On spin-(3/2) systems in Ricci flat space–times

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The Dirac formulation of massless spin-(3/2) fields is discussed. The existence and uniqueness for the solutions of the spin-(3/2) field equations in Dirac form is proven. It is shown that the system of equations can be split into a symmetric hyperbolic system of evolution equations and a set of constraint equations. The constraints are shown to propagate on a curved manifold if and only if it is an Einstein space. The gauge freedom present in the spin-(3/2) system is discussed and it is shown that the complete system "solutions modulo gauge" has a well posed Cauchy problem if and only if the Einstein equations hold. © *1995 American Institute of Physics.*

I. INTRODUCTION

The field equations for massless particles with spin-(3/2) have recently gained renewed interest. This is due to two observations:[1] first, the integrability conditions for the field equations are the vacuum Einstein equations. Second, the charges of the spin-(3/2) fields in flat space form a twistor space. It is this position of spin-(3/2) fields between the vacuum Einstein equations and twistor theory which has triggered interest in the system. It gave rise to the hope of providing a link between the twistor theory which up to now is well defined only on conformally flat space–times and the vacuum Einstein equations. The purpose of the present work is to show that the field equations have a well defined initial value problem on a curved manifold if and only if the Einstein tensor on that manifold vanishes, i.e., $G_{ab}=0$ is not only necessary but also sufficient for the local existence of solutions.

In the first article on higher spin equations, Dirac[2] derived the necessary form for wave equations which are to be satisfied by the wave functions of massless particles with arbitrary spin. A massless particle with spin s has to be described by a spinor field $\psi^{A\cdots B}_{C'\cdots D'}$ with p unprimed and p' primed indices, totally symmetric in the two kinds of indices and so that $p+p'=2s$. The field equation is

$$\nabla_{AA'}\psi^{A\cdots B}_{C'\cdots D'}=0, \tag{1.1}$$

which implies that each component of the spinor field satisfies the wave equation: $\Box\psi^{A\cdots B}_{C'\cdots D'}=0$. Here $\nabla_{AA'}$ is the usual spinor derivative operator on Minkowski space.

Later, Fierz[3,4] investigated these field equations and showed that different splittings of $2s$ into p and p' describe the same one-particle states. One can transform fields with index structure (p,p') into fields with index structure $(p+1,p'-1)$ by taking a derivative contracted over the primed index and symmetrized over the unprimed indices. Fierz constructed expressions for energy momentum and angular momentum for the fields and observed that there exist "trivial" solutions of the field equation (1.1) which can be added to any solution without affecting its energy momentum and angular momentum. These solutions are obtained from spin $s-1$ fields with index structure $(p-1,p'-1)$ which satisfy the field equation (1.1) by taking a derivative and symmetrizing over all indices. He proposed that two solutions of (1.1) should be regarded as equivalent if their difference is one of these "gauge solutions." So we have the view that spin s systems are to be regarded as solutions of (1.1) modulo gauge transformations. Obviously, among all the possible ways to decompose $2s$ into positive integers p and p' there are two distinguished

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ones, namely, those where either of them vanishes. These lead to cases where the resulting field equation has gauge invariant solutions for particles with positive and negative helicity and these equations are usually referred to as the zero rest-mass equations. They have the form

$$\nabla^{A}{}_{A'}\psi_{A\cdots B}=0, \quad \nabla^{A'}{}_{A}\phi_{A'\cdots B'}=0, \tag{1.2}$$

where ψ now is totally symmetric in its $2s$ indices.

Although the fields with different (p,p') are equivalent as states of free particles in flat space, problems arise when one tries to couple the fields to an external electromagnetic field. This was also pointed out by Fierz and together with Pauli[5] he presented a way to construct Lagrangians for massive and for massless fields of arbitrary spin in an external Maxwell field which gave rise to consistent equations. Similar inconsistencies arise when the field equation (1.1) is put onto a curved four-dimensional (pseudo-) Riemannian manifold. The appearance of the so called Buchdahl conditions severely restricts the solution spaces of (1.1) in most cases. Within the class of solutions of (1.1) there are three distinguished cases: $s=(1/2)$, $s=1$, and $s=(3/2)$. The Weyl equation $[s=(1/2)]$ is unrestricted (no Buchdahl condition) and gauge invariant. The case $s=1$ with $p=2$ and $p'=0$ corresponds to the Maxwell equations and is also gauge invariant and without Buchdahl conditions, while $p=p'=1$ corresponds to the version of the Maxwell equations using potentials which is not gauge invariant. Finally, the case $s=(3/2)$ is distinguished among all equations of the type (1.1) by the fact that only for $p=2$ and $p'=1$ the Ricci curvature alone appears in the Buchdahl conditions. In all other cases the Weyl curvature appears also. This establishes a link between the Einstein equation and the spin-(3/2) fields. The corresponding equation is

$$\nabla_{AA'}\rho^{AB}_{B'}=0. \tag{1.3}$$

The gauge solutions are of the form $\rho^{AB}_{B'} = \nabla^{B}_{B'}\nu^{A}$ with $\nabla^{B}_{B'}\nu_{B} = 0$. This form of the field equation is referred to as the Dirac form while there is another formulation to describe the same system which is due to Rarita and Schwinger.[6] The (RS)-formulation regards a spin-(3/2) field as a one-form Ψ_a with values in the space of Dirac spinors. The field equation is $\gamma^a\nabla_a\Psi_b=0$ with the supplementary condition $\gamma^a\Psi_a=0$ and the gauge transformations are $\Psi_a\to\Psi_a+\nabla_a\nu$ where $\gamma^a\nabla_a\nu=0$. Here the γ^a are the Dirac matrices.

