

# A Regularity Theorem for Solutions of the Spherically Symmetric Vlasov–Einstein System

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**Abstract:** We show that if a solution of the spherically symmetric Vlasov–Einstein system develops a singularity at all then the first singularity has to appear at the center of symmetry. The main tool is an estimate which shows that a solution is global if all the matter remains away from the center of symmetry.

## 1. Introduction

This paper is concerned with the long-time behaviour of solutions of the spherically symmetric Vlasov–Einstein system. In [4] a continuation criterion was obtained for solutions of this system, and it was shown that for small initial data the corresponding solution exists globally in time. In the following we investigate what happens for large initial data. In the coordinates used in [4] the equations to be solved are as follows:

$$\partial_t f + e^{\mu-\lambda} \frac{v}{\sqrt{1+v^2}} \cdot \partial_x f - \left( \frac{x \cdot v}{r} \dot{\lambda} + e^{\mu-\lambda} \sqrt{1+v^2} \mu' \right) \frac{x}{r} \cdot \partial_v f = 0, \quad (1.1)$$

$$e^{-2\lambda}(2r\dot{\lambda}' - 1) + 1 = 8\pi r^2 \rho, \quad (1.2)$$

$$e^{-2\lambda}(2r\mu' + 1) - 1 = 8\pi r^2 p, \quad (1.3)$$

$$\rho(t, x) = \int \sqrt{1+v^2} f(t, x, v) dv, \quad (1.4)$$

$$p(t, x) = \int \left( \frac{x \cdot v}{r} \right)^2 f(t, x, v) \frac{dv}{\sqrt{1+v^2}}. \quad (1.5)$$

Here  $x$  and  $v$  belong to  $\mathbb{R}^3$ ,  $r := |x|$ ,  $x \cdot v$  denotes the usual inner product of vectors in  $\mathbb{R}^3$ , and  $v^2 := v \cdot v$ . The distribution function  $f$  is assumed to be invariant under simultaneous rotations of  $x$  and  $v$ , hence  $\rho$  and  $p$  can be regarded as functions of  $t$  and  $r$ . Spherically symmetric functions of  $t$  and  $x$  will be identified with functions of  $t$  and  $r$  whenever it is convenient. In particular  $\lambda$  and  $\mu$  are regarded as functions of  $t$  and  $r$ , and the dot and prime denote derivatives with respect to  $t$  and  $r$  respectively. It is assumed that  $f(t)$  has compact support for each fixed  $t$ . We

are interested in regular asymptotically flat solutions which leads to the boundary conditions that

$$\lambda(t, 0) = 0, \quad \lim_{r \rightarrow \infty} \mu(t, r) = 0. \tag{1.6}$$

for each fixed  $t$ .

The main aim of this paper is to show that if singularities ever develop in solutions of the system (1.1)–(1.5) with the above boundary conditions then the first singularity must be at the center of symmetry. In order to do this we consider solutions of (1.1)–(1.5) on a certain kind of exterior region with different boundary conditions and prove that for the modified problem there exists a global in time solution for any initial data. One of the most interesting implications of the results of [4] is that the naked singularities in spherically symmetric solutions of the Einstein equations coupled to dust can be cured by passing to a slightly different matter model, namely that described by the Vlasov equation, in the case of small initial data. The results of this paper strengthen this conclusion to say that shell-crossing singularities are completely eliminated.

The following remarks put our results in context. For the Vlasov–Poisson system, which is the non-relativistic analogue of the Vlasov–Einstein system, it is known that global existence holds for boundary conditions which are the analogue of the requirement of asymptotic flatness in the relativistic case [2, 3, 6] and also in a cosmological setting [5]. No symmetry assumptions are necessary. For the relativistic Vlasov–Poisson system with an attractive force spherically symmetric solutions with negative energy develop singularities in finite time [1]. It is easy to show that in these solutions the first singularity occurs at the center of symmetry. On the other hand it was also shown in [1] that spherically symmetric solutions of the relativistic Vlasov–Poisson system with a repulsive force never develop singularities. The latter are in one-to-one correspondence with spherically symmetric solutions of the Vlasov–Maxwell system.

The paper is organized as follows. Section 2 contains the main estimates together with a proof that they imply global existence in the case that all the matter remains away from the center. In Sect. 3 a local existence theorem and continuation criterion for the exterior problem are proved. It is then shown that the estimates of Sect. 2 imply a global existence theorem for the exterior problem. Finally, in Sect. 4, these results are combined to give the main theorem.

