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MAKING QUANTUM GRAVITY CALCULABLE

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Abstract

We describe recent attempts at discretizing canonical quantum gravity in four dimensions in terms of a connection formulation. This includes a general introduction, a comparison between the real and complex connection approach, and a discussion of some open problems. (Contribution to the proceedings of the workshop "Recent mathematical developments in classical and quantum gravity", Sintra, Portugal, July 1995.)

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1. Introduction

In this article I want to report on some recent developments in applying discretization methods to quantum gravity in four space-time dimensions. My emphasis will be on explaining some of the main ideas and motivations to the non-expert, rather than going much into the technical details, which for the most part can already be found in the literature.

As is well known, there is as yet no *complete* consistent framework that would deserve to be called a theory of quantum gravity. However, there has been a great deal of activity during the last few years on the canonical quantization of gravity, and most of the material that follows will be related to these new developments. Having no very precise idea of what a complete theory of quantum gravity is eventually going to look like, it is not clear *a priori* what kind of quantities one is interested in calculating (or approximating). One point of view is that the mathematical structure of this hypothetical theory would naturally lead to a set of preferred quantities that take on a particularly easy form, an idea borne out by recent results in the canonical approach to quantization.

Our starting point will be the usual Einstein-Hilbert action for pure gravity,

$$S^{\rm EH}[^{(4)}g] = \int_M d^4x \sqrt{-^{(4)}g} R[^{(4)}g], \tag{1.1}$$

possibly with a cosmological constant term. In (1.1), $g_{\mu\nu}$ is a non-degenerate, Lorentzian four-metric on the manifold M. The two basic approaches to the quantization of gravity, as given by the action (1.1), are the Lagrangian one using a path integral over all possible configurations of the system, and a Hamiltonian one, based on operator algebras of preferred functions or observables. In the first case, one is usually only able to treat four-metrics with *Euclidean* signature, which leaves one with the problem of having to continue the results to the Lorentzian regime. Although the two approaches may look rather different in concrete implementations, they nevertheless share some important fundamental questions in common.

In the Lagrangian formulation, one works with functional integrals which, for example, are of the form

$$< \hat{O} >= \int_{\frac{\rm Riem\, M}{\rm Diff\, M}} \mathcal{D}[g]\, O([g]) e^{-S^{\rm EH}}, \tag{1.2}$$

in order to compute vacuum expectation values of functions O of the four-metric $g_{\mu\nu}$, or rather of diffeomorphism equivalence classes $[g_{\mu\nu}]$ of metrics, often called "geometries". The integration in (1.2) is over some appropriate set of such geometries, with a suitable measure $\mathcal{D}[g]$ that is supposed to depend only on equivalence classes of metrics, i.e., elements of the quotient space Riem M/Diff M. The usual strategy of quantum field theory would be to expand (1.2) perturbatively around the flat Minkowskian (or rather Euclidean) metric $\eta_{\mu\nu}$, by considering metrics of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ that differ only by a small amount $h_{\mu\nu}$ from the flat metric. However, for quantum gravity this strategy fails, since the theory turns out to be non-renormalizable. One therefore has to try to give meaning to the integral in some other, non-perturbative way. The crucial inputs in any such attempt are the space of quantum configurations to be integrated over, and the explicit form of the measure $\mathcal{D}[g]$.

In a Hamiltonian context, the central problem is to identify and quantize a preferred set of classical variables or observables. A typical such set consists of a sufficiently large Poisson algebra of functions which in the quantum theory is represented as an algebra of operators. As in the Lagrangian case, one has to determine the space of all quantum configurations and an inner product on this space in order to be able to compute scalar products and matrix elements of observables O. For example, in the metric approach, one seeks to make well-defined an expression like

$$< \psi([g]), \hat{O}\phi([g]) >= \int_{\frac{\mathrm{Riem}\,\Sigma}{\mathrm{Diff}\,\Sigma}} \mathcal{D}[g]\psi^* O\phi. \tag{1.3}$$

where for simplicity we have assumed that the four-manifold M was of the form $M = \mathbb{R} \times \Sigma$, with compact spatial slices Σ. The approaches to quantum gravity I want to discuss here take the viewpoint that one should start from discretized versions of quantities like (1.2) and (1.3), and then obtain the true theory in an appropriately taken (continuum) limit.