The (RS)-system has been investigated by several authors. Velo and Zwanziger[7] have shown that the Cauchy problem for the field equation is well posed in flat space and have quantized the system. They pointed out that the characteristics of the system become spacelike in strong external electromagnetic fields so that the wave propagation can become acausal. A similar property was shown to be present in curved spaces by Madore.[8] Finally, there is a large number of authors who looked at the (RS)-system from the point of view of super-gravity (see, e.g., Ref. 9).

The aim of the present paper is to show local existence and uniqueness of solutions of the spin-(3/2) system modulo gauge solutions in its Dirac formulation (1.3) on manifolds which are Ricci flat. We show that the equations can be split into two sets of equations: evolution equations which can be put into a symmetric hyperbolic form and constraint equations which are propagated by the evolution equations provided that the Ricci tensor is pure trace. Thus, the Cauchy problem for Eq. (1.3) is locally well posed if the manifold is an Einstein space. This clarifies the role of the Buchdahl conditions in this case. We investigate the role of the gauge solutions and show that they remain "gauge" only if additionally also the scalar curvature vanishes. Thus, the spin-(3/2) system modulo gauge is well defined on Ricci flat manifolds.

The main tool employed in this work will be the formalism of space [or SU(2)] spinors which allows a straightforward decomposition of spinor equations into evolution and constraint equations. This formalism is briefly described in Sec. II. In Sec. III, we discuss the evolution and constraint equations. We treat the gauge solutions in Sec. IV, and conclude the paper with Sec. V.

II. SPACE SPINORS

We want to briefly introduce the formalism of space spinors which is very useful for analyzing the initial value problem for spinor equations.[10,11] This formalism can be set up in all those situations where there exists a distinguished time direction like the normal vector field of a foliation by spacelike hypersurfaces.

Let us first discuss things algebraically. In this work we will use the notation and conventions of Ref. 12. So let (S^A, ϵ_{AB}) and $(S^{A'}, \epsilon_{A'B'})$ be unprimed and primed spin space, i.e., the representation spaces of the fundamental representation of SL(2,$\mathbb{C}$) and its complex conjugate representation, respectively. Then we have the usual spinor–vector correspondence $\alpha^{AA'} \leftrightarrow \alpha^a$ between $S^A \otimes S^{A'}$ and complexified Minkowski vector space $\mathbb{M}$. Let t^a be a real future timelike vector which we assume to be normalized by the condition $t_a t^a = 2$, then $t_{AA'} t^{BA'} = \delta_A^B$. We may consider $t_{AA'}$ as defining a Hermitian form on spin space which we can use to identify $S^{A'}$ with S_A and hence also $S_{A'}$ with S_A. Explicitly, this isomorphism is $\pi_{A'} \mapsto \pi_{A'} t^{A'}{}_A$. It extends canonically to the complete spinor algebra. Formally, each primed index can be converted to an unprimed one, so primed indices never appear. One can also introduce a complex conjugation map on the spinor algebra by first defining $\hat{\pi}_A := \bar{\pi}_{A'} t^{A'}{}_A$ and then extending this map canonically to the full algebra. Note that $\hat{\pi}^A \pi_A \geqslant 0$ for all π_A and that $\hat{\pi}^A \pi_A = 0 \Leftrightarrow \pi_A = 0$. We introduce real spinors by the condition $\hat{\phi}_{A_1 \cdots A_{2n}} = (-1)^n \phi_{A_1 \cdots A_{2n}}$ (it is easily seen that this notion only makes sense for spinors with an even number of indices).

Given a covector $v_a = v_{AA'}$ we can convert the primed to an unprimed index and then decompose the result into irreducible parts: $v_{AA'} t^{A'}{}_B = \frac{1}{2} \epsilon_{AB} v_{CC'} t^{CC'} + v_{A'(A} t^{A'}{}_{B)}$. Thus, we see that if $v_a t^a = 0$ then $v_{AA'} t^{A'}{}_B = v_{BA'} t^{A'}{}_A$. So we can identify the three dimensional subspace of vectors orthogonal to t^a with the space S_{AB} of symmetric two-spinors and there is a canonical isomorphism $\mathbb{C} \oplus S_{AB} \leftrightarrow \mathbb{M}$.

Now suppose that (M,g) is a four-dimensional Lorentz manifold with a spin structure. Furthermore, suppose there exists a congruence of timelike curves on M with tangent vector t^a normalized by $t_a t^a = 2$. Then we can perform all the constructions above at each point of M. The covariant derivative operator $\nabla_{AA'}$ now can be decomposed as

$$t^{A'}{}_B \nabla_{AA'} = \epsilon_{AB} D + D_{AB}. \tag{2.1}$$

Then $2D = t^a \nabla_a$ is the covariant derivative along the congruence while the three (complex) operators D_{AB} act in the directions orthogonal to t^a. Since $\nabla_{AA'}$ is a real operator, we also have $\hat{D} = \bar{D} = D$ and $\hat{D}_{AB} = -D_{AB}$.

Associated with $t_{AA'}$ there are two spinor fields on M, $K_{AB} = t_A{}^{A'} D t_{BA'}$ and $K_{ABCD} = t_D{}^{C'} D_{AB} t_{CC'}$. Since t_a is normalized we have the symmetries $K_{AB} = K_{BA}$ and $K_{ABCD} = K_{(AB)(CD)}$. K_{AB} is the "acceleration" $t^a \nabla_a t_b$ of t_b while K_{ABCD} contains the expansion, twist and shear of the vector field t^a. The congruence is hypersurface orthogonal if and only if $K_{A(B}{}^A{}_{D)} = 0$ and then K_{ABCD} is the extrinsic curvature of these (spacelike) hypersurfaces. Both K_{AB} and K_{ABCD} are real spinor fields.

