

Crushing singularities in spacetimes with spherical, plane and hyperbolic symmetry

Alan D Rendall†

Max-Planck-Institut für Astrophysik, Karl-Schwarzschildstr. 1, Postfach 1523, 85740 Garching, Germany

Received 14 November 1994, in final form 14 March 1995

Abstract. It is shown that the initial singularities in spatially compact spacetimes with spherical, plane or hyperbolic symmetry admitting a symmetric compact constant mean curvature hypersurface are *crushing singularities* when the matter content of spacetime is described by the Vlasov equation (collisionless matter) or the wave equation (massless scalar field). In the spherically symmetric case it is further shown that if the spacetime admits a maximal slice then there are crushing singularities both in the past and in the future. The essential properties of the matter models chosen are that their energy–momentum tensors satisfy certain inequalities and that they do not develop singularities in a given regular background spacetime.

PACS numbers: 0420D, 0440

1. Introduction

The nature of singularities in general solutions of Einstein's equations is still a matter about which very little is known. Up to now the best information obtained on this subject concerns the singularities in special classes of solutions defined by symmetry assumptions. The hope is that the insights obtained in solving these restricted problems will allow the symmetry assumptions to be progressively relaxed and so the study of singularities in solutions of the Einstein equations with various symmetries can be seen as a systematic approach to the general problem of understanding spacetime singularities. The following is intended as a contribution in this direction.

At the moment the study of singularities in solutions of the vacuum Einstein equations is significantly further advanced than in the case of the Einstein equations coupled to matter. This paper is mainly concerned with non-vacuum spacetimes and helps to redress the balance a little. The spacetimes treated have spherical, plane or hyperbolic symmetry and a compact Cauchy hypersurface. (The reader wishing to gain some insight into this class of spacetimes is referred to appendix B, where the vacuum solutions are determined explicitly.) Some of the results are for general matter models which are only restricted by some inequalities on the components of the energy–momentum tensor. However, the main results require a matter model which is well behaved in a certain sense. Roughly speaking, the matter fields should not develop singularities in a given regular spacetime. These results are worked out for two particular matter models, namely collisionless matter and the massless scalar field (theorems 3.2 and 3.3, respectively).

† Present address: Institut des Hautes Etudes Scientifiques, 35 Route de Chartres, 91440 Bures sur Yvette, France.

There are some differences between the results for the different symmetry classes. For plane symmetry it is shown that, if the spacetime is the maximal globally hyperbolic development of constant mean curvature (CMC) initial data on a compact spacelike hypersurface, there exists a foliation of compact hypersurfaces of constant mean curvature which covers either the past or the future of the initial hypersurface. If the mean curvature of the initial hypersurface is H_0 then the range of the mean curvature of this foliation contains the interval $(-\infty, H_0]$ or $[H_0, \infty)$, respectively. The analogous result in the case of hyperbolic symmetry is proved under the additional assumption that the mass function, defined by (2.15), is positive on the initial hypersurface. In the spherically symmetric case the results are as follows. If the initial hypersurface is maximal (i.e. its mean curvature is zero) then it is shown that the spacetime can be covered by a CMC foliation where this time the range of the mean curvature is the whole real line. This should probably be true whatever the mean curvature of the initial hypersurface but all that could be shown is that there is a foliation by compact CMC hypersurfaces whose range includes all positive real numbers or all negative real numbers and this foliation will, in general, not cover the whole spacetime. It does, however, cover the part of the spacetime either to the past or to the future of the initial hypersurface. All these results are dependent on the choice of a well behaved matter model, as indicated above. In the spherically symmetric case they are also dependent on the assumption that the topology of the Cauchy hypersurface is $S^2 \times S^1$ so that there is no centre of symmetry. The differences between the various cases are related to the fact that while an initially expanding solution with plane or hyperbolic symmetry can be expected to expand for ever a spherically symmetric solution can be expected to recollapse.

In [6, 7] conjectures were formulated concerning the existence of CMC foliations. The results just discussed are closely related to conjecture 2.3 of [7] and conjecture C2 of [6] for the class of spacetimes considered here. They show in particular that any of these spacetimes has a crushing singularity in at least one time direction (and in both time directions if it contains a maximal hypersurface). Recall that a crushing singularity is one where there exists a foliation on a neighbourhood of the singularity whose mean curvature tends uniformly to infinity as the singularity is approached [7]. A difficulty in studying spacetime singularities which may exist at the boundary of the maximal Cauchy development is that the definition of the maximal Cauchy development is so abstract (Zorn's lemma is used to show its existence). Having a geometrically defined global time coordinate helps to make it more concrete. It allows the strong cosmic censorship hypothesis to be reformulated as a question on the global existence and asymptotic behaviour of solutions of a system of partial differential equations [6]. In this sense proving the existence of global CMC foliations in a given class of spacetimes is a first step towards proving strong cosmic censorship in this class. It also guarantees that a numerical calculation done using a CMC slicing is, in principle, capable of covering the whole spacetime. (The possibility of using a CMC slicing for numerical studies of cosmological spacetimes has been discussed in [9].)

The most general results on the existence of CMC foliations in any class of spacetimes with compact Cauchy hypersurfaces are due to Isenberg and Moncrief [10]. They showed that Gowdy spacetimes (which are vacuum spacetimes with $U(1) \times U(1)$ symmetry) on the torus can be foliated globally by constant mean curvature slices. For spacetimes with matter less is known. In [13] it was shown that spatially homogeneous spacetimes with matter described by the Vlasov equation can be foliated by CMC hypersurfaces with the mean curvature ranging either over the whole interval $(-\infty, 0)$ or the whole real line, depending on the symmetry type. In [15] a similar result was obtained for perfect fluids and for non-interacting mixtures of perfect fluids.

There is another motivation for the work reported in this paper which has not yet been mentioned. When investigating the global structure of spacetimes it seems essential to introduce some auxiliary elements such as coordinates in order to 'find one's way in spacetime'. If the spacetimes being studied have symmetry properties then it may be advantageous to use elements which exist due to the symmetry. The problem with this, from the point of view of the general programme outlined above, is that this approach is less likely to produce techniques which can be used beyond the given symmetry type. The definition of a CMC hypersurface does not depend on any symmetry assumptions and there is at least a chance that global CMC foliations will exist in rather general spacetimes. One aim of this work was to see whether it was possible to prove something about the properties of singularities working with a CMC foliation from the beginning. This turned out to be the case.

It is plausible that using coordinates adapted to a special symmetry type could be more efficient for proving sharper results than CMC slicing. This is confirmed by the fact that more precise information about the nature of the singularities in the spacetimes considered in this paper has been obtained in certain cases using other coordinate systems. In [14] the case of a plane symmetric scalar field was studied. It was shown that the initial singularity is a curvature singularity and a velocity-dominated singularity in that case. These results provide a model for what one would like to do more generally. For the case of collisionless matter Rein [12] has shown that under certain assumptions on the initial data the initial singularity is a curvature singularity and possesses some attributes of a velocity-dominated singularity.