## 2. The Restricted Regularity Theorem

The goal of this section is to show that a solution may be extended as long as  $f$  vanishes in a neighborhood of the center of symmetry. Consider an initial datum  $f^\circ \geq 0$  which is spherically symmetric,  $C^1$ , compactly supported, and satisfies

$$\int_{|x| < r} \int \sqrt{1 + v^2} f^\circ(x, v) \, dv \, dx < r/2 \tag{2.1}$$

for all  $r > 0$ . By Theorem 3.1 of [4] a regular solution  $(f, \lambda, \mu)$  of the system (1.1)–(1.5) with boundary conditions (1.6) and initial datum  $f^\circ$  exists on  $[0, T] \times \mathbb{R}^6$  for some  $T > 0$ .

**Theorem 2.1.** *Let  $\mathring{f}$  and  $T$  be as above ( $T$  finite). Assume there exists  $\varepsilon > 0$  such that*

$$f(t, x, v) = 0 \text{ if } 0 \leq t < T \text{ and } |x| \leq \varepsilon. \tag{2.2}$$

*Then  $(f, \lambda, \mu)$  extends to a regular solution on  $[0, T'$  for some  $T' > T$ .*

Define

$$P(t) := \sup \{|v| : (x, v) \in \text{supp } f(t)\}, \tag{2.3}$$

then Theorem 3.2 of [4] states that Theorem 2.1 above follows once  $P(t)$  is shown to be bounded on  $[0, T'$ . We need a few other facts from [4]:

$$e^{-2\lambda} = 1 - 2m/r, \tag{2.4}$$

where

$$m(t, r) := 4\pi \int_0^r s^2 \rho(t, s) ds \tag{2.5}$$

(Eqs.(2.11) and (2.12) of [4]). Also

$$\dot{\lambda} = -4\pi r e^{\mu+\lambda} j, \tag{2.6}$$

where

$$j(t, r) := \int \frac{x \cdot v}{r} f(t, x, v) dv \tag{2.7}$$

(Eqs.(3.37) and (3.38) of [4]).

The following notation will be used:

$$u := |v|, \quad w := r^{-1}x \cdot v, \quad F := |x \wedge v|^2 = r^2u^2 - (x \cdot v)^2.$$

Differentiation along a characteristic of the Vlasov equation is denoted by  $D_t$ , so

$$D_t x = e^{\mu-\lambda} \frac{v}{\sqrt{1+u^2}}$$

and

$$D_t v = - \left( \frac{x \cdot v}{r} \dot{\lambda} + e^{\mu-\lambda} \sqrt{1+u^2} \mu' \right) \frac{x}{r}.$$

It follows that

$$D_t F = 0, \tag{2.8}$$

$$D_t r = e^{\mu-\lambda} \frac{w}{\sqrt{1+u^2}}, \tag{2.9}$$

and

$$D_t w = -r^{-2}(D_t r)x \cdot v + r^{-1}(D_t x) \cdot v + r^{-1}x \cdot (D_t v).$$

Substitution for  $D_t r, D_t x$ , and  $D_t v$  and simplification yields

$$D_t w = \frac{F}{r^3 \sqrt{1+u^2}} e^{\mu-\lambda} - w \dot{\lambda} - e^{\mu-\lambda} \sqrt{1+u^2} \mu'. \tag{2.10}$$

The letter  $C$  will denote a generic constant which changes from line to line, and may depend only on  $\mathring{f}, \varepsilon$ , and  $T$ .

Now we make a few preliminary estimates. The values of  $f$  are conserved along characteristics so

$$0 \leq f \leq \sup \mathring{f} = C.$$

Also we claim that

$$\int \rho(t,x) dx = \int \rho(0,x) dx = C . \tag{2.11}$$

To show this multiply (1.1) by  $\sqrt{1+v^2}$  and integrate in  $v$ , which yields, (after simplification)

$$\begin{aligned} 0 &= \partial_t \rho + e^{\mu-\lambda} \operatorname{div} \left( \int f v d v \right) + (\rho + p) \dot{\lambda} + 2j e^{\mu-\lambda} \mu' \\ &= \partial_t \rho + \operatorname{div} \left( e^{\mu-\lambda} \int f v d v \right) + (\rho + p) \dot{\lambda} + j e^{\mu-\lambda} (\mu' + \lambda') . \end{aligned}$$

