have to be transported into this range by multiples of ℓ .

It remains to verify that the operators (3.15) really do satisfy the Virasoro algebra with the correct central charge. To this aim we split the Sugawara operators and introduce the following operators:

$$\tilde{\mathcal{L}}_m^{[\ell]} = L_m^{(1)} + L_m^{(2)} + L_m^{(3)}, \quad (4.8)$$

with

$$\begin{aligned} L_m^{(1)} &:= \frac{1}{2\ell} \sum_{n \in \mathbb{Z}} \sum_{i=1}^{d-2} \times [{}^1A_n^i \quad {}^1A_{m-n}^i \times, \\ L_m^{(2)} &:= \frac{h^\vee}{2\ell(\ell + h^\vee)} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0(\ell)}} \sum_{i=1}^{d-2} \times [{}^{[\ell]}A_n^i \quad {}^{[\ell]}A_{m-n}^i \times, \\ L_m^{(3)} &:= -\frac{1}{2\ell(\ell + h^\vee)} \sum_{\mathbf{r} \in \bar{\Delta}} \sum_{p=1}^{\ell-1} \frac{1}{|\zeta^p - 1|^2} \oint_0 \frac{dw}{2\pi i} \left\{ w^{\ell m - 1} \times e^{i\mathbf{r} \cdot [{}^\ell] \mathcal{Y}_p(w)} \times \right\}. \end{aligned} \quad (4.9)$$

Observe that we have absorbed all terms involving ${}^1A_n^i$ into $L_m^{(1)}$ so that the prefactor is “renormalized” to $(2\ell)^{-1}$ with respect to (3.15), because these operators commute with the DDF oscillators ${}^{[\ell]}A_n^i$ with $n \not\equiv 0(\ell)$. We obviously have (with ${}^1K_0 = \ell$)

$$\begin{aligned} [L_m^{(1)}, L_n^{(1)}] &= (m-n)L_{m+n}^{(1)} + \frac{d-2}{12}(m^3 - m)\delta_{m+n,0}, \\ [L_m^{(1)}, L_n^{(2)}] &= [L_m^{(1)}, L_n^{(3)}] = 0. \end{aligned} \quad (4.10)$$

It is equally straightforward to show that

$$[L_m^{(2)}, L_n^{(2)}] = (m-n) \frac{h^\vee}{\ell + h^\vee} L_{m+n}^{(2)} + \frac{(d-2)(\ell-1)(h^\vee)^2}{(\ell + h^\vee)^2} \left(m^3 + \frac{m}{\ell} \right) \delta_{m+n,0} \quad (4.11)$$

and

$$[L_m^{(2)}, L_n^{(3)}] = (m-n) \frac{h^\vee}{\ell + h^\vee} L_{m+n}^{(3)}. \quad (4.12)$$

The remaining commutator requires more work. For its evaluation we need the operator product

$$\times e^{i\mathbf{r} \cdot [{}^\ell] \mathcal{Y}_p(z)} \times \times e^{i\mathbf{s} \cdot [{}^\ell] \mathcal{Y}_q(w)} \times = \left[\frac{(z_p - w_q)(z - w)}{(z_p - w)(z - w_q)} \right]^{\mathbf{r} \cdot \mathbf{s}} \times e^{i\mathbf{r} \cdot [{}^\ell] \mathcal{Y}_p(z) + i\mathbf{s} \cdot [{}^\ell] \mathcal{Y}_q(w)} \times. \quad (4.13)$$

We write

$$[L_m^{(3)}, L_n^{(3)}] = \sum_{\mathbf{r}, \mathbf{s} \in \bar{\Delta}} \sum_{p, q=1}^{\ell-1} I(\mathbf{r}, \mathbf{s}, p, q),$$

with

$$\begin{aligned} I(\mathbf{r}, \mathbf{s}, p, q) &:= \frac{1}{4\ell^2(\ell + h^\vee)^2} \frac{1}{|\zeta^p - 1|^2 |\zeta^q - 1|^2} \times \\ &\times \oint_0 \frac{dw}{2\pi i} \sum_{a=1}^{\ell} \oint_{w_a} \frac{dz}{2\pi i} \left\{ w^{\ell n - 1} z^{\ell m - 1} \left[\frac{(z_p - w_q)(z - w)}{(z_p - w)(z - w_q)} \right]^{\mathbf{r} \cdot \mathbf{s}} \times e^{i\mathbf{r} \cdot [{}^\ell] \mathcal{Y}_p(z) + i\mathbf{s} \cdot [{}^\ell] \mathcal{Y}_q(w)} \times \right\}. \end{aligned}$$