2. Analysis of the field equations

The spacetimes of interest here are defined on manifolds of the form $M = \mathbf{R} \times S^1 \times F$, where F is a compact orientable surface. Let p denote the projection of the universal cover \tilde{F} onto F . Let $g_{\alpha\beta}$ be a globally hyperbolic metric on M for which each submanifold $\{t\} \times S^1 \times F$ is a Cauchy hypersurface. The spacetime $(M, g_{\alpha\beta})$ is called spherically symmetric if $F = S^2$ and the transformations of M induced by the standard action of $SO(3)$ on S^2 are isometries which leave any matter fields invariant. In the case $F = T^2$ the projection p induces a projection \hat{p} of $\tilde{M} = \mathbf{R} \times S^1 \times \mathbf{R}^2$ onto M . Let $\tilde{g}_{\alpha\beta}$ be the pull-back of $g_{\alpha\beta}$ by \hat{p} . The spacetime is called plane symmetric if $F = T^2$ and if the transformations of \tilde{M} induced by the standard action of the Euclidean group E_2 on \mathbf{R}^2 are isometries which leave (the pull-backs of) any matter fields invariant. In the case where the genus of F is greater than one \tilde{F} can be identified with the hyperbolic plane H^2 . A projection \hat{p} can be defined as in the plane symmetric case. The spacetime is said to have hyperbolic symmetry if the genus of F is greater than one and the transformations of $\tilde{M} = \mathbf{R} \times S^1 \times H^2$ induced by the action of the connected component of the identity of the isometry group of the hyperbolic plane are isometries which leave any matter fields invariant. In the spherically symmetric case $SO(3)$ acts on $S^1 \times S^2$ without fixed points. In other words there are no centres. The group $SO(3)$ also acts on S^3 with fixed points and this leads to a different class of spherically symmetric spacetimes. This other class will not be considered in this paper and so for clarity the spherically symmetric spacetimes considered here will be referred to as spherically symmetric spacetimes without centre. It will be convenient to refer to spherically symmetric spacetimes without centre and spacetimes with plane and hyperbolic symmetry collectively as surface symmetric spacetimes. The surfaces diffeomorphic to F which are defined by the product decomposition will be referred to as surfaces of symmetry.

Now the Einstein equations for surface symmetric spacetimes will be analysed in a certain coordinate system. In order to ensure the existence of such a coordinate system it will be assumed that the spacetime possesses a Cauchy hypersurface of constant mean curvature which is symmetric, in the sense that it is a union of surfaces of symmetry.

Lemma 2.1. Let (M, g) be a non-flat surface symmetric spacetime having a symmetric constant mean curvature Cauchy hypersurface and satisfying the dominant energy and non-negative pressures conditions. Then in a neighbourhood of any point there exist local coordinates adapted to the product decomposition $\mathbf{R} \times S^1 \times F$ such that the metric takes the form

$$-\alpha^2 dt^2 + A^2[(dx + \beta dt)^2 + a^2 d\Sigma^2]. \tag{2.1}$$

The functions α, β and A depend on t and x , a depends only on t and the metric $d\Sigma^2$ has constant curvature ϵ . It may be assumed without loss of generality that $\epsilon = 1$ for spherical symmetry, $\epsilon = 0$ for plane symmetry and $\epsilon = -1$ for hyperbolic symmetry. The time coordinate t may be chosen so that the hypersurface where t has a given value has constant mean curvature equal to that value.

Proof. See appendix A.

Remark. As mentioned in appendix A, a CMC hypersurface is automatically symmetric if $\epsilon = 1$ or 0 . Hence the word ‘symmetric’ in the statement of this lemma and those of a number of other results in this paper is only necessary in the case $\epsilon = -1$.

The neighbourhood in this lemma may be chosen to be the product $I_1 \times I_2 \times U$, where I_1 and I_2 are intervals and U is an open subset of F . The interval I_1 will be denoted by (t_1, t_2) in the following. By a slight extension of the usual notion of coordinates I_2 can be taken to be the closed interval $[0, 2\pi]$, where it is understood that $x = 0$ and $x = 2\pi$ are to be identified. The functions α, β and A are assumed to be functions which have C^∞ extensions to functions which are 2π -periodic in x . The coordinates can be chosen so that $\int_0^{2\pi} \beta(t, x) dx = 0$ for each t . The second fundamental form of each hypersurface $t = \text{constant}$ can be written as

$$A^2(K dx^2 - \frac{1}{2}(K - t)a^2 d\Sigma^2) \tag{2.2}$$

for a function $K(t, x)$ which has the same regularity properties as those demanded of α, β and A above. The Einstein equations will now be written out for the metric (2.1), making use of the variable K defined by (2.2):

$$(A^{1/2})'' = -\frac{1}{8}A^{5/2}\left[\frac{3}{2}(K - \frac{1}{3}t)^2 - \frac{2}{3}t^2 + 16\pi\rho\right] + \frac{1}{4}\epsilon A^{-1/2}a^{-2} \tag{2.3}$$

$$\alpha'' + A^{-1}A'\alpha' = \alpha A^2\left[\frac{3}{2}(K - \frac{1}{3}t)^2 + \frac{1}{3}t^2 + 4\pi(\rho + \text{tr}S)\right] - A^2 \tag{2.4}$$

$$K' + 3A^{-1}A'K - A^{-1}A't = 8\pi jA \tag{2.5}$$

$$\beta' = -a^{-1}\partial_t a + \frac{1}{2}\alpha(3K - t) \tag{2.6}$$

$$\partial_t a = a[-\beta' + \frac{1}{2}\alpha(3K - t)] \tag{2.7}$$

$$\partial_t A = -\alpha KA + (\beta A)' \tag{2.8}$$

$$\partial_t K = \beta K' - A^{-2}\alpha'' + A^{-3}A'\alpha' + \alpha[-2A^{-3}A'' + 2A^{-4}A'^2 + Kt - 8\pi S_1^1 + 4\pi \text{tr}S - 4\pi\rho]. \tag{2.9}$$

The primes here denote derivatives with respect to x . Equation (2.3) is the Hamiltonian constraint while (2.5) is the momentum constraint. The constant mean curvature condition leads to the lapse equation (2.4). Equation (2.6) is a consequence of the choice of spatial coordinate condition. Equations (2.7) and (2.8) come from the definition of the second fundamental form and (2.9) is the one independent Einstein evolution equation which exists in this situation. To give the definition of the matter quantities occurring in (2.9) it is convenient to introduce a (locally defined) orthonormal frame. Let e_0 be the future-pointing unit normal to the hypersurfaces of constant t . Let e_1 be a unit vector tangent to these hypersurfaces which is normal to the surfaces of constant t and x . Complete e_0 and e_1 to an orthonormal frame by adding vectors e_2 and e_3 . Then $\rho = T_{\alpha\beta}e_0^\alpha e_0^\beta$, $j = -T_{\alpha\beta}e_0^\alpha e_1^\beta$, $S_1^1 = T_{\alpha\beta}e_1^\alpha e_1^\beta$ and $\text{tr} S = T_{\alpha\beta}(e_1^\alpha e_1^\beta + e_2^\alpha e_2^\beta + e_3^\alpha e_3^\beta)$.