Now substituting (2.6) and (1.2), (1.3) this becomes

$$0 = \partial_t \rho + \operatorname{div} \left( e^{\mu-\lambda} \int f v d v \right) \tag{2.12}$$

and (2.11) follows. Also by (2.5),

$$0 \leq m(t,r) \leq \int \rho(t,x) dx \leq C, \quad r \geq 0 . \tag{2.13}$$

It follows from (1.2), (1.3) that

$$\mu' + \lambda' \geq 0 ,$$

and from (1.6) and (2.4) that

$$\lim_{r \rightarrow \infty} (\mu + \lambda) = 0 ,$$

so

$$\mu - \lambda \leq \mu + \lambda \leq 0$$

and

$$e^{\mu-\lambda} \leq e^{\mu+\lambda} \leq 1 . \tag{2.14}$$

Note that

$$0 \leq F \leq C$$

on the support of  $f$ , so

$$u^2 = w^2 + \frac{F}{r^2} \leq w^2 + \frac{C}{\varepsilon^2} = w^2 + C . \tag{2.15}$$

Hence we will focus on  $w$ . Define

$$P_i(t) := \inf \{ w : \exists x, v \text{ with } f(t,x,v) \neq 0 \text{ and } w = r^{-1} x \cdot v \} \tag{2.16}$$

and

$$P_s(t) := \sup \{ w : \exists x, v \text{ with } f(t,x,v) \neq 0 \text{ and } w = r^{-1} x \cdot v \} , \tag{2.17}$$

then if  $P_i(t)$  and  $P_s(t)$  are bounded, it follows that  $P(t)$  is bounded. Also note that

$$\operatorname{measure} \{ v : (x,v) \in \operatorname{supp} f(t) \} \leq \pi C \varepsilon^{-2} (P_s(t) - P_i(t)), \quad |x| \geq \varepsilon . \tag{2.18}$$

Next we focus on the characteristic equation for  $w$ . Note first that by (1.3) and (2.4),

$$\begin{aligned} \mu' &= \frac{1}{2r} \left( e^{2\lambda} (8\pi r^2 p + 1) - 1 \right) = \frac{1}{2r} e^{2\lambda} \left( 8\pi r^2 p + 1 - 1 + \frac{2m}{r} \right) \\ &= e^{2\lambda} (r^{-2} m + 4\pi r p) . \end{aligned}$$

Using this and (2.6) in (2.10) yields

$$D_t w = \frac{F}{r^3 \sqrt{1+u^2}} e^{\mu-\lambda} - r^{-2} \sqrt{1+u^2} e^{\mu+\lambda} m + 4\pi r e^{\mu+\lambda} (w j - \sqrt{1+u^2} p). \quad (2.19)$$

By (2.2) and (2.14) we may bound the first term of (2.19) by

$$0 \leq \frac{F}{r^3 \sqrt{1+u^2}} e^{\mu-\lambda} \leq \varepsilon^{-3} C = C$$

on the support of  $f$ . By (2.2), (2.15), (2.14), and (2.13) we may bound the second term of (2.19) by

$$0 \leq r^{-2} \sqrt{1+u^2} e^{\mu+\lambda} m \leq \varepsilon^{-2} \sqrt{C+w^2} C \leq C \sqrt{C+w^2}$$

on the support of  $f$ . Hence, (2.19) becomes

$$\begin{aligned} -C \sqrt{C+w^2} + 4\pi r e^{\mu+\lambda} (w j - \sqrt{1+u^2} p) &\leq D_t w \\ &\leq C + 4\pi r e^{\mu+\lambda} (w j - \sqrt{1+u^2} p). \end{aligned} \quad (2.20)$$

On the support of  $f$ ,

$$0 \leq 4\pi r e^{\mu+\lambda} \leq C,$$

so we must consider the quantity  $w j - \sqrt{1+u^2} p$ . Let us denote

$$\tilde{w} := r^{-1} x \cdot \tilde{v} \cdot \tilde{u} := |\tilde{v}|, \quad \tilde{F} := |x \wedge \tilde{v}|^2 = r^2 \tilde{u}^2 - (x \cdot \tilde{v})^2.$$