III. THE SPIN-(3/2) SYSTEM

Let us now consider the Rarita–Schwinger system in the Dirac form

$$P^B{}_{A'B'} \equiv \nabla_{AA'} \rho^{AB}_{B'} = 0. \tag{3.1}$$

Converting the primed to unprimed indices with respect to some timelike vector $t^{AA'}$ and using the decomposition (2.1) of $\nabla_{AA'}$ we obtain

$$P_{ABC} \equiv t^{A'}{}_A P_{BA'B'} t^{B'}{}_C = D\rho_{ABC} - D^E{}_A \rho_{EBC} - K_C{}^E \rho_{ABE} + K^E{}_{AC}{}^D \rho_{EBD} = 0, \tag{3.2}$$

with $\rho_{ABC}=\rho_{ABC'}t^{C'}{}_{C}$. Now we decompose ρ_{ABC} into its irreducible parts $\rho_{ABC}=r_{ABC}+2\epsilon_{C(A}r_{B)}$, where r_{ABC} is totally symmetric and obtain

$$P_{ABC}=Dr_{ABC}+2\epsilon_{C(A}Dr_{B)}-D_{A}{}^{E}r_{BCE}+D_{AC}r_{B}-\epsilon_{CB}D^{E}{}_{A}r_{E}$$
$$-K_{C}{}^{E}r_{ABE}-2K_{C(A}r_{B)}+K^{E}{}_{AC}{}^{D}r_{EBD}+2K^{E}{}_{AC(E}r_{B)}. \tag{3.3}$$

The irreducible parts of P_{ABC} now automatically yield the decomposition of Eq. (3.1) into evolution and constraint equations. In particular, we can get four systems of equations by taking, respectively, the totally symmetric part and the three possible traces of P_{ABC}. These are

$$0=P_{(ABC)}\equiv Dr_{ABC}-D_{(A}{}^{E}r_{BC)E}+D_{(AB}r_{C)}-K_{(A}{}^{E}r_{BC)E}-2K_{(AB}r_{C)}$$
$$+K^{E}{}_{(A}{}^{D}{}_{B}r_{C)ED}-K_{E(AB}{}^{E}r_{B)}+K^{E}{}_{(ABC)}r_{E}, \tag{3.4}$$

$$0=P_{CB}{}^{C}\equiv-3Dr_{B}+D^{AC}r_{ACB}+D^{E}{}_{B}r_{E}-K^{AC}r_{ACB}-K^{E}{}_{B}r_{E}+K^{E}{}_{A}{}^{AD}r_{EBD}+2K^{E}{}_{A}{}^{A}{}_{(E}r_{B)}, \tag{3.5}$$

$$0=P_{AB}{}^{B}\equiv-3Dr_{A}+3D^{E}{}_{A}r_{E}-K^{BC}r_{ABC}-K^{E}{}_{A}r_{E}-K^{E}{}_{A}{}^{BD}r_{EBD}+K^{E}{}_{A}{}^{B}{}_{E}r_{B}, \tag{3.6}$$

$$0=P_{B}{}^{B}{}_{C}\equiv D^{AB}r_{ABC}-2D^{B}{}_{C}r_{B}-K^{EB}{}_{C}{}^{D}r_{EBD}-2K^{EB}{}_{C(E}r_{B)}. \tag{3.7}$$

We find that (3.4) describes the evolution of r_{ABC} while (3.7) are the constraint equations. Both (3.5) and (3.6) are evolution equations for r_A. The difference between those two equations is proportional to the constraint equations. Therefore, we can take any linear combination α(3.5) $+\beta$(3.7) with complex constants α and β as evolution equations for r_A.

A. The evolution equations

First, we want to discuss the properties of the system of evolution equations which we take to consist of (3.4) together with a linear combination of (3.5) and the constraints (3.7). Our aim is to choose the linear combination in such a way that the resulting system becomes symmetric hyperbolic and is thus amenable to standard methods in the theory of partial differential equations.

Let us briefly review what a symmetric hyperbolic system is. Suppose we have a first order system of semilinear partial differential equations in an open set of $\mathbb{R}^n$ given in the form

$$A^{a}_{\mu\nu}(x)\partial_a\phi^{\nu}=B_{\mu}(x,\phi), \tag{3.8}$$

where the "unknown" $\phi^{\nu}=\phi^{\nu}(x)$ is a column vector of N complex valued functions. B_{μ} is a row vector of N complex valued functions which may depend on the coordinates x as well as on the unknowns ϕ^{ν} and $A^{a}_{\mu\nu}(x)$ is a square matrix with complex valued entries. Then the system (3.8) is symmetric hyperbolic if the following two conditions hold for all x: for all (real) covectors v_a the matrix $v_aA^{a}_{\mu\nu}(x)$ is Hermitian and there exists a covector t_a such that the matrix $t_aA^{a}_{\mu\nu}(x)$ is positive definite. Note that the matrix $v_aA^{a}_{\mu\nu}$ is obtained by replacing the derivative operator in the principal part of (3.8) by v_a. The Hermiticity condition is equivalent to the following statement: for all ϕ^{ν}, ψ^{μ}

$$\overline{\psi^{\mu}}A^{a}_{\mu\nu}v_a\phi^{\nu}=\overline{\overline{\phi^{\mu}}A^{a}_{\mu\nu}v_a\psi^{\nu}}, \tag{3.9}$$

i.e., the matrix $A^{a}_{\mu\nu}v_a$ is Hermitian with respect to the canonical inner product on $\mathbb{C}^N$ defined by $\langle\psi,\phi\rangle = \overline{\psi^{\mu}}\phi_{\mu}$.