Malec and Murchadha [11] have written the constraints in an alternative form which turns out to be very useful in certain circumstances. (In fact they only consider the spherically symmetric case but the extension to plane and hyperbolic symmetry is straightforward.) They use as fundamental variables the expansions θ and θ' of the families of null geodesics orthogonal to the orbits. (The prime in θ' is not a derivative.) In terms of the above coordinates these are given explicitly by

$$\theta = 2A^{-2}A' + K - t \quad \theta' = 2A^{-2}A' - K + t. \tag{2.10}$$

The sign conventions for θ and θ' are those of [11] but the sign convention for the second fundamental form is the opposite of that used in [11]. The normalization of θ and θ' depends on a choice of Cauchy hypersurface. Equation (2.10) applies to the case where the hypersurface chosen is a level surface of the coordinate t . However, the analysis of [11] applies to any Cauchy hypersurface S compatible with the product decomposition of M in the sense that it is a union of surfaces of symmetry. The mean curvature of a Cauchy hypersurface of this kind will be denoted by H and is in general not constant. Let l be a proper length parameter along a curve in S orthogonal to the surfaces of symmetry. When expressed in terms of θ , θ' and l the constraints (2.3) and (2.5) become

$$\begin{aligned} \partial_l \theta &= -8\pi(\rho - j) - \frac{3}{4}\theta^2 - \theta H + \epsilon(aA)^{-2} \\ \partial_l \theta' &= -8\pi(\rho + j) - \frac{3}{4}\theta'^2 + \theta' H + \epsilon(aA)^{-2}. \end{aligned} \tag{2.11}$$

For the following discussion it is useful to introduce the area radius $r = aA$. The idea of [11] is to use (2.11) to obtain bounds for the quantities $r\theta$ and $r\theta'$. It is assumed that the dominant energy condition holds so that $|j| \leq \rho$. From equation (2.11)

$$\begin{aligned} \partial_l(r\theta) &= -8\pi r(\rho - j) - \frac{1}{4r}(\theta^2 r^2 - 4\epsilon + 4\theta Hr^2 + \theta r(\theta r - \theta' r)) \\ \partial_l(r\theta') &= -8\pi r(\rho + j) - \frac{1}{4r}(\theta'^2 r^2 - 4\epsilon - 4\theta' Hr^2 + \theta' r(\theta' r - \theta r)). \end{aligned} \tag{2.12}$$

Now consider one particular Cauchy hypersurface S which is a union of surfaces of symmetry. Denote the maximum value attained by $r\theta$ and $r\theta'$ on this hypersurface by M_+ and the minimum by M_- . Let x_0 be a point where M_+ is attained and suppose without loss of generality that $\theta(x_0) \geq \theta'(x_0)$. Since x_0 is a critical point of $r\theta$ it follows from (2.12) that at that point

$$\theta^2 r^2 + (4Hr)(\theta r) - 4\epsilon \leq 0. \tag{2.13}$$

Working out the roots of the corresponding quadratic equation then shows that

$$-2(|Hr| + \sqrt{\epsilon + H^2 r^2}) \leq M_+ \leq 2(|Hr| + \sqrt{\epsilon + H^2 r^2}). \quad (2.14)$$

The same inequality holds for M_- . For each symmetry type this implies that $r\theta$ and $r\theta'$ can be bounded in terms of H and r . An important quantity in the following is the mass function m which is defined by

$$\epsilon - 2m/r = \frac{1}{4}r^2\theta\theta'. \quad (2.15)$$

The boundedness result just obtained shows that the following holds:

Lemma 2.2. Let (M, g) be a surface symmetric spacetime which is foliated by symmetric compact CMC hypersurfaces with the mean curvature varying in a finite interval (t_1, t_2) . If the dominant energy condition holds and r is bounded then $2m/r$ is bounded.

There is another formulation of the equations which is also useful. This is based on the fact that the field equations can be written as equations on the two-dimensional manifold of symmetry orbits. In the following lower case Latin indices take the values 0 and 1 and are used to express tensor equations on this quotient manifold. In particular, g_{ab} and T_{ab} denote the tensors on the quotient manifold naturally related to the spacetime metric and the energy-momentum tensor. The equations of importance in the following are

$$\nabla_a \nabla_b r = \frac{m}{r^2} g_{ab} - 4\pi r (T_{ab} - \text{tr} T g_{ab}) \quad (2.16)$$

$$\nabla_a m = 4\pi r^2 (T_{ab} - \text{tr} T g_{ab}) \nabla^b r. \quad (2.17)$$

Note that these equations do not contain ϵ explicitly. The expression for m in this formulation is

$$m = \frac{1}{2}r(\epsilon - \nabla^a r \nabla_a r). \quad (2.18)$$

The following is a generalization of an argument of Burnett [2].

Lemma 2.3. Let (M, g) be a surface symmetric spacetime and let S be a compact Cauchy hypersurface of the form $\bar{S} \times F$ for a curve \bar{S} in $\mathbb{R} \times S^1$. Let r_{\min} and r_{\max} denote the minimum and maximum values of r on S , respectively. Define R_ϵ to be r_{\min} , 0 or $-r_{\max}$ for ϵ equal to 1, 0 or -1 , respectively. Then $2m \geq R_\epsilon$ on S .

Proof. Let U be the set of $x \in S$ where $2m < R_\epsilon$. Then $\nabla^a r$ is spacelike on U . Let s_a be the projection of the gradient of r onto S . Then

$$s^a \nabla_a m = (T_{ab} - \text{tr} T g_{ab}) s^a \nabla^b r. \quad (2.19)$$

If the dominant energy condition holds then the right-hand side of (2.19) is non-negative. Thus on U the mass m increases in the direction in which r increases. Now the restriction of r to S cannot have a stationary point in U . This means, in particular, that $S \setminus U$ is non-empty. If x_0 is a point of U then moving from x_0 in the direction of increasing r leads to an increase in m . Eventually a point of the boundary of U must be reached. At that point $m = R_\epsilon/2$. Thus $2m(x_0) \leq R_\epsilon$. Moving away from x_0 in the opposite direction gives the reverse inequality. Hence $2m = R_\epsilon$ on U . However, this contradicts the definition of U unless U is empty.

In the spherical case lemma 2.3 implies that the minimum of $2m$ on S is greater than or equal to the minimum of r there. For plane symmetry it implies that the mass is non-negative. The latter conclusion can be strengthened using (2.11) to get a kind of positive mass theorem.

Lemma 2.4. Let (M, g) be a plane symmetric spacetime which satisfies the dominant energy condition. Then if $m = 0$ at any point the spacetime is flat.