2. Lagrangian approaches to discretization

Two of the currently active research programs within the path integral approach are those of Regge calculus and dynamical triangulations. Quantum Regge calculus has developed from the ground-breaking idea of Regge [1] to do "general relativity without coordinates", by approximating a Riemannian manifold M by a piecewise linear, simplicial manifold $\mathcal{M}$. Since the manifold is piecewise flat, its curvature is concentrated on lower-dimensional hinges (in $d = 4$, on the two-dimensional triangles). A simplicial manifold $\mathcal{M}$ is characterized by an incidence matrix which describes the way in which individual simplices are glued together. In Regge calculus, one fixes an incidence matrix, i.e. a triangulation, and allows the edge-lengths of the 1-simplices to vary. Edge-length configurations, subject to certain restrictions, then give rise to (discretized) Riemannian metrics.

In the quantum theory, one tries to approximate the functional integral by the following product over all 1-simplices,

$$< \hat{O} >= \int \prod_i d\mu(l_i) O(l) e^{-S^{\mathrm{Regge}}(l)}, \tag{2.1}$$

where S^{Regge} is a discretized version of the Einstein-Hilbert action, depending on the link lengths l_i (see, for example, [2]). One central issue also here is the question of the right measure $d\mu(l)$. Since many edge length configurations correspond to the same metric, one expects the appearance of a non-trivial Jacobian J in $d\mu(l) = \prod_i dl_i J(l)$. In two dimensions, progress has been reported recently in determining the measure factor analytically [3], however, no similar results are known in higher dimensions. In numerical applications, the trivial measure $\prod_i dl_i$, or its scale-invariant version, $\prod_i \frac{dl_i}{l_i}$, have been used, in the hope that they would lead to reasonable "effective measures" for the functional integral.

The second approach, that of "dynamical triangulations", may be regarded as a rigid version of Regge calculus. One starts from a set of equilateral simplices, with all edge lengths set to 1, say, and then considers all ways in which they may be glued together to obtain a manifold of a fixed topology (this implies restrictions on the possible glueings). The main observation then is that distinct such triangulations $\mathcal{T}$ give rise to different metric structures. One therefore introduces the functional integral

$$< \hat{O} >= \sum_{\mathcal{T}, |\mathcal{T}|=M} \rho(\mathcal{T}) O(\mathcal{T}) e^{-S^{DT}(\mathcal{T})}, \tag{2.2}$$

where the Einstein-Hilbert Lagrangian (with cosmological constant) is discretized to $S^{\mathrm{DT}} = c_4 N_4(\mathcal{T}) - c_2 N_2(\mathcal{T})$. N_2 and N_4 denote the number of 2- and 4-simplices in the triangulation $\mathcal{T}$, and the c_i are bare coupling constants. This method was first applied in two dimensions, where a great deal of analytical results are available, and then generalized to higher dimensions. In this case, hardly any analytical results exist, and also the numerical analysis is much harder. The weight factor $\rho(\mathcal{T})$ is usually set to 1. Relevant questions in the dynamical triangulations program are over which (inequivalent) triangulations the sum (2.2) should be performed, and whether in this way one obtains a uniform sample of Riemannian geometries (see [4] for recent reviews).

In either approach there are as yet no definite results about the existence of phase transitions and continuum limits in four dimensions, and therefore also the question of the restoration of the diffeomorphism symmetry (which is not present in the discretized formulation) in such a limit remains open.