In our case there are six unknowns ϕ^{ν} given by the pair of spinors (r^{ABC},r^{A}). Let us define the sesqui-linear form $\langle\psi,\phi\rangle=\hat{s}^{ABC}r_{ABC}+\hat{s}^{A}r_{A}$ on the space of these pairs. This form can easily be shown to be positive definite, thus defining an inner product. Now we have

$$\overline{\psi^\mu A^a_{\mu\nu} v_a \phi^\nu} = \hat{s}^{ABC}(v r_{ABC} - v_A{}^E r_{BCE} + v_{AB} r_C) + \hat{s}^A(-3\alpha v r_A + (\alpha+\beta) v^{BC} r_{ABC} + (\alpha - 2\beta) v^B{}_A r_B). \tag{3.10}$$

Here we have replaced the derivative operators in the principal part of the system with (v, v_{AB}) and we have taken an arbitrary linear combination of (3.5) and (3.7) as evolution equation for r_a. The complex numbers α and β have to be determined so that the equation (3.9) holds. From the rules of complex conjugation of SU(2) spinors we have $\hat{\hat{r}}_{ABC} = -r_{ABC}$ and $\hat{\hat{r}}_A = -r_A$ and hence the right hand side of (3.9) becomes

$$\overline{\phi^\mu A^a_{\mu\nu} v_a \psi^\nu} = -r^{ABC}(\bar{v}\hat{s}_{ABC} - \hat{v}_A{}^E \hat{s}_{BCE} + \hat{v}_{AB}\hat{s}_C) - r^A \times(-3\bar{\alpha}\bar{v}\hat{s}_A + (\bar{\alpha}+\bar{\beta})\hat{v}^{BC}\hat{s}_{ABC} - (\bar{\alpha}-2\bar{\beta})\hat{v}_{AB}\hat{s}_C). \tag{3.11}$$

The reality of v_a implies $v = \bar{v}$ and $v_{AB} = -\hat{v}_{AB}$ and so we find that (3.9) will be satisfied if we can arrange that

$$3\bar{\alpha} v r^A \hat{s}_A + r^{ABC} v_{AB}\hat{s}_C + (\bar{\alpha}+\bar{\beta}) r^A v^{BC}\hat{s}_{ABC} - (\bar{\alpha}-2\bar{\beta}) r^A v_{AB}\hat{s}^B = -3\alpha v \hat{s}^A r_A + (\alpha+\beta)\hat{s}^A v^{BC} r_{ABC} + \hat{s}^{ABC} v_{AB} r_C - (\alpha-2\beta)\hat{s}^A v_{AB} r^B. \tag{3.12}$$

This will be the case if we choose $(\alpha+\beta) = -1$ and $\bar{\alpha} = \alpha$. Furthermore, from (3.10) with $\psi^\mu = \phi^\mu$ we get for matrices with $v_{AB} = 0$

$$\overline{\phi^\mu A^a_{\mu\nu} v_a \phi^\nu} = v \hat{r}^{ABC} r_{ABC} - 3 v \alpha \hat{r}^A r_A. \tag{3.13}$$

So if we choose $\alpha < 0$ then the matrix $v_a A^a_{\mu\nu}$ will be positive definite for those v_a with $v_{AB} = 0$.

Thus we have shown that the system

$$D r_{ABC} - D_{(A}{}^E r_{BC)E} + D_{(AB} r_{C)} = K_{(A}{}^E r_{BC)E} + K_{(AB} r_{C)} - K^E{}_{(A}{}^D{}_B r_{C)ED} + K_{E(AB}{}^E r_{B)} - K^E{}_{(ABC)} r_E, \tag{3.14}$$

$$-3\alpha D r_B - D^{AC} r_{ABC} + (3\alpha+2) D^E{}_B r_E = \alpha(K^{AC} r_{ACB} + K^E{}_B r_E) + \alpha(K_B{}^{CDE} r_{CDE} + K_{BA}{}^A{}_C r^C) - K^{ECD}{}_B r_{CDE} - 2K^{CE}{}_{EB} r_C \tag{3.15}$$

is symmetric hyperbolic for any choice of $\alpha < 0$.

This property of the system of evolution equations implies existence and uniqueness of solutions. The precise procedure depends on the concrete situation one is interested in. Without going too much into details we want to present here how to get local existence and uniqueness of solutions. We choose a spacelike hypersurface Σ in M and introduce Gauss coordinates (t, x^i) with respect to Σ to cover an open neighborhood U of Σ. Then the surfaces t=const. foliate U which is topologically of the form $U = \Sigma \times \mathbb{R}$. We can use the congruence of normal vectors $t^a = \sqrt{2}\partial_t^a$ to introduce SU(2) spinors. We introduce an adapted spin basis (o_A, ι_A) by fixing a spinor o_A so that $t^{AA'} o_A o_{A'} = 1$ and then defining $\iota_A = \hat{o}_A$. Then (o_A, ι_A) is a normalized spinor dyad which can be used to expand r_{ABC} and r_A in terms of the coordinates. It can also be used to define an orthonormal frame e_1, e_2, e_3 on Σ:

$$e_1^{AB}=\frac{i}{\sqrt{2}}(o^A o^B+\iota^A \iota^B),$$

$$e_2^{AB}=\frac{1}{\sqrt{2}}(o^A o^B-\iota^A \iota^B),$$

$$e_3^{AB}=\frac{1}{\sqrt{2}}(o^A \iota^B+\iota^A o^B). \tag{3.16}$$

Furthermore, we need to express the spatial derivative operator D_{AB} in terms of the partial derivatives with respect to the Gauss coordinates. This can be done with the observation that the derivative operator D_{AB} is related to the SU(2) spin connection $\tilde{\nabla}_{AB}$ of the induced metric on Σ by the formula[13]

$$D_{AB}\lambda_C=\tilde{\nabla}_{AB}\lambda_c+K_{ABC}{}^D\lambda_D. \tag{3.17}$$