Proof. Consider any compact Cauchy hypersurface S of the form $\bar{S} \times F$. If m vanishes at some point x_0 of S then (for $\epsilon = 0$) either θ or θ' must vanish there. Suppose without loss of generality that it is θ . Equation (2.11) gives

$$\partial_t \theta = -(H + \frac{3}{4}\theta)\theta - 8\pi(\rho - j). \quad (2.20)$$

This equation is similar to one which arises in a similar context for Gowdy spacetimes [5] and can be treated in exactly the same way. In fact if l is an arc length parameter which is zero at x_0 then the solution of (2.20) is

$$\theta(l) = -8\pi \int_0^l (\rho - j)(u) \exp \left[\int_u^l (-H - \frac{3}{4}\theta)(v) dv \right] du. \quad (2.21)$$

From this formula it is clear that θ is everywhere non-positive and that it can only become zero for some positive l if $\rho - j$ vanishes identically on the interval $[0, l]$. In that case θ also vanishes on that interval. Since $\theta(l)$ is a periodic function it follows that it must be identically zero and that $\rho = j$ everywhere on S . The mass is also zero on S . When θ is zero the rate of change of r along \bar{S} is given by θ' . Since the restriction of r to \bar{S} must have a critical point somewhere, θ' must vanish somewhere. Applying to θ' the argument previously applied to θ shows that $\theta' = 0$ and that $\rho = j = 0$. By the dominant energy condition $\rho = 0$ implies the vanishing of the whole energy-momentum tensor on S . When the dominant energy condition holds the vanishing of the energy-momentum tensor on a Cauchy hypersurface implies that it vanishes everywhere. Thus the spacetime is a vacuum. This in turn implies that m is zero on the whole spacetime. It follows that θ and θ' are identically zero and that r is constant. In vacuum the Gaussian curvature of the metric g_{ab} is $K = r^{-1}\Delta r$ and so in the present case $K = 0$ and g_{ab} is flat. It is easily seen that this and the constancy of r imply that the spacetime is flat.

In the case of hyperbolic symmetry the following analogue of lemma 2.4 holds.

Lemma 2.5. Let (M, g) be a spacetime with hyperbolic symmetry which satisfies the dominant energy condition. Then $\nabla^a r$ is timelike.

Proof. $\nabla^a r$ must be timelike somewhere on a symmetric Cauchy surface, as shown in the proof of lemma 2.3. Hence it suffices to show that $\nabla_a r \nabla^a r$ never vanishes. This follows immediately from the analogue of (2.21).

Non-flat spacetimes with plane symmetry and all spacetimes with hyperbolic symmetry have the property that the gradient of r is either everywhere past-pointing timelike or everywhere future-pointing timelike. Spacetimes where the first possibility is realized may be called 'expanding' models those where the second is realized 'contracting' models. This terminology is justified by the following considerations. If the gradient of r is past-pointing then θ is positive and θ' is negative. From equation (2.10) it then follows that $t < K$ everywhere. If t were non-negative then this would mean that $|K|$ was everywhere greater

than $|t|$. However, this is inconsistent with the existence of a compact CMC hypersurface, as can be seen by integrating the Hamiltonian constraint (2.3). Thus if the gradient of r is past-pointing in a region foliated by compact CMC hypersurfaces then t must be negative. Similarly, if the gradient of r is future-pointing, t must be positive. By possibly replacing t by $-t$ it can be assumed without loss of generality that the model is expanding, i.e. that the r increases monotonically with t along any causal curve. In that case $t < 0$ in the whole spacetime so that $t_1 \leq 0$. In this case it will be assumed that in fact $t_2 < 0$.

The information obtained so far in this section implies bounds for various geometrical quantities in a surface symmetric spacetime without centre defined on a finite time interval (t_1, t_2) . Now as many other quantities as possible will be bounded. Suppose that the energy-momentum tensor satisfies the non-negative pressures condition. Then equation (2.17) implies that when $\nabla_a r$ is timelike the rates of change of r and m along an integral curve of $\nabla_a r$ have opposite signs. It follows that in the cases $\epsilon = 0$ and $\epsilon = -1$ the radius r is bounded below by a positive constant and the mass bounded above on the interval (t_3, t_2) while the radius is bounded above and the mass bounded below on the interval (t_1, t_3) , where t_3 is any time with $t_1 < t_3 < t_2$. If $\epsilon = 0$ or if $\epsilon = -1$ and it is assumed that m is positive for $t = t_3$ then the mass is bounded below by a positive constant on (t_1, t_3) . In the case $\epsilon = 1$ results of Burnett [2] show that the radius is bounded above and the mass bounded below by a positive constant on both these intervals. Using the upper bound for $2m/r$ obtained earlier it can be seen that on an interval where the radius is bounded above and the mass bounded below by a positive constant the mass is bounded above and the radius is bounded away from zero.

An estimate for the lapse function α can be obtained from (2.4). Considering a point where α attains its maximum and using the fact that $\rho + \text{tr} S \geq 0$ shows that $\alpha \leq 3/t^2$. Hence if $t_1 \neq 0$ and $t_2 \neq 0$ it follows that α is bounded on (t_1, t_2) . An interval which satisfies $t_1 \neq 0$ and $t_2 \neq 0$ and where r is bounded above and m is bounded below by a positive constant will be called admissible. The volume of the slice $t = \text{constant}$ is $V(t) = C \int_0^{2\pi} a^2 A^3$ and its time evolution is given by

$$dV/dt = -Ct \int_0^{2\pi} \alpha a^2 A^3. \quad (2.22)$$

In the case of an admissible interval this shows that $V(t)$ and its inverse are bounded. Now $V = a^{-1} \int_0^{2\pi} r^3$ and so if $t_1 \neq 0$, $t_2 \neq 0$ there are positive constants C_1 and C_2 such that $C_1 \leq a \leq C_2$ and $C_1 \leq A \leq C_2$. Now integrate (2.3) from 0 to 2π . This results in the inequality

$$\int_0^{2\pi} 16\pi \rho A^{5/2} \leq 2 \int_0^{2\pi} A^{-1/2} a^{-2} + \frac{2}{3} \int_0^{2\pi} t^2 A^{5/2} \quad (2.23)$$

which implies that on an admissible interval $\int_0^{2\pi} \rho$ is bounded. The dominant energy condition then gives bounds for $\int j$ and $\int \text{tr} S$. The bounds for $r\theta$ and $r\theta'$ now show that A' and K are bounded on any admissible time interval. Integrating (2.6) over the circle allows a_r to be bounded and then (2.6) itself gives a bound for β' . A bound for β can be deduced using the condition $\int \beta = 0$. Equation (2.8) gives a bound for A_r . Integrating equation (2.4) from a point where $\alpha' = 0$ and using (2.23) provides a bound for α' .

Theorem 2.1. Let a solution of the Einstein equations with surface symmetry be given and suppose that when coordinates are chosen which cast the metric into the form (2.1) with

constant mean curvature time slices the time coordinate takes all values in the finite interval (t_1, t_2) . Further suppose that:

- (i) the dominant energy and non-negative pressures conditions hold
- (ii) neither t_1 or t_2 is zero
- (iii) if ϵ is 0 or 1 then $t_1 < 0$
- (iv) if $\epsilon = -1$ then the mass function is positive on the initial hypersurface.

Let t_3 satisfy $t_1 < t_3 < t_2$. Then the following quantities are bounded on (t_1, t_3) :

$$\alpha, \alpha', A, A^{-1}, A', K, \beta, a, a^{-1}, \partial_t a \tag{2.24}$$

$$\partial_t A, K', \beta' \tag{2.25}$$

3. The matter fields

Theorem 2.1 provides some information on the boundedness of certain geometrical quantities in a spacetime with spherical, plane or hyperbolic symmetry without any assumptions on the matter content except the dominant energy and non-negative pressures conditions. To get further bounds and hence to proceed towards showing that the spacetime can be extended it is necessary to use the matter field equations. In the following, two examples will be treated, namely the collisionless gas and the massless scalar field.