3. Discretizing in the canonical picture

Let us now turn to the Hamiltonian approach, which will be the subject of the rest of this article. It will be indispensible for us to use a connection form A_a^i as our basic variable, instead of the usual three-metric g_{ab} of the ADM approach. The basic pair of canonically conjugate variables will consist of an $SO(3)$-connection $A_a^i(x)$ and a dreibein $\tilde{E}_i^a$. They are the exact analogues of the basic variables on the phase space of a Yang-Mills theory with gauge group G, where we would call the A_a^i's gauge potentials and the $\tilde{E}_i^a$'s generalized electric fields. The far-reaching insight that also canonical gravity might be profitably described by gauge theory-like variables is due to Ashtekar [5]. The relation with the three-metric of the ADM canonical variable pair $(g_{ab}, \tilde{\pi}^{ab})$ is given by $\tilde{\tilde{g}}^{ab} = \tilde{E}_i^a \tilde{E}^{bi}$. Note that, unlike in Yang-Mills theory, the Einstein-Hilbert action cannot be rewritten as a functional $S[^{(4)}A]$ of a four-dimensional connection only. The gauge-theoretic analogy therefore holds only after the 3+1-decomposition.

In this formulation, and at a kinematical level, canonical gravity looks like a Yang-Mills theory with additional constraints, coming from the diffeomorphism symmetry not present in a gauge theory. Although the basic variables carry an internal index i, associated with the local $SO(3)$-symmetry, physical quantities, like in Yang-Mills theory, have to be gauge-invariant. Denoting by $\mathcal{A}$ the space of all connections on Σ, this means that all physically relevant information is already contained in the quotient space $\mathcal{A}/\mathcal{G}$, the space of gauge equivalence classes of connections.

Another input from Yang-Mills theory is the use of functionals of A labelled by closed curves γ in the manifold Σ. These are the so-called Wilson loops, the traces of path-ordered exponentials of a connection A along a curve γ. Their importance comes from the fact that they are explicitly gauge-invariant and form an (over-complete) set of variables on the quotient space $\mathcal{A}/\mathcal{G}$. Their non-local nature, that may be regarded as a defect from a gauge-theoretical point of view, turns out to be an asset in gravity. This happens because spatial loops are extended objects with diffeomorphism-invariant properties: they can link, knot and intersect.

For our purposes, Wilson loops are interesting objects because they are readily discretized in a lattice framework. Consider a cubic, three-dimensional $N \times N \times N$-lattice (with periodic boundary conditions), made up of vertices and one-dimensional, oriented links or edges l_i. The link holonomy U_l of a link l can be thought of as the exponentiated integral of the connection A along that link. Since the A's are non-commuting matrices in the internal space, one has to take the *path-ordered* exponential of A. Note that no metric structure is necessary to perform the integration since A is a spatial one-form. We write $U_l(A) = \mathrm{P} \exp \int_l A$, where one has to keep in mind that U_l takes values in the gauge group $SO(3)$ and still has a non-trivial transformation behaviour under gauge transformations at the beginning and end point of the link l. To get rid of this dependence, one may combine a set of oriented lattice links so that they form a closed loop γ on the lattice. Taking the trace of the resulting object, one obtains the *lattice Wilson loop* $\mathrm{Tr}\, U_\gamma := \mathrm{Tr}\, U_{l_1} U_{l_2} \ldots U_{l_k}$.

4. Real vs. complex connection formulation

We do not want to derive the Ashtekar variables from scratch, but rather introduce them in a form that exhibits some special features. Starting from the Arnowitt-Deser-Misner variables $(g_{ab}, \tilde{\pi}^{ab})$, one can derive an equivalent version of canonical gravity based on a pair $(\tilde{P}_i^a, K_a^i)$, with an internal $SO(3)$-index i, where $\tilde{P}_i^a$ is again a dreibein variable, and the canonically conjugate momentum K_a^i is closely related to the intrinsic curvature of Σ. Compared with the usual ADM formulation, one obtains three additional first-class constraints, corresponding to the local $SO(3)$-symmetry.