The Ricci rotation coefficients of $\tilde{\nabla}_{AB}$ with respect to the orthonormal frame (3.16) are then used to express the derivative operators in terms of the partial derivatives with respect to the Gauss coordinates. Inserting all these representations into the system we obtain a symmetric hyperbolic system on $\mathbb{R}^4$. Using results from Ref. 14 we have the following

Theorem III.1: *Let (M,g) be a manifold of class $\mathscr{C}^\infty$ and let Σ be a spacelike $\mathscr{C}^\infty$ hypersurface. Then there exists a neighborhood $U\sim\Sigma\times\mathbb{R}$ such that the system of evolution equations* (3.14), (3.15) *has a unique $\mathscr{C}^\infty$ solution for arbitrary $\mathscr{C}^\infty$ initial data of compact support.*

Finally, we want to determine the characteristics of the system. Recall,[14] that a hypersurface is characteristic for a first order system of partial differential equations $A^a_{\mu\nu}\partial_a\phi^\nu+b_\mu=0$ if it cannot be used as an initial data surface, i.e., if one cannot determine the outward derivative of the unknown functions from data given on the surface using the system. Let n_a be a normal covector to the hypersurface. Then we are led to compute the determinant $D(n)$ of the matrix $A(n)\equiv n_a A^a_{\mu\nu}$. If $D(n)=0$ then n_a is the conormal to a characteristic hypersurface.

In the present case, we write n_a as $n_{AA'}=nt_{AA'}+t^B_{A'}n_{AB}$. Then we choose an adapted spin frame as above and define $n_0\equiv n_{AB}o^A o^B$, $n_1\equiv n_{AB}o^A\iota^B$ and $n_2\equiv n_{AB}\iota^A\iota^B$, so that n_1 is real, $n_0=-\overline{n_2}$ and $n_a n^a=2(n^2-n_1^2+n_0n_2)$. Note, that $(n_1^2-n_0n_2)\geqslant 0$ and that it vanishes iff $n_0=n_1=n_2=0$. The matrix $A(n)$ for the present system with respect to the basis

$$\{(o_Ao_Bo_C,0),(3o_{(A}o_B\iota_{C)},0),(3o_{(A}\iota_B\iota_{C)},0),(\iota_A\iota_B\iota_C,0),(0,o_A),(0,\iota_A)\}$$

is

$$\begin{pmatrix} n+n_1 & n_2 & 0 & 0 & n_2 & 0 \\ -n_0 & 3n+n_1 & 2n_2 & 0 & -2n_1 & n_2 \\ 0 & -2n_0 & 3n-n_1 & n_2 & n_0 & -2n_1 \\ 0 & 0 & -n_0 & n-n_1 & 0 & n_0 \\ -n_0 & -2n_1 & -n_2 & 0 & -3\alpha(n+n_1)-2n_1 & -(3\alpha+2)n_2 \\ 0 & -n_0 & -2n_1 & -n_2 & (3\alpha+2)n_0 & -3\alpha(n-n_1)+2n_1 \end{pmatrix}.$$

The determinant of this matrix is found to be

$$D(n)=9(n^2-(n_1^2-n_0n_2))^2(9\alpha^2n^2-(\alpha+2)^2(n_1^2-n_0n_2)) \tag{3.18}$$

which implies that there are two sets of characteristic conormal cones, one of which coincides with the null cone in the cotangent space of each point and is doubly degenerate. The other cone depends on the parameter α, whose value determines its opening angle. Suppose, n_a lies on the second cone, then $n^2=\lambda^2(n_1^2-n_0n_2)$ with $\lambda\equiv(\alpha+2)/3\alpha$ and thus $n_an^a=n^2-(n_1^2-n_0n_2)=(\lambda^2-1)(n_1^2-n_0n_2)$. Therefore, n_a will be spacelike iff $\lambda^2<1$ and it will be timelike iff $\lambda^2>1$.

To each of the conormal cones there correspond a ray cone and a ray conoid (see Ref. 14). In the case of the null conormal cone at a point $p\in M$ the corresponding ray cone is the null cone in T_pM and the ray conoid is the light cone $\mathscr{C}_p$, i.e., the set of all null geodesics emanating from p. For the other conormal cone the ray cone is again a cone in the tangent space T_pM which is a timelike (resp. spacelike) hypersurface (apart from the vertex) if $\lambda^2<1$ (resp. $\lambda^2>1$). The corresponding ray conoid $\mathscr{C}_p(\alpha)$ is generated by the bicharacteristics emanating from p. These are curves in M which are projections from integral curves of the Hamiltonian vector field on the cotangent bundle corresponding to the quadratic form which defines the conormal cone. At each of its points the ray conoid is tangent to the local ray cone. Now if α ranges over the negative real numbers then λ^2 takes on all positive values. This shows that the ray conoid $\mathscr{C}_p(\alpha)$ can be made timelike or spacelike at will by choosing α appropriately.