Inspection of the term in curly brackets reveals poles of order 1, 2 and 4, depending on the values of $\mathbf{r} \cdot \mathbf{s}$ and $p, q \in \{1, \dots, \ell - 1\}$. As usual, there is no contribution for $\mathbf{r} \cdot \mathbf{s} = 0$. Another useful observation is that

$$I(\mathbf{r}, \mathbf{s}, p, q) = I(\mathbf{r}, -\mathbf{s}, p, \ell - q) \quad \forall \mathbf{r}, \mathbf{s}, p, q, \quad (4.14)$$

It is therefore sufficient to consider the cases $\mathbf{r} \cdot \mathbf{s} = -1$ ($\Leftrightarrow \mathbf{r} + \mathbf{s} \in \bar{\Delta}$) and $\mathbf{r} \cdot \mathbf{s} = -2$ ($\Leftrightarrow \mathbf{s} = -\mathbf{r}$); these lead to the following results:

1. If $\mathbf{r} \cdot \mathbf{s} = -1$ and $p = q$ (pole of order 2), then

$$\sum_{\mathbf{r} \in \bar{\Delta}} \sum_{p=1}^{\ell-1} I(\mathbf{r}, \mathbf{s}, p, p) = (n - m) \frac{1}{4(\ell + h^\vee)} L_{m+n}^{(3)},$$

after partial integration. For simply laced algebras and given $\mathbf{r} \in \bar{\Delta}$, there are always $2h^\vee - 4$ roots \mathbf{s} such that $\mathbf{r} \cdot \mathbf{s} = -1$ (or $+1$)⁵. Hence

$$\sum_{\substack{\mathbf{r}, \mathbf{s} \in \bar{\Delta} \\ \mathbf{r} \cdot \mathbf{s} = -1}} \sum_{p=1}^{\ell-1} I(\mathbf{r}, \mathbf{s}, p, p) = (m - n) \frac{2 - h^\vee}{2(\ell + h^\vee)} L_{m+n}^{(3)},$$

2. If $\mathbf{r} \cdot \mathbf{s} = -1$ and $p \neq q$ (poles of order 1), then

$$I(\mathbf{r}, \mathbf{s}, p, q) = -I(\mathbf{s}, \mathbf{r}, q, p),$$

which vanishes upon (symmetric) summation over $\mathbf{r}, \mathbf{s}, p, q$.

3. If $\mathbf{r} \cdot \mathbf{s} = -2$ and $p = q$ (pole of order 4), then

$$\sum_{\mathbf{r} \in \bar{\Delta}} \sum_{p=1}^{\ell-1} I(\mathbf{r}, -\mathbf{r}, p, p) = (m - n) \frac{\ell}{2(\ell + h^\vee)} L_{m+n}^{(2)} + \frac{(d-2)\ell(\ell-1)h^\vee}{24(\ell + h^\vee)^2} \left(m^3 + \frac{m}{\ell} \right) \delta_{m+n,0},$$

after partial integration and use of Lemma 1.

4. If $\mathbf{r} \cdot \mathbf{s} = -2$ and $p \neq q$ (poles of order 2), then

$$\sum_{\mathbf{r} \in \bar{\Delta}} \sum_{\substack{p, q=1 \\ p \neq q}}^{\ell-1} I(\mathbf{r}, -\mathbf{r}, p, q) = (m - n) \frac{\ell - 2}{2(\ell + h^\vee)} L_{m+n}^{(3)},$$

after partial integration.