Consider now the canonical transformation

$$\tilde{E}_i^a = -\frac{1}{\beta} \tilde{P}_i^a, \quad A_a^i = \Gamma^i_{\ a} + \beta K_a^i,\ \beta \neq 0, \tag{4.1}$$

where Γ is the $SO(3)$-connection compatible with the triad $\tilde{P}_i^a$. The *Ashtekar variables* A_a^i and $\tilde{E}_i^a$ obey the classical commutation relations

$$\{A_a^i(x), \tilde{E}_j^b(y)\} = \delta_j^i \delta_a^b \delta^3(x, y), \tag{4.2}$$

since $\Gamma^i_{\ a}$ is a function of $\tilde{E}_i^a$ only. In this new canonical formulation, the first-class constraints read (up to constant factors)

$$\tilde{G}_i := \nabla_a \tilde{E}^a_i = 0, \tag{4.3a}$$
$$\tilde{H}_a := F^i_{ab} \tilde{E}^b_i = 0, \tag{4.3b}$$
$$\tilde{\tilde{H}} = -\zeta \beta^2 \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F_{ab\,k} + 2(\beta^2 \zeta - 1) \tilde{E}^a_{[i} \tilde{E}^b_{j]} (A^i_a - \Gamma^i_{\ a})(A^j_b - \Gamma^j_{\ b}), \tag{4.3c}$$

where $\tilde{G}_i$ are the three Gauss law constraints ($\nabla = \nabla(A)$ is the covariant derivative with respect to A), the $\tilde{H}_a$ are the three spatial diffeomorphism constraints ($F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i_{\ jk} A^j_a A^k_b$ is the field strength of the connection A), and $\tilde{\tilde{H}}$ is the Hamiltonian constraint. The algebraic expression for $\tilde{\tilde{H}}$ contains two free parameters, β and ζ. There are two possible values for ζ, where

$$\zeta = \begin{cases} +1 & \Leftrightarrow \quad \text{Euclidean signature;} \\ -1 & \Leftrightarrow \quad \text{Lorentzian signature.} \end{cases}$$

For special choices of β one can drastically simplify the form of the Hamiltonian $\tilde{\tilde{H}}$. In fact, if $(\beta^2\zeta - 1) = 0$, the second term in (4.3c) drops out and one is left with a *polynomial* Hamiltonian function. This leads to the following choices for β:

$$(\beta^2 \zeta - 1) = 0 \Leftrightarrow \begin{cases} \beta = \pm 1, & \text{Euclidean signature;} \\ \beta = \pm i, & \text{Lorentzian signature.} \end{cases}$$

Our primary interest lies of course with the Lorentzian case. Nevertheless it is interesting to observe how the difference between a Lorentzian and a Euclidean signature for the space-time metric is reflected in the canonical formalism. The derivation for $\tilde{\tilde{H}}$ in this compact form is a slightly modified version of the one given by Barbero [6]. It follows that if (in the Lorentzian case) we want to simplify the Hamiltonian constraint, we have to use a complex canonical transformation (i.e. (4.1) with $\beta = \pm i$). This is the choice originally made by Ashtekar [5], and it leads to a *complex* $SO(3)$-connection A^i_a as one of the basic variables. This gives rise to complications in the sense that one has to keep track of a set of reality conditions, to ensure that real, and not complex gravity is described. On the bonus side, *all* of the constraints are now polynomial in the basic variables A and $\tilde{E}$. This is important in the quantization, because polynomiality of a classical phase space function simplifies the search for a corresponding quantum operator. (Miraculously, if the real world *were* described by Euclidean metrics, one could achieve this simplification without ever leaving the real domain.)