Then

$$w j - \sqrt{1+u^2} p = \int f(t, x, \tilde{v}) \left( w \tilde{w} - \sqrt{1+u^2} \frac{\tilde{w}^2}{\sqrt{1+\tilde{u}^2}} \right) d\tilde{v}. \quad (2.21)$$

The next step is to use (2.20) and (2.21) to derive an upper bound for  $w$  (on the support of  $f$ ), and hence for  $P_s(t)$ . This may be done without a bound on  $P_i(t)$ . Then (2.20), (2.21), and the bound for  $P_s(t)$  will be used to derive a lower bound for  $w$ , and hence for  $P_i(t)$ . Then by (2.15) and (2.3) a bound for  $P(t)$  follows, and the solution may be extended.

To bound  $P_s(t)$  suppose

$$P_s(t) > 0$$

and consider  $w$  (in  $\text{supp } f$ ) with

$$w > 0.$$

For  $\tilde{w} \leq 0$  we have

$$w \tilde{w} - \sqrt{1+u^2} \frac{\tilde{w}^2}{\sqrt{1+\tilde{u}^2}} \leq 0.$$

For  $\tilde{w} > 0$  we have

$$\begin{aligned} w\tilde{w} - \sqrt{1+u^2} \frac{\tilde{w}^2}{\sqrt{1+\tilde{u}^2}} &= \frac{\tilde{w}}{\sqrt{1+\tilde{u}^2}} \frac{w^2(1+\tilde{u}^2) - \tilde{w}^2(1+u^2)}{w\sqrt{1+\tilde{u}^2} + \tilde{w}\sqrt{1+u^2}} \\ &= \frac{\tilde{w}}{\sqrt{1+\tilde{u}^2}} \frac{w^2(1+\tilde{F}r^{-2}) - \tilde{w}^2(1+Fr^{-2})}{w\sqrt{1+\tilde{u}^2} + \tilde{w}\sqrt{1+u^2}}. \end{aligned}$$

Note that in the last step a term of “ $w^2\tilde{w}^2$ ” canceled, which is crucial. Hence

$$w\tilde{w} - \sqrt{1+u^2} \frac{\tilde{w}^2}{\sqrt{1+\tilde{u}^2}} \leq \frac{\tilde{w}}{\sqrt{1+\tilde{u}^2}} \frac{w^2(1+C\varepsilon^{-2})}{w\sqrt{1+\tilde{u}^2}} \leq C \frac{w\tilde{w}}{1+\tilde{w}^2}.$$

Now using the above and (2.18) we have

$$\begin{aligned} \int f \left( w\tilde{w} - \sqrt{1+u^2} \frac{\tilde{w}^2}{\sqrt{1+\tilde{u}^2}} \right) d\tilde{v} &\leq \int_{0 < \tilde{w} < P_s(t)} f C w \frac{\tilde{w}}{1+\tilde{w}^2} d\tilde{v} \\ &\leq \pi C \varepsilon^{-2} \int_0^{P_s(t)} C w \frac{\tilde{w}}{1+\tilde{w}^2} d\tilde{w} \\ &= C w \ln(1+P_s^2(t)) \leq C P_s(t) \ln(1+P_s^2(t)), \end{aligned}$$

and by (2.21) and (2.20)

$$D_t w \leq C + C P_s(t) \ln(1+P_s^2(t)), \tag{2.22}$$

for  $w > 0$  in the support of  $f$ . Denote the values of  $w$  along a characteristic by  $w(\tau)$  and let

$$t_0 := \inf \{ \tau \geq 0 : w(s) \geq 0 \text{ for } s \in ]\tau, t[ \}.$$

Then either  $t_0 = 0$  or  $w(t_0) = 0$  and in either case

$$w(t_0) \leq C.$$

Hence by (2.22),

$$\begin{aligned} w(t) &\leq w(t_0) + \int_{t_0}^t (C + C P_s(\tau) \ln(1+P_s^2(\tau))) d\tau \\ &\leq C + C \int_{t_0}^t P_s(\tau) \ln(1+P_s^2(\tau)) d\tau. \end{aligned}$$

Defining

$$\bar{P}_s(\tau) := \max\{0, P_s(\tau)\},$$

we may write

$$w(t) \leq C + C \int_0^t \bar{P}_s(\tau) \ln(1+\bar{P}_s^2(\tau)) d\tau,$$

and hence

$$\bar{P}_s(t) \leq C + C \int_0^t \bar{P}_s(\tau) \ln(1+\bar{P}_s^2(\tau)) d\tau.$$

We assumed that  $P_s(t) > 0$ , but note that the last inequality is valid in all cases. It now follows that

$$P_s(t) \leq \bar{P}_s(t) \leq \exp(e^{ct}) \leq C \tag{2.23}$$

on  $t \in [0, T[$ .