Hence

$$\begin{aligned} [L_m^{(3)}, L_n^{(3)}] &= (m - n) \frac{\ell}{\ell + h^\vee} L_{m+n}^{(2)} + (m - n) \frac{\ell - h^\vee}{\ell + h^\vee} L_{m+n}^{(3)} \\ &\quad + \frac{(d-2)\ell(\ell-1)h^\vee}{12(\ell + h^\vee)^2} \left(m^3 + \frac{m}{\ell} \right) \delta_{m+n,0}. \end{aligned} \quad (4.15)$$

Adding up all contributions we get

$$[\tilde{\mathcal{L}}_m^{[\ell]}, \tilde{\mathcal{L}}_n^{[\ell]}] = (m - n) \tilde{\mathcal{L}}_{m+n}^{[\ell]} + \left(\frac{c}{12} m^3 + bm \right) \delta_{m,0},$$

with

$$b := -\frac{d-2}{12} + \frac{(d-2)(\ell-1)h^\vee}{12\ell(\ell+h^\vee)} = -\frac{(d-2)(\ell^2+h^\vee)}{12\ell(\ell+h^\vee)},$$

the central charge c given by

$$\begin{aligned} c &= d - 2 + \frac{(d-2)(\ell-1)(h^\vee)^2}{(\ell+h^\vee)^2} + \frac{(d-2)\ell(\ell-1)h^\vee}{(\ell+h^\vee)^2} \\ &= \frac{(d-2)\ell(1+h^\vee)}{\ell+h^\vee} = \frac{\ell \dim \bar{\mathfrak{g}}}{\ell+h^\vee}, \end{aligned}$$

⁵By Weyl invariance it is sufficient to prove the statement for the highest root $\boldsymbol{\theta}$. From the definition of the Coxeter number and the Weyl vector we have

$$2(h^\vee - 1) = 2\bar{\rho} \cdot \boldsymbol{\theta} = \sum_{\mathbf{s} \in \bar{\Delta}_+} \mathbf{s} \cdot \boldsymbol{\theta}.$$

The only contributions in the sum arise from the terms with $\mathbf{s} \cdot \boldsymbol{\theta} = 1$ (whose number we wish to compute) and with $\mathbf{s} \cdot \boldsymbol{\theta} = 2$. However, the only positive root for which $\mathbf{s} \cdot \boldsymbol{\theta} = 2$ is $\mathbf{s} = \boldsymbol{\theta}$, whence the result.

in agreement with (2.24). Since this Virasoro algebra has not yet the standard form, we have to shift $\tilde{\mathcal{L}}_0^{[\ell]}$. Doing this we arrive at the desired result,

$$\begin{aligned}\mathcal{L}_m^{[\ell]} &:= \tilde{\mathcal{L}}_m^{[\ell]} + \frac{c + 12b}{24} \delta_{m,0} \\ &= \tilde{\mathcal{L}}_m^{[\ell]} + \frac{(\ell^2 - 1)(d - 2)h^\vee}{24\ell(\ell + h^\vee)} \delta_{m,0}.\end{aligned}$$

Finally we would like to mention that expressions analogous to (3.15) in terms of DDF oscillators also exist for the step operators ${}^{[1]}E_m^{\mathbf{r}}$. They read

$${}^{[\ell]}E_m^{\mathbf{r}} = \oint_0 \frac{dz}{2\pi i} \left\{ \times z^{m+1+\mathbf{r}\cdot\mathbf{p}} e^{i[\mathbf{r}-(\mathbf{r}\cdot\mathbf{p}+1)\mathbf{k}_\ell]\cdot[\ell]\mathcal{X}(z)} \times \right\}. \quad (4.16)$$

One can show that these operators indeed satisfy the commutation relations (2.14) and (2.15). In addition, they allow a direct verification of the semidirect product (2.25).

5 Examples

In this section we present some examples. As already mentioned we will restrict attention to the Lie algebras E_8 , E_9 and E_{10} .

When expanding the exponential operator in the new formula (3.15), we notice that $\mathcal{L}_n^{[\ell]}$ involves linear combinations of the form

$$\sum_{\substack{m_1, \dots, m_M \neq 0(\ell) \\ m_1 + \dots + m_M = \ell n}} \frac{1}{m_1 \cdots m_M} \mathcal{T}_{j_1 \dots j_M}(\mathbf{a}; \langle m_1 \rangle, \dots, \langle m_M \rangle) {}^{[\ell]}A_{-m_1}^{j_1} \cdots {}^{[\ell]}A_{-m_M}^{j_M}, \quad (5.1)$$