The alternative that emerges from this picture (for $\zeta = -1$) is to set $\beta = -1$, say, so that $A_a^i(x)$ is a real $SO(3)$-connection, and the basic Poisson bracket relations (4.2) have the standard form. As shown in [6], the Hamiltonian can be rewritten as

$$\tilde{\tilde{H}} = \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b \left(F_{ab\,k}(A) - 2R_{ab\,k}(\Gamma)\right) = 0, \tag{4.4}$$

where R is the curvature of the connection Γ, $R_{ab}{}^i = \partial_a \Gamma_b^i - \partial_b \Gamma_a^i + \epsilon^i{}_{jk} \Gamma_a^j \Gamma_b^k$. Because of the Γ-dependence, $\tilde{\tilde{H}}$ is non-polynomial in the basic variables. It is rather similar in structure to the usual ADM Hamiltonian,

$$\tilde{H}^{\mathrm{ADM}}[g, \tilde{\pi}] = -\sqrt{g}\,{}^{(3)}R + \frac{1}{\sqrt{g}}(\tilde{\pi}^{ab}\tilde{\pi}_{ab} - \frac{1}{2}\tilde{\pi}^2) = 0, \tag{4.5}$$

in that it can be brought into an equivalent form which is polynomial *modulo powers of the square root of the determinant of the metric*, $\sqrt{g} = \sqrt{|\tilde{\tilde{E}}|}$. However, as is well known, its non-polynomiality is one important reason that makes the ADM Hamiltonian $\tilde{H}^{\mathrm{ADM}}$ so hard to quantize, and has lead to an impasse in the canonical quantization in the metric approach. So it seems as if nothing had been gained by using this real connection formulation. We will be returning to this issue in due course.

Let us summarize the two basic alternatives for the connection formulation of Lorentzian gravity: either one works with a simple, polynomial Hamiltonian, leading to complex connection variables and the need for reality conditions, or one stays within the real framework, but has to live with a more complicated, non-polynomial Hamiltonian constraint.

5. The lattice discretization

Having set the stage for the connection formulation à la Ashtekar in the continuum, we now come back to the discretized lattice formulation. The basic ingredients are those of the customary Kogut-Susskind approach of Hamiltonian lattice gauge theory [7]. For simplicity, we will work with the double cover $SU(2)$ of $SO(3)$, which in any case is necessary when considering matter coupling. For the complex case, the gauge group should of course be $SU(2)^{\mathbb{C}} = SL(2, \mathbb{C})$. The lattice analogues of the variables $(A_a^i, \tilde{E}_i^a)$ are the link holonomies U_l introduced earlier, and the link momenta $p_i(l)$. We label vertices of the cubic lattice by a triplet $n = (n_1, n_2, n_3)$, $n_i = 0, 1, \ldots, N \equiv 0$, and links l by their origin n and a direction $\hat{a}$. With this notation, the algebra of the basic quantum operators is given by

$$
\begin{aligned}
&[\hat{U}_A{}^B(n,\hat{a}), \hat{U}_C{}^D(m,\hat{b})] = 0 \\
&[\hat{p}_i(n,\hat{a}), \hat{U}_A{}^C(m,\hat{b})] = -\frac{i}{2}\,\delta_{nm}\delta_{\hat{a}\hat{b}}\,\tau_{iA}{}^B \hat{U}_B{}^C(n,\hat{a}) \\
&[\hat{p}_i(n,\hat{a}), \hat{p}_j(m,\hat{b})] = i\,\delta_{nm}\delta_{\hat{a}\hat{b}}\,\epsilon_{ijk}\,\hat{p}_k(n,\hat{a}),
\end{aligned}
\tag{5.1}
$$

where the indices $A, B, \ldots$ are those of the defining representation in terms of 2×2-matrices. The commutators (5.1) are the exact quantization of the corresponding classical Poisson bracket relations. The holonomy operators $\hat{U}_A{}^B$ act by multiplication, $\hat{U}_A{}^B\psi(g) = U_A{}^B\psi(g)$, and the link momenta are represented by the differential operators $\hat{p}_i = -i\tau_{iA}{}^B U_B{}^C(\frac{\partial}{\partial U})_C{}^A$. The τ_i are the usual Pauli σ-matrices, rescaled by a factor i, and g stands for a group element (not to be confused with the same symbol for the metric used elsewhere in this article).