The choice of α does not affect the property of the system being symmetric hyperbolic as long as α remains negative but it does affect the size of the domain of dependence of a point $p\in M$. In fact, given a point p in the future of an initial surface Σ we draw all the backward ray conoids at p defined by the system until they intersect Σ in two sets S and $S(\alpha)$. Then the domain of dependence of p is contained inside the intersection of the interiors of S and $S(\alpha)$. Since the ray conoid $\mathscr{C}_p(\alpha)$ is either spacelike or timelike it lies either completely outside or inside the light cone of p. Thus, either S is in the interior of $S(\alpha)$ or vice versa, so that the size of the domain of dependence is constrained by S [if $\mathscr{C}_p(\alpha)$ is spacelike] or by $S(\alpha)$ [if $\mathscr{C}_p(\alpha)$ is timelike]. Therefore, in order to obtain the maximal possible domain of dependence we need to make $\mathscr{C}_p(\alpha)$ spacelike. This will be the case if we choose $\alpha<-(1/2)$. Then $\lambda^2>1$, $\mathscr{C}_p(\alpha)$ is spacelike and the domain of dependence is given by the light cone $\mathscr{C}_p$ of p. The natural choice to be made for α is $\alpha=-(1/2)$ which makes the second characteristic coincide with the light cone, but for the moment we want to leave α undetermined to see its effect in the constraint equations.

B. The constraint equations

Now we want to discuss the constraint equations. Our aim is to show that the constraints are propagated by the evolution equations under certain circumstances. Before going into the details we need some definitions. As we have seen in (3.2) the complete system of equations is given by $P_{ABC}=0$. Let us define $E_{ABC}\equiv P_{(ABC)}$, $E_A\equiv-\alpha P_{AC}{}^C-P_C{}^C{}_A$ and $Z_A\equiv P_C{}^C{}_A$. Then the equations $E_{ABC}=0$ and $E_A=0$ are the Eqs. (3.14) and (3.15) which constitute the symmetric hyperbolic system and $Z_A=0$ gives the constraint equations (3.7). We have the decomposition

$$P_{ABC}=E_{ABC}-\frac{2}{3\alpha}\,\epsilon_{C(B}E_{A)}-\frac{\alpha+2}{3\alpha}\,\epsilon_{C(B}Z_{A)}+\frac{1}{2}\,\epsilon_{AB}Z_C. \tag{3.19}$$

We want to derive a system of equations obeyed by the constraints. To this end we compute

$$\nabla^E{}_{E'}Z_E=\nabla^E{}_{E'}P_A{}^A{}_E=\nabla^A{}_{E'}P_A{}^E{}_E-\nabla_{DE'}P_A{}^{DA}=\nabla^A{}_{E'}P_A{}^E{}_E+\nabla_{DE'}\nabla^{B'}_E\,\rho^{ED}_{B'}. \tag{3.20}$$

Here we have used the "epsilon-identity" $\epsilon_{A[B}\epsilon_{BC]}=0$ for the second equality. The second term on the right-hand side can be rewritten by commuting the covariant derivatives as follows

$$\begin{aligned}\nabla_{DE'}\nabla_A{}^{B'}\rho^{AD}_{B'}&=[\nabla_{DE'},\nabla_A{}^{B'}]\rho^{AD}_{B'}+\nabla_A{}^{B'}P^A{}_{E'B'}\\&=\Box_{AB}\rho^{AB}_{E'}-\nabla^{BB'}(t^A_{E'}t^C_{B'}P_{ABC})\end{aligned}$$

$$=\Box_{AB}\rho^{AB}_{E'}-\nabla^{BB'}(t^A_{E'}t^C_{B'})P_{ABC}-t^A_{E'}t^C_{B'}\nabla^{BB'}P_{ABC}. \tag{3.21}$$

Here we have introduced the unprimed spinor curvature derivation $\Box_{AB}$ as defined in Ref. 12. Rewriting the second term in terms of K_{AB} and K_{ABCD} we obtain the equation

$$t^{E'}_E\nabla^A{}_{E'}Z_A-t^{E'}_E\nabla^A{}_{E'}P_A{}^C{}_C+t^{B'C}\nabla^B{}_{B'}P_{EBC}=-K^A{}_EP_{AB}{}^B+K^{BCA}{}_EP_{ABC}+K^{BC}P_{EBC}$$
$$+K^{BDC}{}_DP_{EBC}+t^{E'}_E\Box_{AB}\rho^{AB}_{E'}. \tag{3.22}$$

Now assume that the evolution equations are satisfied in some space–time region. Then from (3.19) we have $P_{ABC}=-(\alpha+2)/3\alpha\epsilon_{C(B}Z_{A)}+1/2\epsilon_{AB}Z_C$ and after inserting this into (3.22) we obtain the equation

$$DZ_A+\frac{\alpha+2}{3\alpha}D_{AB}Z^B=-\frac{\alpha+2}{6\alpha}K_{AC}{}^{AC}Z_B+\frac{2\alpha-2}{3\alpha}K_{AB}Z^B-t^{B'}_A\Box_{CD}\rho^{CD}_{B'}. \tag{3.23}$$

The last term in this equation can be written as

$$t^{B'}_A\Box_{CD}\rho^{CD}_{B'}=t^{B'}_A\Phi_{CDB'}{}^{A'}\rho^{CD}_{A'} \tag{3.24}$$

by expressing the spin curvature derivation in terms of the curvature spinors. In this case only the Ricci spinor $\Phi_{ABA'B'}$ appears. If $\Phi_{ABA'B'}=0$ then the constraints satisfy a linear homogeneous first order system of partial differential equations which can easily be seen to be symmetric hyperbolic. The characteristic conormal cone is given by the equation

$$9\alpha^2n^2-(\alpha+2)^2(n_1^2-n_0n_2)=0, \tag{3.25}$$

so the ray cone at $p\in M$ determined by (3.23) is $\mathscr{C}_p(\alpha)$, the same as the second ray cone determined from the evolution equations.