with

$$\mathcal{T}_{j_1 \dots j_M}(\mathbf{a}; \langle m_1 \rangle, \dots, \langle m_M \rangle) := \sum_{p=1}^{\ell-1} \sum_{\mathbf{r} \in \bar{\Delta}} \frac{\zeta^{p\mathbf{r}\cdot\mathbf{a}}}{|\zeta^p - 1|^2} (\zeta^{pm_1} - 1) \cdots (\zeta^{pm_M} - 1) r^{j_1} \cdots r^{j_M}; \quad (5.2)$$

here $\langle m \rangle$ is a coset representative for m , i.e., $m = \langle m \rangle + k\ell$ for some $k \in \mathbb{Z}$, $\langle m \rangle \in \{1, \dots, \ell - 1\}$, and r^j denotes the j -th component of the root \mathbf{r} with respect to some basis $\{\mathbf{e}_j | 1 \leq j \leq d - 2\}$ of \mathfrak{h}^* . Since $M \geq 2$, the tensors can be simplified by writing

$$\begin{aligned}N(\mathbf{r}\cdot\mathbf{a}; \langle m_1 \rangle, \dots, \langle m_M \rangle) &:= \sum_{p=1}^{\ell-1} \frac{\zeta^{p\mathbf{r}\cdot\mathbf{a}}}{|\zeta^p - 1|^2} (\zeta^{pm_1} - 1) \cdots (\zeta^{pm_M} - 1) \\ &= - \sum_{p=1}^{\ell-1} \sum_{k_1=0}^{\langle m_1 \rangle - 1} \sum_{k_2=0}^{\langle m_2 \rangle - 1} \zeta^{p(\mathbf{r}\cdot\mathbf{a} + k_1 + k_2 + 1)} (\zeta^{pm_3} - 1) \cdots (\zeta^{pm_M} - 1).\end{aligned} \quad (5.3)$$

Invoking (4.6) we conclude that the numbers $N(\mathbf{a}\cdot\mathbf{r}; \langle m_1 \rangle, \dots, \langle m_M \rangle)$ are always real integers. Hence further evaluation of the tensors $\mathcal{T}_{j_1 \dots j_M}$ necessitates the computation of weighted sums over tensor products of real roots of the following type:

$$\sum_{\mathbf{r} \in \bar{\Delta}} N(\mathbf{r}\cdot\mathbf{a}) \mathbf{r} \otimes \dots \otimes \mathbf{r}, \quad (5.4)$$

with $N(\mathbf{r}\cdot\mathbf{a}) \in \mathbb{Z}$. Such sums have not been considered in the literature so far, except in the simplest situation where $N = \text{const}$. In this case the sums become invariant tensors w.r.t. the full Weyl group of the finite Lie algebra under consideration. E.g., for E_8 we have the following formula for the unweighted sums over tensor products up to six tensor factors:

$$\sum_{\mathbf{r} \in \bar{\Delta}} r^{j_1} \cdots r^{j_{2k}} = 2^{4-k} (7 + 2^{2k-3}) \delta_{(j_1 j_2 \cdots j_{2k-1} j_{2k})} \quad \text{for } k=1, 2, 3, \quad (5.5)$$

where (...) denotes symmetrization with strength one and tensors with an odd number of indices vanish (this is no longer true for the weighted sums (5.4)). The simple result (5.5) is explained by the absence of invariant tensors other than δ_{ij} for $k \leq 3$ for the Weyl group $\mathfrak{W}(E_8) = D_4(2) \otimes \mathbb{Z}_2^2$. For $k \geq 4$, new invariants appear in accordance with the general theory since the exponents of E_8 are 1, 7, 11, 13, 17, 19, 23, 29 [17].

It is clear that the presence of the factor $N(\mathbf{r} \cdot \mathbf{a})$ in $\mathcal{T}(\mathbf{a})$ breaks the symmetry under the full affine Weyl group down to that subgroup which preserves \mathbf{a} ; this is just the (finite) little Weyl group $\mathfrak{W}(\mathbf{a}, \boldsymbol{\delta})$ introduced in [7]. As a consequence, the results will be the more cumbersome the smaller $\mathfrak{W}(\mathbf{a}, \boldsymbol{\delta})$ becomes. For the examples to be presented below we have therefore evaluated the relevant sums (5.4) on the computer. Inspection of the explicit examples suggests that it may be difficult to find closed (or at least more elegant) general expressions for them, and we have not tried to do so.