This is the right moment to discuss some of the special features of lattice *gravity*, as opposed to lattice gauge theory. Firstly, in the real connection formulation, the total lattice Hilbert space $\mathcal{H}$ is given by the direct product over all links of the square-integrable functions on the group, $\mathcal{H} = \otimes_l L^2(SU(2), dg)$, with dg the Haar measure on $SU(2)$. However, in the complex $SL(2,\mathbb{C})$-case, we immediately encounter the problem that there is no analogue of the left- and right-invariant Haar measure, because the group is non-compact. Secondly, we have already mentioned that gravity possesses an extra symmetry, the spatial diffeomorphisms. This symmetry is badly “broken” by the lattice discretization (as it is by any other discretization), i.e. it is no longer present. Usually this leads one to the requirement that the symmetry should be restored in the continuum limit, when all link lengths shrink uniformly to zero, and/or the number of vertices goes to infinity.

Until recently, no possible resolutions to the scalar product problem in the complex case were known. (The real case has so far not been considered.) Some investigations of lattice versions of Ashtekar gravity were made in [8-10]. Generally speaking, the lattice computations are rather lengthy, and not many results have been obtained concerning the algebra of lattice constraints, both classically and quantum-mechanically (in the latter case one also has to check for possible anomalies), and the continuum limit. Renteln was able to show that a particular discretized version of the quantum algebra of spatial diffeomorphism constraints reproduces the continuum algebra [9].

What comes to our aid in defining a scalar product for the $SL(2,\mathbb{C})$-case is a construction by Hall [11], namely, an integral transform

$$C_t : L^2(SU(2), dg) \to L^2(SL(2, \mathbb{C}), d\nu_t)^{\mathrm{hol}}$$
$$[C_t(f)](g_{\mathbb{C}}) := \int_{SU(2)} dg\, f(g) \rho_t(g^{-1} g_{\mathbb{C}}) \tag{5.2}$$

where the target space consists only of the *holomorphic* square-integrable functions on the complexified group ((5.2) is to be thought of as a non-linear analogue of the Segal-Bargmann transform $L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{C}^n, d\mu)^{\mathrm{hol}}$, where $d\mu$ is the Gaussian measure). The measure in the holomorphic representation is the heat kernel measure, depending on a positive, real parameter t, and ρ_t is the heat kernel itself.

These results have been used in [12] to set up a holomorphic lattice representation. Since the Hall transform is an isomorphism of Hilbert spaces, it induces a scalar product on $SL(2, \mathbb{C})$-lattice wave functions (an explicit coordinate expression for $d\nu_t$ is not known), and it turns out that Wilson loop states are indeed square-integrable with respect to this inner product. The next step then is to find an appropriate lattice discretization H^{discr} and look for solutions to the Wheeler-DeWitt equation on the lattice, $\hat{H}^{\mathrm{discr}} \psi(\gamma) = 0$, and observables $\hat{O}$ satisfying $[\hat{H}^{\mathrm{discr}}, \hat{O}] = 0$. By $\psi(\gamma)$ we denote a generic linear combination of Wilson loop states on the lattice. If one thinks of taking eventually a continuum limit by letting the lattice spacing a go to zero, one has to solve these equations only to lowest order in a. In spite of the finite dimensionality of the lattice discretization, it is rather non-trivial to make progress on these questions.