With our choice of $\alpha<-(1/2)$ from the last section, we obtain the statement that the constraints are satisfied everywhere in the space–time region where the evolution equations hold if they are satisfied on an initial surface, provided the Ricci tensor of the manifold is pure trace. Finally, we can state the following theorem the proof of which is the contents of the previous two sections:

Theorem III.2: *Let (M,g) be an Einstein space with spin structure, then the spin-(3/2) system*

$$\nabla_{AA'}\rho^{AB}_{B'}=0 \tag{3.26}$$

admits a well-posed Cauchy problem.

IV. GAUGE TRANSFORMATIONS

As Fierz[3] pointed out, there are "trivial" solutions of the field equations for higher spin fields on flat space which have no energy momentum and angular momentum. He proposed to regard solutions of the field equations to be defined only up to the addition of these trivial solutions, thus introducing "gauge transformations." In the special case of spin-(3/2) the trivial solutions are of the form

$$\rho^{AB}_{B'}=\nabla^{(B}_{B'}\nu^{A)}, \tag{4.1}$$

where ν_A is a solution of the Weyl (anti) neutrino equation $\nabla^A_{A'}\nu_A=0$. The question arises under what conditions solutions of this form persist to be "gauge solutions" on curved spaces. The first point to discuss is the Cauchy problem for the Weyl equation. It is well known and easy to see that the Weyl equation can be put into the form of a symmetric hyperbolic system of evolution equations without constraints. Therefore, there are no problems related to the existence and uniqueness of solutions of the Weyl equation on curved manifolds from the point of view of the initial value formulation. Locally, there will always exist unique smooth solutions, given appropriate initial data. The second point of discussion is the question under what conditions will fields of the form (4.1) be solutions of (1.3). This is easily checked.[1] For $\rho_{B'}^{AB}$ of the form above to be a solution of (1.3) we need

$$0=\nabla_{AA'}\nabla_{B'}^{(B}\nu^{A)}=[\nabla_{AA'},\nabla_{B'}^{(B}]\nu^{A)}+\nabla_{B'}^{(B}\nabla_{AA'}\nu^{A)}=\tfrac{1}{2}\epsilon_{A'B'}(\Box_{A}{}^{B}+\nu^{A}+\Box\,\nu^{B})=\tfrac{3}{2}\epsilon_{A'B'}\Lambda\,\nu^{B}.$$

Here, we used the fact that solutions of the Weyl equation also satisfy the wave equation. Therefore, if the scalar curvature Λ of the manifold vanishes, then fields of the form (4.1) will still be solutions of the field equation (1.3).

Fierz proceeded further to claim that he could use the gauge freedom to put the fields into a special form. In terms of SU(2) spinors the field $\rho_{B'}^{AB}$ is represented by the pair (r_{ABC},r_A) of a spin-(3/2) part and a spin-(1/2) part. The gauge transformation in this representation is $r_{ABC}\mapsto r_{ABC}+D_{(AB}\nu_{C)}$, $r_A\mapsto r_A+(1/2)D\nu_A+(1/6)D_{AB}\nu^B$, with $D\nu_A+D_{AB}\nu^B=0$. In flat space we can choose t_a to be covariantly constant so that $K_{AB}=0$ and $K_{ABCD}=0$. Fierz claimed that it was always possible to put $r_A=0$ by a suitable gauge transformation. To do this, we use the Weyl equation to eliminate $D\nu_A$ from the transformation for r_A in favor of $D_{AB}\nu^B$. Then we can solve the equation $G_A\equiv r_A-(1/3)D_{AB}\nu^B=0$ for ν^B on the initial surface because this is a linear elliptic system. We can use this solution ν^A to gauge away the r_A component *on the initial surface.* However, problems arise with the propagation of G_A. In contrast to Fierz' statement it does not seem possible to maintain the gauge $r_A=0$ away from the initial surface because the equation $G_A=0$ is not preserved under the time evolution. Put differently, we can solve the equation $G_A=0$ on each time slice for ν_A. But then the resulting spinor field will not be a solution of the Weyl equation. The reason is that r_A itself is not a Weyl field. Since in flat space with t_a covariantly constant, the operators D and D_{AB} commute it follows from $G_A=0$ that r_A would have to satisfy the Weyl equation. Instead, r_A satisfies the equation (3.6):

$$-3Dr_A+D^E{}_A r_E=0. \tag{4.2}$$

Obviously, the situation does not improve in curved space with $R_{ab}=0$ for several reasons. Suppose it were possible to choose a gauge with $r_A=0$ in a space–time region where the evolution and constraint equations (3.4)–(3.7) are satisfied. Then (3.5) implies the algebraic constraint

$$K^{BC}r_{ABC}-K_A{}^{EBC}r_{EBC}=0. \tag{4.3}$$

This equation can be satisfied in general only if $K_{AB}=0$ and $K_{ABCD}=0$. But this implies $\nabla_a t_b=0$, i.e., that t_a is covariantly constant which severely restricts the curvature of the manifold. In fact, only pp-waves are allowed by this condition. But even if one can find a covariantly constant timelike vector then one would still be left with the incompatibility of r_A with the Weyl equation.

We saw in the last section that there are two characteristic surfaces for the evolution equations one of which has multiplicity two. Thus, there are four "components" of the spin-(3/2) field whose propagation is locked onto the light cone, while there are two further "components" whose propagation is determined by a "floppy" cone which can be made timelike or spacelike by choosing a parameter appropriately. It is tempting to attribute these "floppy parts" in the spin-(3/2) system to the gauge freedom. However, it is not obvious how they are affected by gauge transformations and furthermore, how to eliminate them globally by a gauge transformation. The problem lies in the fact, that these "components" are in fact only well defined objects on the projective spin bundle over M, thus containing information not only from the space-time point but also from all possible directions along which propagation is possible.