Let us illustrate the new formula and the above remarks with some examples for the exceptional Lie algebra $\bar{\mathfrak{g}} = E_8$ with affine extension $\mathfrak{g} = E_9$ and hyperbolic extension $\hat{\mathfrak{g}} = E_{10}$. In this (unique) case the finite root lattice is selfdual, and consequently the extended affine root lattice coincides with the weight lattice of E_{10} which is just the unique even selfdual Lorentzian lattice $II_{9,1}$. As a partial check on our results, we have recalculated (and reobtained) (5.6) below directly by means of formula (4.60) in [7] (the Lie algebra analog of (2.20)), i.e., without use of (3.15). Doing the calculation in this "old" way is impossible without massive use of algebraic computer programs, whereas the new formula requires substantially less effort. In fact, knowing the tensors (5.2) (for this we must still rely on the computer), the calculation can be done by hand.

As our first example, we choose the fundamental dominant weight $\boldsymbol{\Lambda}_1 = 2\mathbf{r}_{-1} + \mathbf{r}_0 + 3\boldsymbol{\delta}$ of level $\ell = 2$ with associated tachyonic vector $\mathbf{a}_1 = \boldsymbol{\Lambda}_1 - 2\boldsymbol{\delta}$. We identify the polarization vectors $\boldsymbol{\xi}_i$ with the orthonormal basis vectors \mathbf{e}_i . The little Weyl group is $\mathfrak{W}(\boldsymbol{\Lambda}_1, \boldsymbol{\delta}) = \mathfrak{W}(E_7) \otimes \mathbb{Z}_2 = C_3(2) \otimes \mathbb{Z}_2^2$. An exceptional property of the level-2 sector is the vanishing of all tensors with an odd number of indices; this feature will be lost for higher levels $|\ell| > 2$. With the notation $A_{-m}^i \equiv {}^{[2]}A_{-m}^i$ we find

$$\begin{aligned}
\mathcal{L}_{-1}^{[2]}|\mathbf{a}_1\rangle &= \left\{ \frac{3}{16} \sum_{i=1}^7 A_{-1}^i A_{-1}^i + \frac{7}{16} A_{-1}^8 A_{-1}^8 + \frac{1}{2} \sqrt{2} A_{-2}^8 \right\} |\mathbf{a}_1\rangle, \\