What is still needed in this context is a systematic investigation of all possible discretized Hamiltonians and scalar products. The choice of the classical H^{discr} is non-unique because the criterion that $H^{\mathrm{discr}} \stackrel{a \to 0}{\longrightarrow} H^{\mathrm{cont}}$ in the limit does not fix H^{discr} uniquely. Additional ambiguities arise through different operator orderings of the quantum Hamiltonian $\hat{H}^{\mathrm{discr}}$. As for the scalar product, there are indications that it is too simple to reflect in a direct way the reality conditions $A + A^{\dagger} = 2\,\Gamma(\tilde{E})$ of the classical phase space formulation. Rather, it corresponds to reality conditions of the form $A^{\dagger} = A + 2\tilde{E}$. The implicit metric dependence of this latter expression could lead to problems with the continuum limit in lattice gravity. Such a dependence is also present in the holomorphic continuum formulation [13], where one has to introduce an auxiliary function that depends on the lengths (!) of loop segments. Still this does not necessarily mean that the holomorphic Hilbert space cannot be put to some use in complex Ashtekar gravity.

For specific choices of the discretized Hamiltonian, some solutions to the Wheeler-DeWitt equation have been found. The simplest ones depend on so-called Polyakov loops [12], which are straight closed lattice loops winding around the lattice in one of the three directions (this is possible because of periodicity). More recently, other solutions have been constructed [14], which are labelled by multiple plaquette loops (a plaquette on the lattice is a smallest loop, made up of four links). Although they solve the Wheeler-DeWitt equation, they are not square-integrable with respect to the holomorphic scalar product. There may of course exist other scalar products in which those solutions have finite norm.

6. Computing volumes

Apart from the issue of the scalar product, there is another reason why the solutions to the discretized Wheeler-DeWitt equation found so far are probably not very interesting physically. In order to understand this, we need to study the so-called *volume operator*. It comes from the classical volume function,

$$\mathcal{V}(\mathcal{R}) = \int_{\mathcal{R}} d^3x\, \sqrt{g} = \int_{\mathcal{R}} d^3x\, \sqrt{\frac{1}{3!}|\epsilon_{abc}\,\epsilon^{ijk}\,\tilde{E}^a_i \tilde{E}^b_j \tilde{E}^c_k|}, \tag{6.1}$$

which measures the volume of a spatial region $\mathcal{R} \subset \Sigma$. Note that $\mathcal{V}$ is not an observable in pure gravity in the sense that it does not Poisson-commute with the Hamiltonian constraint $\tilde{\tilde{H}}$). It however becomes an observable if the region $\mathcal{R}$ is defined intrinsically in some way by the presence of matter fields [15]. This quantity can be studied also on the lattice, because it can be discretized straightforwardly and written as a function of the lattice momenta,

$$\mathcal{V}^{\rm latt} = \sum_{n\in\mathcal{R}} \sqrt{\frac{1}{3!}|\epsilon_{abc}\,\epsilon^{ijk}\, p_i(n,\hat{a})p_j(n,\hat{b})p_k(n,\hat{c})|}, \tag{6.2}$$

where the integral has been substituted by a sum over all vertices lying within a lattice region $\mathcal{R}$. Because of the presence of both the modulus and the square root, it is not *a priori* obvious how to quantize (6.2). However, in this case we are lucky, because both in the real and the holomorphic representation the momenta $\hat{p}$ are selfadjoint operators [16]. One can therefore go to a basis where the terms cubic in $\hat{p}$ under the modulus are diagonal operators, and the modulus and square root can be defined in terms of their eigenvalues.

In order to diagonalize the volume operator, it is very convenient to use a particular set of gauge-invariant states in the loop representation, the (generalized) *spin network states*, as was first pointed out in [15]. A spin network state may be represented as a certain antisymmetrized linear combination of Wilson loop states. In order to describe a spin network on the lattice, one needs to associate an "occupation number" $j(l) \in \mathbb{Z}^+$ with each lattice link (subject to a number of restrictions). This can be represented by drawing $j(l)$ lines through the link l. To obtain a multiple Wilson loop, one has to connect these lines at the lattice vertices in a gauge-invariant manner. The contractors at the vertices are the second ingredient in the definition of a spin network state. Details of this construction can be found in [15-17].