These considerations of (im)possible gauges does not affect the truth of the

Theorem IV.1: *The spin-*(3/2) *system on a Lorentz manifold consisting of equation* (1.3) *up to gauge transformations* (4.1) *admits a well-posed initial value problem if and only if the manifold is Ricci flat.*

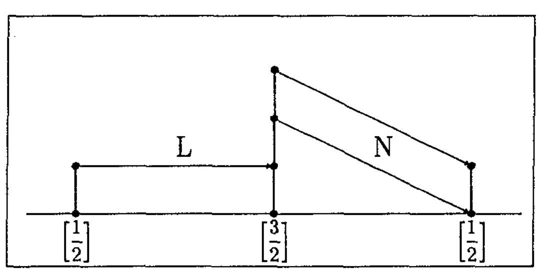
We want to make a final comment. Instead of Eq. (1.3) we consider the weaker equation

$$\nabla_{A(A'}\rho^{AB}_{B')}=0. \tag{4.4}$$

This equation falls into a class of higher spin systems that has recently been discussed.[15] It was shown that the equation is constraint free but can not be written in symmetric hyperbolic form. Using the existence theory of Leray and Ohya for equations with degenerate characteristics[16] it was possible to show that the Cauchy problem of this equation is well posed on arbitrary smooth manifolds. Solutions are in Gevrey classes whose index depended on the Gevrey index of the metric of the manifold. It should be possible to improve these smoothness results. Defining the solution space of (4.4) by the symbol $[\frac{3}{2}]$ and the solution space of the Weyl equation by $[\frac{1}{2}]$ we can summarize the results of this section in the sequence

$$[\tfrac{1}{2}]\xrightarrow{L}[\tfrac{3}{2}]\xrightarrow{N}[\tfrac{1}{2}], \tag{4.5}$$

where L is the map $\nu^A \mapsto \nabla^{(B}_{B'}\nu^{A)}$ and N is the map $\rho^{AB}_{B'} \mapsto \nabla^{B'}_{B}\rho^{AB}_{B'}$. Then we have $N \circ L=0$. The sequence above is not exact. In fact, it is exactly the difference between the image of L and the kernel of N which accounts for the "true degrees of freedom" of the spin-(3/2) fields. The structure of the maps in the sequence above can be visualized in the following diagram:



To discuss this diagram we first restrict ourselves to flat space. Then solutions of (4.4) are mapped by N into the solution space of the Weyl equation. If we use the appropriate function spaces then this map is actually surjective. We will not go into these details here. The kernel of N symbolized by the lower two thirds of the line above $[\frac{3}{2}]$ is the solution space of the equation (3.1). On the other hand, the map L maps the solution space of the Weyl equation injectively into the solution space of (4.4), again modulo questions of the degree of differentiability. In fact, the image of L symbolized by the lower third of the line above $[\frac{3}{2}]$ is contained in the kernel of N. The middle third above $[\frac{3}{2}]$, i.e., $\ker N/\mathrm{im}\, L$ is the part in the solution space of (4.4) which is "pure spin-(3/2)." It corresponds to the two degrees of freedom in a solution of the massless field equation.

Now it is interesting to note that the structure of this diagram is preserved if and only if the manifold is Ricci flat. This comes about in a rather symmetric way: if the Ricci tensor is pure trace, $R_{ab}=\frac{1}{4}Rg_{ab}$ then the maps L and N are well defined. If in addition also $R=0$ then we have the property $\mathrm{im}\, L \subset \ker N$, i.e., that gauge solutions are divergence free.

V. CONCLUSION

We have shown that the spin-(3/2) system defined by the solutions of $\nabla_{AA'}\rho_{B'}^{AB} = 0$ modulo solutions of the Weyl equation is well defined on curved manifolds if and only if they are Ricci flat. This is an interesting property of that system for two reasons. First, it opens up a relationship between the vacuum Einstein equations and spin-(3/2) fields. This has been observed again and again over the years and is well known from super-gravity. The reason for this is that among all the relativistic wave equations derived by Dirac the spin-(3/2) system is the only one (apart from those which do not involve curvature at all) for which the Weyl curvature does not appear in the consistency conditions. Julia[17] has pointed out that one could regard the spin-(3/2) system as a linear system associated with the Einstein equations in analogy with the Lax pair of integrable systems. However, this analogy is not as far reaching as one would hope. (I am grateful to Lionel Mason for pointing this out.)

The spin-(3/2) system is interesting for another reason: the two possible formulations for spin-(3/2) systems (up to the sign of the helicity) corresponding to the splitting ($p=3$, $p'=0$) (the zero rest-mass equation) and ($p=2$, $p'=1$) (the equations considered here) are equivalent in flat space as was pointed out by Fierz. However, the two formulations are radically different on curved spaces. While the ($p=2$, $p'=1$) formulation is well defined on Ricci flat manifolds the zero rest-mass formulation is well defined only on conformally flat manifolds. This is because the field equation $\nabla_{A'}^{A}\psi_{ABC} = 0$ implies the Buchdahl condition $\Psi_{ABCD}\psi^{ABC}=0$. The field equation can be split into a symmetric hyperbolic system of evolution equations and constraint equations which are propagated by the time evolution only if the Weyl curvature vanishes.

The relation between the spin-(3/2) formulation on Ricci flat spaces on the one hand and the massless field formulation with its associated twistor space of charges is the starting point of a recent research programme to obtain a twistor formulation for Ricci flat manifolds, i.e., for the full vacuum Einstein equations.[1]

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