The collisionless gas is described by a distribution function f which is a non-negative real-valued function on the mass shell. It is supposed to satisfy the Vlasov equation which in the class of spacetimes considered here takes the form

$$\begin{aligned} \frac{\partial f}{\partial t} + \left(\alpha A^{-1} \frac{v^1}{v^0} - \beta \right) \frac{\partial f}{\partial x} + \left[-A^{-1} \alpha' v^0 + \alpha K v^1 + \alpha A^{-2} A' \frac{(v^2)^2 + (v^3)^2}{v^0} \right] \frac{\partial f}{\partial v^1} \\ - \alpha \left[A^{-2} A' \frac{v^1}{v^0} + \frac{1}{2}(K - t) \right] v^B \frac{\partial f}{\partial v^B} = 0. \end{aligned} \tag{3.1}$$

Here the mass shell has been coordinatized using components in an orthonormal frame, where the first vector in the spatial frame is proportional to $\partial/\partial x$. The component v^0 is then given by the expression $\sqrt{1 + (v^1)^2 + (v^2)^2 + (v^3)^2}$. The upper case Latin indices take the values 2 and 3. The distribution function depends on t, x, v^1, v^2 and v^3 . In fact the symmetry requires that its dependence on the last two quantities is only a dependence on the combination $(v^2)^2 + (v^3)^2$. This will be assumed for the initial data and is then also satisfied by the solution. The initial data is assumed to be compactly supported and then the solution has compact support at each fixed time. That the initial value problem for the Vlasov-Einstein system is well posed was shown by Choquet-Bruhat [3]. The matter quantities occurring in the field equations are given by

$$\begin{aligned} \rho &= \int f v^0 dv & j &= \int f v^1 dv \\ S_j^1 &= \int f (v^1)^2 / v^0 dv & \text{tr } S &= \int f [(v^0)^2 - 1] / v^0 dv. \end{aligned} \tag{3.2}$$

For a solution which evolves from initial data given at $t = t_0$ let

$$P(t) = \sup\{|v| : f(s, x, v) \neq 0 \text{ for some } (s, x, v) \text{ with } s \in I\} \tag{3.3}$$

where I is the interval $[t_0, t]$ if $t \geq t_0$ and the interval $[t, t_0]$ if $t \leq t_0$. The maximum of f is time independent and so all the matter quantities defined in (3.2) can be bounded by $C(1 + P(t))^4$. The quantity $P(t)$ itself can be controlled by studying the characteristics of (3.1) since this equation says that f is constant along these characteristics. It follows that $P(t)$ will be bounded on a given interval provided the coefficients in (3.1) are bounded. The geometrical quantities which occur in these coefficients are α , A^{-1} , β , K and t . Theorem 2.1 shows that all of these are bounded on an admissible time interval. Considering a point where α attains its minimum on a given hypersurface $t = \text{constant}$ leads to a bound for α^{-1} . Hence the following is obtained.

Theorem 3.1. Let a solution of the Vlasov–Einstein system with surface symmetry be given and suppose that when coordinates are chosen which cast the metric into the form (2.1) with constant mean curvature time slices the time coordinate takes all values in the finite interval (t_1, t_2) . Suppose further that conditions (ii)–(iv) of theorem 2.1 are satisfied. Then all the quantities in (2.24) and (2.25) are bounded on (t_1, t_3) , as are P , α^{-1} and

$$\rho, j, S_1^1, \text{tr} S. \tag{3.4}$$

Notice that while theorem 2.1 is to a large extent independent of the matter model used this is not the case for theorem 3.1. It is probable that the analogous statement would be false if the collisionless matter was replaced by dust since shell-crossing singularities would presumably provide counterexamples. It will now be shown by induction that under the hypotheses of theorem 3.1 all derivatives of the solution are bounded.

Lemma 3.1. If the hypotheses of theorem 2.1 are satisfied and if all derivatives with respect to x of order up to n of the quantities in (2.24) and (3.4) are bounded then all derivatives with respect to x of order up to $n + 1$ of the quantities in (2.24) are bounded.

Proof. In the following D_x denotes a derivative with respect to x . Note first that the boundedness of the derivatives with respect to x up to order $n + 1$ of the quantities α , A , A^{-1} , a and a^{-1} follows immediately from the hypotheses of the lemma. Equations (2.3)–(2.6) can be solved for the quantities A'' , α'' , K' and β' . Differentiating the resulting equations n times with respect to x allows $D_x^{n+1}(A')$, $D_x^{n+1}(\alpha')$, $D_x^{n+1}K$ and $D_x^{n+1}\beta$ to be bounded.

Lemma 3.2. If the hypotheses of lemma 3.1 are satisfied by a solution of the Vlasov–Einstein system then all derivatives with respect to x of order up to $n + 1$ of the quantities in (3.4) are bounded. Moreover all derivatives of f of order up to $n + 1$ with respect to x and v are bounded.

Proof. The hypotheses imply that the coefficients of the characteristic system are bounded together with their derivatives with respect to x up to order $n + 1$. Differentiating this system $n + 1$ times with respect to x and v gives an inhomogeneous linear system of ordinary differential equations for the derivatives of order $n + 1$ of the unknowns. The coefficients of this system are bounded, as long as attention is confined to the support of f . Hence these derivatives are bounded on the support of f . It follows from this that the derivatives of f up to order $n + 1$ are bounded. Differentiating (3.2) then gives the desired conclusion for the quantities in (3.4).

Lemma 3.3. If the hypotheses of theorem 2.1 are satisfied by a solution of the Vlasov–Einstein system and if all derivatives of the quantities in (2.24) and (3.4) of the form $D_t^k D_x^n$ with n arbitrary and $k \leq m$ are bounded and if all higher derivatives of f with at most m

time derivatives are bounded then the derivatives of the form $D_t^{m+1}D_x^n$ of the quantities in (3.4) are bounded. Moreover the higher derivatives of f with at most $m + 1$ time derivatives are bounded.

Proof. Use the Vlasov equation to bound an extra time derivative of f and substitute the result into (3.2).

Lemma 3.4. If the hypotheses of lemma 3.1 are satisfied and if:

- (i) all derivatives of the quantities in (2.24) and (3.4) of the form $D_t^k D_x^n$ with n arbitrary and $k \leq m$ are bounded
- (ii) all derivatives of the quantities in (3.4) of the form $D_t^{m+1} D_x^n$ are bounded then the derivatives of the quantities in (2.24) of the form $D_t^{m+1} D_x^n$ are bounded.

Proof. The conclusion for a and a^{-1} follows immediately from the assumptions. The conclusion for A, A^{-1}, A' and K follows from (2.8) and (2.9). Now differentiate (2.4) n times with respect to x and $m + 1$ times with respect to t . The result is an equation of the form

$$\begin{aligned} (D_t^{m+1} D_x^n \alpha)'' + A^{-1} A' (D_t^{m+1} D_x^n \alpha)' \\ = A^2 \left[\frac{3}{2} (K - \frac{1}{3} t)^2 + \frac{1}{3} t^2 + 4\pi(\rho + \text{tr} S) \right] D_t^{m+1} D_x^n \alpha + B \end{aligned}$$

where the remainder term B is bounded. Examining the points where $D_t^{m+1} D_x^n \alpha$ has maximum modulus gives $|D_t^{m+1} D_x^n \alpha| \leq 3A^{-2} B / t^2$. Next, the equation (2.7), integrated with respect to x , allows the conclusion to be obtained for $\partial_t a$. Finally, the conclusion for β follows from (2.6).