To bound  $P_i(t)$  from below suppose

$$P_i(t) < 0$$

and consider  $w$  (in support  $f$ ) with

$$P_i(t) < w < 0.$$

For  $\tilde{w} \leq 0$  we have

$$\begin{aligned} w\tilde{w} - \sqrt{1+u^2} \frac{\tilde{w}^2}{\sqrt{1+\tilde{u}^2}} &= \frac{\tilde{w}}{\sqrt{1+\tilde{u}^2}} \frac{w^2(1+\tilde{F}r^{-2}) - \tilde{w}^2(1+Fr^{-2})}{w\sqrt{1+\tilde{u}^2} + \tilde{w}\sqrt{1+u^2}} \\ &= \frac{|\tilde{w}|}{\sqrt{1+\tilde{u}^2}} \frac{w^2(1+\tilde{F}r^{-2}) - \tilde{w}^2(1+Fr^{-2})}{|w|\sqrt{1+\tilde{u}^2} + |\tilde{w}|\sqrt{1+u^2}} \\ &\geq \frac{|\tilde{w}|}{\sqrt{1+\tilde{u}^2}} \frac{(-\tilde{w}^2)(1+Fr^{-2})}{|\tilde{w}|\sqrt{1+u^2}} \\ &\geq \frac{-\tilde{w}^2(1+C\varepsilon^{-2})}{\sqrt{1+\tilde{w}^2}\sqrt{1+w^2}} \\ &\geq -C \frac{|\tilde{w}|}{\sqrt{1+w^2}}. \end{aligned}$$

For  $w < 0 < \tilde{w} \leq P_s(t)$  we have (using (2.23))

$$\begin{aligned} w\tilde{w} - \sqrt{1+u^2} \frac{\tilde{w}^2}{\sqrt{1+\tilde{u}^2}} &\geq P_s(t)w - \sqrt{1+w^2+Fr^{-2}} \frac{\tilde{w}^2}{\sqrt{1+\tilde{w}^2}} \\ &\geq Cw - |\tilde{w}|\sqrt{1+C\varepsilon^{-2}+w^2} \\ &\geq Cw - P_s(t)\sqrt{C+w^2} \\ &\geq -C\sqrt{C+w^2}. \end{aligned}$$

Hence (using (2.18) as before)

$$\begin{aligned} &\int f \left( w\tilde{w} - \sqrt{1+u^2} \frac{\tilde{w}^2}{\sqrt{1+\tilde{u}^2}} \right) d\tilde{v} \\ &\geq \int_{\tilde{w} \leq 0} f \left( -C \frac{|\tilde{w}|}{\sqrt{1+w^2}} \right) d\tilde{v} + \int_{\tilde{w} > 0} f \left( -C\sqrt{C+w^2} \right) d\tilde{v} \\ &\geq -\frac{C}{\sqrt{1+w^2}} \pi C\varepsilon^{-2} \int_{-P_i(t)}^0 |\tilde{w}| d\tilde{w} - C\sqrt{C+w^2} \pi C\varepsilon^{-2} \int_0^{\bar{P}_s(t)} d\tilde{w} \\ &= -CP_i^2(t) \frac{1}{\sqrt{1+w^2}} - C\bar{P}_s(t)\sqrt{C+w^2}. \end{aligned}$$

Now by (2.21), (2.20), and (2.23),

$$\begin{aligned} D_t w &\geq -C\sqrt{C+w^2} - CP_i^2(t)\frac{1}{\sqrt{1+w^2}} - C\bar{P}_s(t)\sqrt{C+w^2} \\ &\geq -C\sqrt{C+w^2} - CP_i^2(t)\frac{1}{\sqrt{1+w^2}} . \end{aligned}$$