It can be shown that the spin network states span the space of all gauge-invariant lattice functions [18,17]. The operator $\hat{\mathcal{V}}^{\text{latt}}$ is "almost diagonal" in terms of the spin networks, i.e. the diagonalization is reduced to diagonalizations within finite-dimensional eigenspaces of $\hat{\mathcal{V}}^{\text{latt}}$, and its spectrum is discrete [15]. In determining the eigenvalues of the volume operator explicitly, it suffices to study the action of the operator $\hat{D}(n) := \epsilon_{abc}\,\epsilon^{ijk}\,\hat{p}_i(n,\hat{a})\hat{p}_j(n,\hat{b})\hat{p}_k(n,\hat{c})$ around a vertex n. Moreover it is useful to consider separately cases of different valence, where the valence counts the number of links with non-vanishing $j(l)$ meeting at n.

Because of the antisymmetry of the operator $\hat{D}(n)$, it annihilates spin networks which are two-valent at n. Although it is not immediately obvious, one can construct a general argument that also trivalent vertices are annihilated, i.e. spin network states that possess at most trivalent intersections have "zero volume" [16]. This is the reason why we considered the solutions for the Hamiltonian constraint equation described above as probably uninteresting. One obtains the first non-trivial eigenvalues for four-valent intersections. Some explicit spectral formulae for such spin networks were derived in [19]. One technical problem one encounters in the diagonalization is the occurrence of linearly dependent spin network states (because not all ways of contracting flux lines at vertices are independent). These states have to be eliminated to avoid spurious eigenvalues.

7. Outlook, and a new perspective

Apart from yielding interesting information about the volume operator itself, the construction described in the previous section also throws new light on the *real* connection formulation. We mentioned earlier that Barbero's Hamiltonian can be written in polynomial form up to powers of $\sqrt{g}$. Now observe that in the diagonalized spin network basis, where the volume operator is diagonal, also the determinant g (which may be written as a cubic polynomial in the momenta) is diagonal. This means that in this basis it is straightforward to define all operators of the kind $\widehat{\sqrt{g}^k}$. It turns out that this enables one to quantize a discretized version of the Hamiltonian (4.4) on the lattice [20].

Of course this quantum Hamiltonian is still more complicated than the one of the complex Ashtekar formulation, $\tilde{\tilde{H}} = \epsilon^{ijk}\tilde{E}_i^a\tilde{E}_j^b F_{ab\,k}$, in that it has an additional potential term. On the other hand, there are no problems with the choice of a scalar product which is simply given by the ordinary Haar measure on $SU(2)$, as in lattice Yang-Mills theory. However, contrary to the ADM formulation, there is now an obvious way to quantize arbitrary functions of the determinant of the metric.

If one wants to pursue this line of thought further, and try to solve the Wheeler-DeWitt equation, say, it is clear that one a) needs an efficient general method for generating volume eigenstates, and b) must eliminate the zero-volume states because of the inverse powers of the metric occurring in the Barbero Hamiltonian. Thus, unlike in complex Ashtekar gravity, in the real context one cannot extrapolate smoothly to the case where the (quantized) three-metric is allowed to be degenerate.

To summarize, there has been progress in Hamiltonian lattice gravity, although many details remain to be worked out. Crucial in all recent developments has been the use of connection variables, and the associated gauge-invariant Wilson loops. There are two basic alternatives to proceed in the lattice framework. The first one is that corresponding to complex A; there it remains to find solutions "with volume" to $\hat{H}\psi = 0$ (although it is not strictly necessary to work in terms of spin network states), and to investigate different Hamiltonians and scalar products. The second way is that based on a real connection A; as we have seen, in this case the use of a diagonalized spin network basis is mandatory; the search for solutions to the Wheeler-DeWitt equation has only just begun [20]. Once some solutions have been found, it would be very interesting to compare the results of both the real and the complex formulation, and to see whether they are equivalent. Other issues that remain to be addressed are whether and how the continuum limit should be taken, and whether and how diffeomorphism invariance should be incorporated in the lattice picture. Also it would be most interesting to try to study matter couplings in the same framework.

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