Putting together the conclusions of theorem 3.1 and lemmas 3.1–3.4 we see that under the hypotheses of theorem 3.1 all derivatives of all metric coefficients and of the distribution function are bounded.

When all the derivatives of the solution are bounded on a given interval, it can be extended smoothly to the closure of that interval. This results in a new initial data set and applying the local existence and uniqueness result discussed in appendix A gives an extension of the solution to an interval which strictly contains the original one. Thus we have the following theorem.

Theorem 3.2. Let (M, g, f) be a C^∞ solution of the Vlasov–Einstein system with surface symmetry which is the maximal globally hyperbolic development of data given on a symmetric hypersurface of constant mean curvature H_0 . Then:

- (i) If $\epsilon = 1$ and $H_0 = 0$ then the whole spacetime can be covered by a foliation of CMC hypersurfaces where the mean curvature takes all real values.
- (ii) If $\epsilon = 1$ or $\epsilon = 0$ and $H_0 < 0$ then the part of the spacetime to the past of the initial hypersurface can be covered by a foliation of CMC hypersurfaces where the mean curvature takes all values in the interval $(-\infty, H_0]$.
- (iii) If $\epsilon = -1, H_0 < 0$ and the mass function is positive on the initial hypersurface then the part of the spacetime to the past of the initial hypersurface can be covered by a foliation of CMC hypersurfaces where the mean curvature takes all values in the interval $(-\infty, H_0]$.

Now another example, the massless scalar field, will be discussed. The massless scalar field is described by a real-valued function ϕ satisfying $\nabla_\alpha \nabla^\alpha \phi = 0$. The energy–momentum tensor is

$$T_{\alpha\beta} = \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} (\nabla_\gamma \phi \nabla^\gamma \phi) g_{\alpha\beta} . \tag{3.5}$$

The dominant energy condition is satisfied but the non-negative pressures condition does not hold for a general ϕ . Hence theorem 2.1 does not apply to this case. However, as has been remarked by Burnett [2], it is true that $T_{\alpha\beta}x^\alpha x^\beta \geq 0$ for spacelike vectors x^α orthogonal to the surfaces of symmetry and it turns out that this fact and the condition that $\rho + \text{tr} S \geq 0$ are the only ones which are needed in the proof of the theorem. They are satisfied by the scalar field (see below). There is also a potential problem with applying the analysis of appendix A in this case, since there the non-negative pressures condition was also used. It was needed for the argument using the implicit function theorem but only in the case that the initial hypersurface is maximal. Hence the analogue of theorem 2.1 holds for the scalar field under the extra hypothesis that the interval (t_1, t_2) does not contain zero.

To proceed further it is useful to introduce the null vectors

$$\begin{aligned} e_+ &= \alpha^{-1} \left(\frac{\partial}{\partial t} - \beta \frac{\partial}{\partial x} \right) + A^{-1} \frac{\partial}{\partial x} \\ e_- &= \alpha^{-1} \left(\frac{\partial}{\partial t} - \beta \frac{\partial}{\partial x} \right) - A^{-1} \frac{\partial}{\partial x}. \end{aligned} \tag{3.6}$$

Let $\phi_+ = e_+ \phi$ and $\phi_- = e_- \phi$. Then the wave equation for ϕ can be written as

$$\alpha e_+(\phi_-) = \alpha t(\phi_+ + \phi_-) + (A'\alpha' + 2\alpha A^{-2}A')(\phi_+ - \phi_-) + \alpha[e_-, e_+]\phi \tag{3.7}$$

$$\alpha e_-(\phi_+) = \alpha t(\phi_+ + \phi_-) + (A'\alpha' + 2\alpha A^{-2}A')(\phi_+ - \phi_-) + \alpha[e_+, e_-]\phi. \tag{3.8}$$

Now $\alpha[e_+, e_-] = b_+e_+ + b_-e_-$, where the coefficients b_+ and b_- are polynomials in the quantities (2.24) and (2.25). In addition, the definitions of e_+ and e_- imply that

$$\partial\phi/\partial t - \beta\partial\phi/\partial x = \frac{1}{2}\alpha(e_+ + e_-). \tag{3.9}$$

The matter quantities occurring in the field equations are given by

$$\begin{aligned} \rho &= \frac{1}{4}(\phi_+^2 + \phi_-^2) & j &= \frac{1}{4}(-\phi_+^2 + \phi_-^2) & S_1^1 &= \frac{1}{4}(\phi_+^2 + \phi_-^2) \\ \text{tr} S &= \frac{1}{4}(\phi_+ + \phi_-)^2 + \frac{1}{2}\phi_+\phi_- & \rho + \text{tr} S &= \frac{1}{2}(\phi_+ + \phi_-)^2. \end{aligned} \tag{3.10}$$

Let

$$\Phi(t) = \|\phi(t)\|_\infty + \|\phi_+(t)\|_\infty + \|\phi_-(t)\|_\infty. \tag{3.11}$$

The equations (3.7)–(3.9) and the boundedness of the quantities (2.24) and (2.25) imply that

$$\Phi(t) \leq \Phi(0) + C \int_0^t \Phi(s) ds. \tag{3.12}$$

Hence ϕ , ϕ_+ and ϕ_- are bounded on an admissible interval. It then follows from (3.10) that the matter quantities in the field equations are bounded. Hence the analogue of Theorem 3.1 with the Vlasov–Einstein system replaced by the Einstein–scalar system and P replaced by Φ holds provided the interval (t_1, t_2) does not contain zero.

Lemma 3.5. If the hypotheses of lemma 3.1 are satisfied by a solution of the Einstein–scalar system then all derivatives with respect to x of order up to $n + 1$ of the quantities in (3.4) are bounded. Moreover all derivatives of ϕ of order up to $n + 1$ and of $\partial_t\phi$ up to order n with respect to x are bounded.

Proof. Note first that the results of lemma 3.1 can be improved slightly by using (2.6) and (2.8) to bound the spatial derivatives of order up to order $n + 1$ of the quantities β^i and $\partial_t A$. Next, differentiating the equations (3.7)–(3.9) gives an inhomogeneous linear hyperbolic system for $D_x^{n+1}\phi_+$, $D_x^{n+1}\phi_-$ and $D_x^{n+1}\phi$ with a bounded right-hand side. This gives the desired bounds for derivatives of ϕ . Putting these into (3.10) gives the bounds for the components of the energy–momentum tensor.

Lemma 3.6. If the hypotheses of theorem 2.1 are satisfied by a solution of the Einstein–scalar system and if all derivatives of the quantities in (2.24) and (3.4) of the form $D_t^k D_x^n$ with n arbitrary and $k \leq m$ are bounded and if all higher derivatives of ϕ with at most $m + 1$ time derivatives are bounded then the derivatives of the form $D_t^{m+1} D_x^n$ of the quantities in (3.4) are bounded. Moreover, the higher derivatives of ϕ with at most $m + 2$ time derivatives are bounded.

Proof. This result follows immediately from (3.7)–(3.9) and (3.10).