Since we have assumed  $0 > w > P_i(t)$ , it is convenient to write this as

$$\begin{aligned} D_t(w^2) &= 2wD_t w \\ &\leq C(-w)\sqrt{C+w^2} + CP_i^2(t)\frac{(-w)}{\sqrt{1+w^2}} \\ &\leq C|P_i(t)|\sqrt{C+P_i^2(t)} + CP_i^2(t) \\ &\leq C + CP_i^2(t) . \end{aligned}$$

As before define

$$t_1 := \inf \{ \tau \geq 0 : w(s) \leq 0 \text{ for } s \in ]\tau, t[ \} ,$$

then

$$0 \geq w(t_1) \geq -C ,$$

so

$$\begin{aligned} w^2(t) &\leq C + \int_{t_1}^t (C + CP_i^2(\tau))d\tau \\ &\leq C + C \int_0^t P_i^2(\tau)d\tau . \end{aligned}$$

It follows that

$$P_i^2(t) \leq C + C \int_0^t P_i^2(\tau)d\tau \tag{2.24}$$

if  $P_i(t) < 0$ . But if  $P_i(t) \geq 0$  then

$$0 \leq P_i(t) \leq P_s(t) \leq C ,$$

so (2.24) holds in this case, too. Now by Gronwall's inequality it follows that

$$P_i^2(t) \leq e^{ct} \leq C$$

on  $[0, T[$ . Finally a bound for  $P(t)$  follows from (2.15) and (2.3), and the proof is complete.

### 3. The Exterior Problem

If  $r_1$  and  $T$  are positive real numbers, define the exterior region

$$W(T, r_1) := \{ (t, r) : 0 \leq t < T, r \geq r_1 + t \} .$$

In this section the initial value problem for (1.1)–(1.5) will be studied on a region of this kind. Consider an initial datum  $\mathring{f}(x, v)$  defined on the region  $|x| \geq r_1$  which is non-negative, compactly supported,  $C^1$ , and spherically symmetric. The first of the boundary conditions (1.6) cannot be used in the case of an exterior region, and so it will be replaced as follows. Let  $m_\infty$  be any number greater than  $4\pi \int_{r_1}^\infty r^2 \rho(0, r) dr$ . For any solution of (1.1)–(1.5) on  $W(T, r_1)$  define

$$m(t, r) := m_\infty - 4\pi \int_r^\infty s^2 \rho(t, s) ds . \tag{3.1}$$

Provided  $m_\infty$  satisfies the above inequality, the quantity  $m(0, r)$  is everywhere positive. The replacement for (1.6) is:

$$e^{-2\lambda(t,r)} = 1 - 2m(t, r)/r, \quad \lim_{r \rightarrow \infty} \mu(t, r) = 0 . \tag{3.2}$$

The first of these conditions is a combination of (1.2) with a choice of boundary condition. Note that if a solution of the original problem with boundary conditions (1.6) is restricted to  $W(T, r_1)$ , then it will satisfy (3.2) provided  $m_\infty$  is chosen to be equal to the ADM mass of the solution on the full space. Just as in the local existence theorem in [4], a further restriction must be imposed on the initial datum. In this case it reads

$$m_\infty - \int_{|x| \geq r} \int \sqrt{1 + v^2} \mathring{f}(x, v) dv dx < r/2, \quad r \geq r_1 . \tag{3.3}$$

This is of course necessary if (3.2) is to hold on the initial hypersurface. The nature of the solutions to be constructed is encoded in the following definition.

**Definition.** A solution  $(f, \lambda, \mu)$  of (1.1)–(1.5) on a region of the form  $W(T, r_1)$  is called regular if

- (i)  $f$  is non-negative, spherically symmetric, and  $C^1$ , and  $f(t)$  has compact support for each  $t \in [0, T[$ ,
- (ii)  $\lambda \geq 0$ , and  $\lambda, \mu, \lambda',$  and  $\mu'$  are  $C^1$ .

A local existence theorem can now be stated.

**Theorem 3.1.** Let  $m_\infty > 0$  be a fixed real number. Let  $\mathring{f} \geq 0$  be a spherically symmetric function on the region  $|x| \geq r_1$  which is  $C^1$  and has compact support. Suppose that (3.3) holds for all  $r \geq r_1$  and that

$$\int_{|x| \geq r_1} \int \sqrt{1 + v^2} \mathring{f}(x, v) dv dx < m_\infty . \tag{3.4}$$

Then there exists a unique regular spherically symmetric solution of (1.1)–(1.5) on a region  $W(T, r_1)$  with  $f(0) = \mathring{f}$  and satisfying (3.2).