\mathcal{L}_{-1}^{[2]} \mathcal{L}_{-1}^{[2]}|\mathbf{a}_1\rangle &= \left\{ \frac{11}{64} \sum_{i=1}^7 A_{-3}^i A_{-1}^i + \frac{11}{256} \sum_{i,k=1}^7 (A_{-1}^i)^2 (A_{-1}^k)^2 + \frac{3}{16} \sqrt{2} \sum_{i=1}^7 A_{-2}^8 (A_{-1}^i)^2 \right. \\
&\quad \left. + \frac{15}{128} \sum_{i=1}^7 (A_{-1}^8)^2 (A_{-1}^i)^2 + \frac{35}{64} A_{-3}^8 A_{-1}^8 + \frac{1}{2} (A_{-2}^8)^2 + \frac{7}{16} \sqrt{2} A_{-2}^8 (A_{-1}^8)^2 \right. \\
&\quad \left. + \frac{35}{256} (A_{-1}^8)^4 + \frac{1}{2} \sqrt{2} A_{-4}^8 \right\} |\mathbf{a}_1\rangle, \\
\mathcal{L}_{-2}^{[2]}|\mathbf{a}_1\rangle &= \left\{ \frac{7}{16} \sum_{i=1}^7 A_{-3}^i A_{-1}^i + \frac{1}{4} \sum_{i=1}^8 A_{-2}^i A_{-2}^i - \frac{1}{64} \sum_{i,k=1}^7 (A_{-1}^i)^2 (A_{-1}^k)^2 \right. \\
&\quad \left. + \frac{3}{32} \sum_{i=1}^7 (A_{-1}^8)^2 (A_{-1}^i)^2 + \frac{29}{48} A_{-3}^8 A_{-1}^8 + \frac{5}{192} (A_{-1}^8)^4 + \frac{1}{2} \sqrt{2} A_{-4}^8 \right\} |\mathbf{a}_1\rangle, \\
\mathcal{L}_{-3}^{[2]}|\mathbf{a}_1\rangle &= \left\{ \frac{9}{20} \sum_{i=1}^7 A_{-5}^i A_{-1}^i + \frac{1}{2} \sum_{i=1}^8 A_{-4}^i A_{-2}^i + \frac{11}{48} \sum_{i=1}^7 A_{-3}^i A_{-3}^i - \frac{1}{48} \sum_{i,k=1}^7 A_{-3}^i A_{-1}^i (A_{-1}^k)^2 \right. \\
&\quad \left. + \frac{1}{16} \sum_{k=1}^7 A_{-3}^8 A_{-1}^8 (A_{-1}^k)^2 + \frac{1}{16} \sum_{i=1}^7 A_{-3}^i A_{-1}^i (A_{-1}^8)^2 - \frac{1}{64} \sum_{i,k=1}^6 (A_{-1}^i)^2 (A_{-1}^k)^2 (A_{-1}^7)^2 \right. \\
&\quad \left. + \frac{1}{64} \sum_{i,k=1}^6 (A_{-1}^i)^2 (A_{-1}^k)^2 (A_{-1}^8)^2 + \frac{1}{32} \sum_{i=1}^6 (A_{-1}^i)^2 (A_{-1}^7)^2 (A_{-1}^8)^2 - \frac{1}{96} \sum_{i,k=1}^6 (A_{-1}^i)^2 (A_{-1}^k)^4 \right. \\
&\quad \left. + \frac{1}{48} \sum_{i=1}^6 (A_{-1}^i)^4 (A_{-1}^7)^2 - \frac{1}{192} \sum_{i=1}^6 (A_{-1}^i)^2 (A_{-1}^7)^4 + \frac{1}{64} \sum_{i=1}^6 (A_{-1}^i)^2 (A_{-1}^8)^4 \right. \\
&\quad \left. + \frac{1}{64} (A_{-1}^7)^2 (A_{-1}^8)^4 + \frac{1}{64} (A_{-1}^8)^2 (A_{-1}^7)^4 + \frac{1}{120} \sum_{i=1}^6 (A_{-1}^i)^6 + \frac{11}{20} A_{-5}^8 A_{-1}^8 + \frac{37}{144} A_{-3}^8 A_{-3}^8 \right\} |\mathbf{a}_1\rangle,
\end{aligned}$$