Putting together the analogue of theorem 3.1 for the scalar field and the lemmas 3.1, 3.4, 3.5 and 3.6, we see that under the hypotheses of the analogue of theorem 3.1 and assuming that $H_0 \neq 0$ all derivatives of all metric coefficients and of the scalar field are bounded. If $H_0 = 0$ then we still get a neighbourhood of the initial hypersurface foliated by CMC hypersurfaces. If there exists a hypersurface belonging to this foliation with non-zero mean curvature both to the past and to the future of the initial hypersurface then the problem of getting bounds is reduced to the case $H_0 \neq 0$. If all the CMC hypersurfaces belonging to the foliation which are to the past, say, of the initial hypersurface are maximal then the second fundamental form and the Ricci tensor contracted twice with the normal vector are both zero on that region. The latter implies that the gradient of ϕ is tangent to the CMC hypersurfaces. Under these circumstances the wave equation reduces to the Laplace equation and ϕ must be spatially constant. If ϕ varied from one spacelike hypersurface to the next then $\nabla^\alpha \phi$ would be timelike, contradicting what has been said already. Hence ϕ is constant, the energy–momentum tensor is zero and the vacuum Einstein equations are satisfied. These considerations show that the following analogue of theorem 3.2 holds for the scalar field:

Theorem 3.3. Let (M, g, ϕ) be a C^∞ solution with surface symmetry of the Einstein equations coupled to a massless scalar field which is the maximal globally hyperbolic development of data given on a symmetric hypersurface of constant mean curvature H_0 . Then:

- (i) If $\epsilon = 1$ and $H_0 = 0$ then the whole spacetime can be covered by a foliation of CMC hypersurfaces where the mean curvature takes all real values.
- (ii) If $\epsilon = 1$ or $\epsilon = 0$ and $H_0 < 0$ then the part of the spacetime to the past of the initial hypersurface can be covered by a foliation of CMC hypersurfaces where the mean curvature takes all values in the interval $(-\infty, H_0]$.
- (iii) If $\epsilon = -1$, $H_0 < 0$ and the mass function is positive on the initial hypersurface then the part of the spacetime to the past of the initial hypersurface can be covered by a foliation of CMC hypersurfaces where the mean curvature takes all values in the interval $(-\infty, H_0]$.

Appendix A

The purpose of this appendix is to prove lemma 2.1 and a local existence and uniqueness theorem for (2.3)–(2.9). Let $(M, g_{\alpha\beta})$ be a surface symmetric spacetime, as defined in

section 2. Thus, in particular M is of the form $\mathbf{R} \times S^1 \times F$. Let \tilde{M} be the universal cover of M and let $\tilde{g}_{\alpha\beta}$ the pull-back of $g_{\alpha\beta}$ to \tilde{M} . Suppose that $(M, g_{\alpha\beta})$ contains a CMC Cauchy hypersurface S . It follows from the fact [10] that all globally defined Killing vectors must be tangential to a compact CMC hypersurface that a compact CMC Cauchy hypersurface S in a spacetime with spherical or plane symmetry is of the form $\tilde{S} \times F$. For in that case there are enough global Killing vectors to generate the surfaces of symmetry. This is not true in the case of hyperbolic symmetry. It is a standard fact that a neighbourhood of a compact CMC hypersurface can be foliated by compact CMC hypersurfaces unless the original hypersurface is such that its second fundamental form vanishes and the Ricci tensor contracted twice with the normal vector is zero. Furthermore, the mean curvature of these hypersurfaces can be used as a time coordinate in this neighbourhood. Even if this condition fails there is still a neighbourhood of the initial hypersurface foliated by CMC hypersurfaces, although in that case the mean curvature cannot be used as a time coordinate [1]. When the dominant energy and non-negative pressures conditions are satisfied the condition can only fail if the spacetime is a vacuum on S . From what was said earlier the CMC hypersurfaces must be symmetric in the cases of spherical and plane symmetry. In fact they must also be symmetric in the hyperbolic case. To see this note that the existence of these hypersurfaces is proved by using the inverse function theorem. However, it is possible to apply the inverse function theorem in the class of symmetric deformations of the initial hypersurface (provided this initial hypersurface is itself symmetric) and then the CMC hypersurfaces obtained are by construction symmetric. Consider now the exceptional case where the spacetime is a vacuum and the second fundamental form of the initial hypersurface vanishes. In the cases of spherical and hyperbolic symmetry integrating (2.3) from 0 to 2π gives a contradiction. In the plane symmetric case the vanishing of the second fundamental form implies that spacetime is flat. Hence unless the spacetime is flat the mean curvature can be used as a time coordinate t in a neighbourhood of S . The inverse image of t under the projection $p : \tilde{M} \rightarrow M$ will also be denoted by t .

It is elementary to see that the metric g_{ab} of the hypersurface $t = \text{constant}$ in M can be written locally in the form

$$A^2 dx^2 + B^2 d\Sigma^2 \tag{A.1}$$

where $d\Sigma^2$ is a metric of constant curvature. What is less clear is that that this can be done globally in such a way that $x \in [0, 2\pi]$ and the functions A and B are 2π -periodic. Consider one of the hypersurfaces $t = \text{constant}$ in \tilde{M} . Let γ be a geodesic in this hypersurface which starts orthogonal to one of the group orbits \mathcal{O}_1 at a point p . It continues to be orthogonal to the orbits. After a finite time it must hit an orbit \mathcal{O}_2 which projects to the same orbit in M as \mathcal{O}_1 . Suppose it meets \mathcal{O}_2 at a point q . Let q' be the unique point of \mathcal{O}_1 which projects to the same point of M as q . Any isometry which fixes p must fix q' . It follows in the plane and hyperbolic cases that $p = q'$ and in the spherical case that either $p = q'$ or p and q' are antipodal points on the sphere. If $p = q'$ then A and B can be made 2π -periodic, as desired. In the case where the antipodal map occurs the same thing can be arranged by allowing x to go twice around the circle.

Let

$$a = 2\pi \left[\int_0^{2\pi} B/A \right]^{-1}. \tag{A.2}$$

Then the new coordinate x' defined by

$$x' = a \int_0^x B/A \tag{A.3}$$

satisfies $x(0) = 0$ and $x(2\pi) = 2\pi$. Define A' by the relation $A'(x') \frac{dx'}{dx}(x) = A(x)$. After transforming to the new coordinate and dropping the primes the metric takes the form

$$A^2(dx^2 + a^2 d\Sigma^2) \tag{A.4}$$

where A is a positive function of x with $A(0) = A(2\pi)$ and a is a constant. Doing this construction on each hypersurface of constant time gives the coordinate system whose existence is asserted by lemma 2.1.