*Proof.* This is similar in outline to the proof of Theorem 3.1 of [4] and thus will only be sketched, with the differences compared to that proof being treated in more detail. Define  $\lambda_0(t, r)$  and  $\mu_0(t, r)$  to be zero. If  $\lambda_n$  and  $\mu_n$  are defined on the region  $W(T_n, r_1)$ , then  $f_n$  is defined to be the solution of the Vlasov equation with  $\lambda$  and  $\mu$  replaced by  $\lambda_n$  and  $\mu_n$  respectively and initial datum  $\mathring{f}$ . In order that this solution be uniquely defined it is necessary to know that no characteristic can enter a region of the form  $W(T, r_1)$  except through the initial hypersurface which is guaranteed

if  $\lambda_n \geq 0$  and  $\mu_n \leq 0$  on the region of interest. This will be proved by induction. If  $f_n$  is given, then  $\lambda_{n+1}$  and  $\mu_{n+1}$  are defined to be the solutions of the field equations with  $\rho_n$  and  $p_n$  constructed from  $f_n$  rather than  $f$ . The quantities  $\lambda_{n+1}$  and  $\mu_{n+1}$  are defined on the maximal region  $W(T_{n+1}, r_1)$  where  $0 < m_n(t, r) < r/2$  so that  $\lambda_{n+1}$  can be defined by the first equation in (3.2) and is positive on  $W(T_{n+1}, r_1)$ ; note that  $0 < m_n(t, r) < r/2$  for  $r$  large and  $0 < m_n(0, r) < r/2, r \geq r_1$  by assumption of  $\hat{f}$  so that  $T_{n+1} > 0$  by continuity. It can be shown straightforwardly that the iterates  $(\lambda_n, \mu_n, f_n)$  are well-defined and regular for all  $n$ . The most significant difficulty in proving the corresponding statement in [4] was checking the differentiability of various quantities at the center of symmetry, and in the exterior problem the center of symmetry is excluded. The next step is to show that there exists some  $T > 0$  so that  $T_n \geq T$  for all  $n$  and that the quantities  $\lambda_n, \dot{\lambda}_n,$  and  $\mu'_n$  are uniformly bounded in  $n$  on the region  $W(T, r_1)$ . Let  $L_n(t, r) := r^{-1}(1 - e^{-2\lambda_n(t, r)})^{-1}$ . For  $t \in [0, T_n[$  define

$$P_n(t) := \sup\{|v| : (x, v) \in \text{supp } f_n(t)\} ,$$

$$Q_n(t) := \|e^{2\lambda_n(t)}\|_\infty + \|L_n(t)\|_\infty .$$

Now it is possible to carry out the same sequence of estimates as in [4] to get a differential inequality for  $\tilde{P}_n(t) := \max_{0 \leq k \leq n} P_k(t)$  and  $\tilde{Q}_n(t) := \max_{0 \leq k \leq n} Q_k(t)$  which is independent of  $n$ , and this gives the desired result. The one point which is significantly different from what was done in [4] is the estimate for  $\dot{\lambda}_n$ . There a partial integration in  $r$  must be carried out, and in the exterior problem the limits of integration are changed; nevertheless the basic idea goes through. The remainder of the proof is almost identical to the proof of Theorem 3.1 of [4]. It is possible to bound  $\mu'_n$  and  $\dot{\lambda}'_n$  on the region  $W(T, r_1)$  and then show that the sequence  $(\lambda_n, \mu_n, f_n)$  converges uniformly to a regular solution of (1.1)–(1.5) on that region. Moreover this solution is unique. □

Next a continuation criterion will be derived. By the maximal interval of existence for the exterior problem is meant the largest region  $W(T, r_1)$  on which a solution exists with given initial data and the parameters  $r_1$  and  $m_\infty$  fixed.