$$+ \frac{5}{144} A_{-3}^8 (A_{-1}^8)^3 - \frac{1}{320} (A_{-1}^7)^6 - \frac{1}{2880} (A_{-1}^8)^6 - \frac{1}{4} \prod_{i=1}^6 A_{-1}^i + \frac{1}{2} \sqrt{2} A_{-6}^8 \} | \mathbf{a}_1 \rangle. \quad (5.6)$$

Our second example is the fundamental dominant weight Λ_8 of level $\ell = 3$ with associated tachyonic vector $\mathbf{a}_8 = \Lambda_8 - 2\delta$ and little Weyl group $\mathfrak{W}(\Lambda_8, \delta) = \mathfrak{W}(A_8) = S_8$. In terms of our standard basis of orthonormal polarization vectors we found the results

$$\begin{aligned} \mathcal{L}_{-1}^{[3]} | \mathbf{a}_8 \rangle &= \left\{ \frac{1}{6} \sum_{i=1}^8 A_{-3}^i + \frac{7}{22} \sum_{i=1}^8 A_{-2}^i A_{-1}^i - \frac{1}{264} \sum_{i,j,k=1}^8 (1 - 6\delta_{ij} + 12\delta_{ij}\delta_{jk}\delta_{ki}) A_{-1}^i A_{-1}^j A_{-1}^k \right\} | \mathbf{a}_8 \rangle, \\ \mathcal{L}_{-2}^{[3]} | \mathbf{a}_8 \rangle &= \left\{ \frac{1}{6} \sum_{i=1}^8 A_{-6}^i + \frac{17}{55} \sum_{i=1}^8 A_{-5}^i A_{-1}^i + \frac{27}{88} \sum_{i=1}^8 A_{-4}^i A_{-2}^i + \frac{1}{6} \sum_{i=1}^8 (A_{-3}^i)^2 \right. \\ &\quad - \frac{1}{352} \sum_{i,j,k=1}^8 (1 - 4\delta_{ij} - 2\delta_{jk} + 12\delta_{ij}\delta_{jk}) A_{-4}^i A_{-1}^j A_{-1}^k \\ &\quad + \frac{1}{2112} \sum_{i,j,k=1}^8 (1 - 6\delta_{ij} + 12\delta_{ij}\delta_{jk}) A_{-2}^i A_{-2}^j A_{-2}^k \\ &\quad - \frac{1}{704} \sum_{i,j,k,l=1}^8 (\delta_{ij} + \delta_{kl} + 4\delta_{jk} - 12\delta_{ij}\delta_{jk} - 12\delta_{jk}\delta_{kl} \\ &\quad + 24\delta_{ij}\delta_{jk}\delta_{kl} - 4\delta_{ij}\delta_{kl} - 8\delta_{ik}\delta_{jl}) A_{-2}^i A_{-2}^j A_{-1}^k A_{-1}^l \\ &\quad + \frac{1}{2816} \sum_{i,j,k,l,m=1}^8 (1 - 8\delta_{ij} - 12\delta_{jk} + 24\delta_{ij}\delta_{jk} + 16\delta_{jk}\delta_{kl} - 32\delta_{ij}\delta_{jk}\delta_{kl} - 8\delta_{jk}\delta_{kl}\delta_{lm} \\ &\quad - 16\delta_{ij}\delta_{jk}\delta_{kl}\delta_{lm} + 64\delta_{ij}\delta_{kl} + 16\delta_{jk}\delta_{lm} + 72\delta_{ij}\delta_{jk}\delta_{lm} + 48\delta_{ij}\delta_{kl}\delta_{lm}) A_{-2}^i A_{-1}^j A_{-1}^k A_{-1}^l A_{-1}^m \\ &\quad \left. - \frac{1}{42240} \sum_{i,j,k,l,m,n=1}^8 (1 - 15\delta_{ij} + 40\delta_{ij}\delta_{jk} - 60\delta_{ij}\delta_{jk}\delta_{kl} + 144\delta_{ij}\delta_{jk}\delta_{kl}\delta_{lm} - 144\delta_{ij}\delta_{jk}\delta_{kl}\delta_{lm}\delta_{mn} \right. \\ &\quad \left. + 80\delta_{ij}\delta_{kl}\delta_{mn} + 320\delta_{ij}\delta_{jk}\delta_{lm}\delta_{mn}) A_{-1}^i A_{-1}^j A_{-1}^k A_{-1}^l A_{-1}^m A_{-1}^n \right\} | \mathbf{a}_8 \rangle, \quad (5.7) \end{aligned}$$

where now $A_{-m}^i \equiv {}^{[3]}A_{-m}^i$. This example illustrates that terms with an odd number of DDF oscillators need not vanish in general. The invariance of under the little Weyl group S_8 can be made manifest by switching to a non-orthonormal and S_8 invariant basis of polarization vectors. This leads to slightly simpler expressions. However, this simplicity is an artefact caused by the size of the little Weyl group, which generically becomes trivial. In addition we note that, by (5.1), even the operator $\mathcal{L}_{-1}^{[\ell]}$ will involve oscillator number $m_1 + \dots + m_M = \ell$ and thus an exponentially growing number of terms with increasing level ℓ .

6 Outlook

The above results nicely display the increasing ‘‘anisotropy’’ of the root space representations in terms of the DDF basis with increasing level, a feature which we have already stressed before and which can be traced to the decrease (and eventual triviality) of the little Weyl group at higher level. While the further exploration of higher-level root spaces by direct methods as in [7] seems prohibitively difficult, prospects are much brighter with our new formula (3.15). What is still missing at this point is an analogous and similarly explicit expression for the full coset generators

$$\mathcal{K}_m^{[\ell]} := (\mathcal{L}_m^{[1]} \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1}) + \dots + (\mathbf{1} \otimes \dots \otimes \mathbf{1} \otimes \mathcal{L}_m^{[1]}) - \mathcal{L}_m^{[\ell]}. \quad (6.1)$$

While the last term involves transversal DDF operators only by (3.15), we have checked that the other terms will introduce a dependence on the longitudinal DDF operators. Since the operators $\mathcal{K}_m^{[\ell]}$ commute with the affine subalgebra this might also shed some light on the long-standing problem of finding explicit expressions for

the (non-polynomial) higher order Casimir invariants of affine algebras. We hope to come back soon to these issues in another publication.

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