Consider now the question of local existence and uniqueness of solutions of equations (2.3)–(2.9) with given initial data on a hypersurface $t = \text{constant}$. In order to have a well-posed initial value problem it is necessary to have some matter equations such that the resulting Einstein–matter system has a well-posed Cauchy problem in the context of C^∞ data and solutions. More precisely we assume that the solution of the initial value problem for the reduced equations in harmonic coordinates exists and is unique so that the general theory of the maximal Cauchy development [4] can be applied. (If it were desired to consider gauge theories, where solutions are only unique up to gauge transformations, then some more work would be required.) It is not obvious that this theory applies to kinetic theory models, where the matter fields are defined on the mass shell rather than on spacetime. However, the analogous results do hold in that case [3]. An initial data set consists of periodic functions A and K , a constant a and matter data which satisfy the constraint equations (2.3) and (2.5). The matter data are assumed to have the necessary symmetry properties. These properties are most easily expressed on the covering manifold. The data set on the covering manifold has a maximal Cauchy development on the manifold \tilde{M} . The maximal Cauchy development inherits the symmetries of the data and so the original initial data set has a surface symmetric Cauchy development. In this surface symmetric spacetime coordinates can be introduced as above. Thus a solution of (2.3)–(2.9) (and the matter equations) on some interval (t_1, t_2) is obtained. It remains to show that solutions of these equations are uniquely determined by initial data. Suppose there exist two solutions with the same initial data on the interval (t_1, t_2) . Then by the general theory of the Cauchy problem there must exist embeddings ϕ_1 and ϕ_2 of M into the maximal Cauchy development of the given initial data set such that ϕ_1 is a matter-preserving isometry for the first solution and ϕ_2 a matter-preserving isometry for the second. The uniqueness of compact constant mean curvature hypersurfaces implies that the images of any hypersurface of constant t under ϕ_1 and ϕ_2 are identical. In particular, this means that the images of M under ϕ_1 and ϕ_2 are identical, so that there exists a diffeomorphism $\phi_{12} : M \rightarrow M$ such that $\phi_1 = \phi_{12} \circ \phi_2$. The diffeomorphism ϕ_{12} maps the one solution into the other and preserves the hypersurfaces of constant time. Suppose temporarily that the initial data do not have Robertson–Walker symmetry. Then a unique two-plane is defined by the isotropy group of the universal cover. This plane must be preserved by ϕ_{12} as must its orthogonal complement. Thus, if ϕ_{12} is written in terms of coordinates adapted to the first metric in the form $(t, x, y) \mapsto (t', x', y')$. Then $t' = t$ and y' depends only on t and y . By composing with an isometry it can be reduced to the identity on the initial hypersurface. The form of the shift vector then implies that it is the identity everywhere. In a given spacetime the coordinate x is defined up to a translation. Hence $x' = x + c$ and since ϕ_{12} is the identity on the initial hypersurface it follows that the two solutions are identical. In the case where the data have Robertson–Walker symmetry the solutions must have Robertson–Walker symmetry and in that case uniqueness for the reduced equations is obvious.

Appendix B

In this appendix the vacuum solutions with the symmetry properties considered in this paper will be determined. This is done using the following lemma:

Lemma B.1. Consider the ordinary differential equation $d^2u/dx^2 = f(u)$, where $f : (0, \infty) \rightarrow \mathbb{R}$ is Lipschitz. Suppose that $f(u_0) = 0$ for some u_0 , $f(u) < 0$ for $0 < u < u_0$ and $f(u) > 0$ for $u > u_0$. Then any periodic solution is constant.

Proof. Let u be a periodic solution. By periodicity there exists a point x_0 where d^2u/dx^2 vanishes. At that point $u = u_0$. If $du/dx(x_0)$ is positive then it is easy to show that du/dx remains positive for $x > x_0$, contradicting periodicity. Similarly the assumption $du/dx(x_0) < 0$ leads to a contradiction. Hence, in fact, $du/dx(x_0) = 0$. By uniqueness for solutions of the ordinary differential equation it follows that u is constant.

Now consider vacuum solutions of (2.3)–(2.9). The momentum constraint can be solved explicitly, giving $K - \frac{1}{3}t = CA^{-3}$ for some constant C . Substituting this into the Hamiltonian constraint gives

$$(A^{1/2})'' = -\frac{3}{16}C^2A^{-7/2} + \frac{1}{12}t^2A^{5/2} + \frac{1}{4}\epsilon a^{-2}A^{-1/2}. \tag{B.1}$$

It can be checked straightforwardly that this ordinary differential equation for $A^{1/2}$ satisfies the hypotheses of the lemma and so A is constant. Then the same lemma may be applied to the lapse equation to show that α is constant. The constancy of A implies that of K and the equation for β then gives $\beta = 0$. Hence every vacuum solution of (2.3)–(2.9) is spatially homogeneous. These solutions will now be identified with known exact solutions. This will be done by examining the Cauchy data on one spacelike hypersurface. Suppose that constants t , a and K are given and satisfy the following sign condition, which is necessary for the constraints to have a solution:

- (i) if $\epsilon = 1$ then $\frac{3}{2}(K - \frac{1}{3}t)^2 - \frac{2}{3}t^2 > 0$;
- (ii) if $\epsilon = 0$ then $\frac{3}{2}(K - \frac{1}{3}t)^2 - \frac{2}{3}t^2 = 0$;
- (iii) if $\epsilon = -1$ then $\frac{3}{2}(K - \frac{1}{3}t)^2 - \frac{2}{3}t^2 < 0$.

Suppose that $t = 0$. Then the sign condition is incompatible with $\epsilon = -1$. It is only compatible with $\epsilon = 0$ if $K = 0$. In that case the data give rise to flat space, identified in a simple way. If $\epsilon = 1$ then the Hamiltonian constraint can be solved for A in terms of a and K . For $t \neq 0$ the sign condition can readily be analysed by dividing the expression of interest by t^2 and studying the resulting quadratic expression in K/t .

The case $\epsilon = 0$ is the simplest. There are two possible values for K/t , namely $-\frac{1}{3}$ and 1. These solutions of the constraints can be realized by the $\tau = \text{constant}$ hypersurfaces in the Kasner solution

$$-d\tau^2 + b^2\tau^{2p}dx^2 + \tau^{1-p}(dy^2 + dz^2) \tag{B.2}$$

where $p = -\frac{1}{3}$ or $p = 1$ and b is a positive constant. In the case $\epsilon = 1$ the quantity K/t takes all values in the intervals $(-\infty, -\frac{1}{3})$ and $(1, \infty)$ and these solutions of the constraints can be realized by the $\tau = \text{constant}$ hypersurfaces in the following metric, which is obtained by identifying the part of the Schwarzschild solution inside the horizon:

$$-(2m/\tau - 1)^{-1}d\tau^2 + b^2(2m/\tau - 1)dx^2 + \tau^2d\Sigma^2. \tag{B.3}$$

Here $d\Sigma^2$ is the standard metric on the sphere. Similarly, the solutions with $\epsilon = -1$ produce all values of K/t in the interval $(-\frac{1}{3}, 1)$ and these solutions of the constraints can be realized by the $\tau = \text{constant}$ hypersurfaces in the following pseudo-Schwarzschild metric:

$$-(2m/\tau + 1)^{-1} d\tau^2 + b^2(2m/\tau + 1) dx^2 + \tau^2 d\Sigma^2. \quad (\text{B.4})$$

In this case $d\Sigma^2$ is a metric of constant negative curvature on a compact manifold obtained by identifying the hyperbolic plane by means of a discrete group of isometries. The general theorems proved in this paper imply, in particular, that for $m > 0$ the initial singularity in this solution, which occurs at $t = 0$ is a crushing singularity. It is worth remarking that for $m \leq 0$ the initial singularity, which occurs at $t = -2m$ is also crushing, even though the theorems do not apply.

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