**Theorem 3.2.** *Let  $(f, \lambda, \mu)$  be a regular solution of the reduced system (1.1)–(1.5) on  $W(T, r_1)$  with compactly supported initial datum  $\hat{f}$ . If  $T < \infty$  and  $W(T, r_1)$  is the maximal interval of existence then  $P$  is unbounded.*

*Proof.* Note first that, just as in the case of the problem in the whole space, the reduced equations (1.1)–(1.5) imply that the additional field equation (2.6) is satisfied. The conservation law (2.12) can be rewritten in the form

$$\partial_t(r^2 \rho) + \partial_r(e^{\mu - \lambda} r^2 j) = 0 . \tag{3.5}$$

Integrating (3.5) in space shows that the minimum value of  $m$  at time  $t$  (which occurs at the inner boundary of  $W(T, r_1)$ ) is not less than at  $t = 0$ . Hence  $L$  is bounded on the whole region. Suppose now that  $P$  is bounded. From (2.6) it follows that  $\dot{\lambda}$  is bounded, and if  $T$  is finite this gives an upper bound for  $\lambda$ . Combining this with the lower bound for  $\lambda$  already obtained shows that  $Q$  is bounded. This means that the quantities which influence the size of the interval of existence are all bounded on  $W(T, r_1)$ , and it follows that the solution is extendible. □

Combining Theorem 3.2 with the estimates of Sect. 2 gives a proof that the solution of the exterior problem corresponding to an initial datum of the kind assumed in Theorem 3.1 exists globally in time, i.e. that  $T$  can be chosen to be infinity in the conclusion of Theorem 3.1. For we know that it suffices to bound  $P$ , and the estimates bound the momentum  $v$  along any characteristic on which  $r$  is bounded below. Moreover the inequality  $r \geq r_1$  holds on  $W(\infty, r_1)$ .

#### 4. The Regularity Theorem

This section is concerned with the initial value problem for (1.1)–(1.5) on the whole space with boundary conditions (1.6).

**Theorem 4.1.** *Let  $(f, \lambda, \mu)$  be a regular solution of the reduced system (1.1)–(1.5) on a time interval  $[0, T[$ . Suppose that there exists an open neighborhood  $U$  of the point  $(T, 0)$  such that*

$$\sup\{|v| : (t, x, v) \in \text{supp } f \cap (U \times \mathbb{R}^3)\} < \infty. \tag{4.1}$$

*Then  $(f, \lambda, \mu)$  extends to a regular solution on  $[0, T'[$  for some  $T' > T$ .*

*Proof.* Suppose that the condition (4.1) is satisfied. Since the equations are invariant under time translations, it can be assumed without loss of generality that  $T$  is as small as desired. Choosing  $T$  sufficiently small ensures that  $U$  contains all points with  $(T - t)^2 + r^2 < 4T^2$  and  $0 \leq t < T$ . Now let  $r_1 < T$ . Then  $[0, T[ \times \mathbb{R}^3 \subset U \cup W(T, r_1)$ . Let  $\mathring{f}$  be the restriction of  $f$  to the hypersurface  $t = 0$ . Restricting  $\mathring{f}$  to the region  $|x| \geq r_1$  gives an initial datum for the exterior problem on  $W(T, r_1)$ . Let  $m_\infty$  be the ADM mass of  $\mathring{f}$ . There are now two cases to be considered, according to whether  $f$  vanishes in a neighborhood of the point  $(T, 0)$  or not. If it does then by doing a time translation if necessary it can be arranged that the matter stays away from the center on the whole interval  $[0, T[$  so that the results of Sect. 2 are applicable. It can be concluded that the solution extends to a larger time interval in this case. If  $f$  does not vanish in a neighborhood of  $(T, 0)$ , then by doing a time translation it can be arranged that (3.4) holds. In this case the results of the previous section show that there exists a global solution on  $W(\infty, r_1)$  satisfying (3.2). The extended solution must agree on  $W(T, r_1)$  with the solution we started with. Thus a finite upper bound is obtained for  $|v|$  on the part of the support of  $f$  over  $W(T, r_1)$ . But by assumption (4.1) we already have a bound of this type on the remainder of the support of  $f$ . Hence

$$\sup\{|v| : (t, x, v) \in \text{supp } f\} < \infty.$$

Applying Theorem 3.2 of [4] now shows that the solution is extendible to a larger time interval in this case too. □

There is another way of looking at this result. Suppose that  $(f, \lambda, \mu)$  is a regular solution of (1.1)–(1.5) on the whole space and that  $[0, T[$  is its maximal interval of existence. Then by a similar argument to the above the solution extends in a  $C^1$  manner to the set  $([0, T] \times \mathbb{R}^3) \setminus \{(T, 0)\}$ . Thus if a solution of (1.1)–(1.5) develops a singularity at all the first singularity must be at the center.

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