E_{10} for beginners*

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December 9, 1994

In this contribution, we summarize a recent attempt to understand hyperbolic Kac Moody algebras in terms of the string vertex operator construction [12] (which readers are also advised to consult for a comprehensive list of references). As is well known, Kac Moody algebras (see e.g. [21], [15]) fall into one of three classes corresponding to whether the associated Cartan matrices are positive definite, positive semi-definite and indefinite. Of these, the first two are well understood, leading to finite and affine Lie algebras (the latter being equivalent to current algebras in two space-time dimensions). Virtually nothing is known, however, about the last class of Kac Moody algebras, based on indefinite Cartan matrices. Nonetheless these have been repeatedly suggested as natural candidates for the still elusive fundamental symmetry of string theory (see e.g. [24], [28] for recent proposals in this direction). Being vastly larger than affine Kac Moody algebras, they are certainly "sufficiently big" for this purpose, but an even more compelling argument supporting such speculations is the intimate link that exists between Kac Moody algebras and the vertex operator construction of string theory which has been known for a long time. More specifically, it has been established that the elements making up a Kac Moody algebra of finite or affine type can be explicitly realized in terms of tachyon and photon emission vertex operators of a compactified open bosonic string [9], [14]. On the basis of these results, it has been conjectured that Kac Moody algebras of indefinite type might not only furnish new symmetries of string theory, but might themselves be understood in terms of string vertex operators associated with the higher excited (massive) states of a compactified bosonic string [14], [8].

Of what nature are these new symmetries then? In [17], it was argued that in the ultrahigh-energy limit of string theory, where the Planck mass goes to zero, an infinite number of linear relations exists between the scattering amplitudes of different string states that are valid order by order in perturbation theory. This suggests an enormous

* Invited talk presented by H. Nicolai at the Feza Gürsey Memorial Conference, Istanbul, June 1994.
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symmetry which is restored at high energies. It may seem unreasonable to study such a queer limit but, in fact, it is a very conservative approach. For example, if we had not known the spontaneously broken symmetry of the electroweak interactions, we could have in principle discovered it at high energies where all gauge particles become massless again. In agreement with this analogy it is indeed a reasonable hope to find hints of the unbroken string gauge algebra by studying the relations between high energy scattering amplitudes. But since the latter, according to the above result, are essentially unique for a given number of scattered physical states, it is tempting to regard the Lie algebra of physical states itself as part of some universal gauge algebra. Note that we obtain, by construction, different Lie algebras of physical states when we consider inequivalent string backgrounds. Moreover, due to the presence of infinitely many massive physical states, each Lie algebra would have to be spontaneously broken almost completely. If we take this picture for granted then our task will be to make a clever choice for the specific string background in order to find a Lie algebra of physical states as large as possible. 'Clever' here apparently means 'as symmetric as possible', and one is therefore naturally led to Minkowskian torus compactifications where *all* spacetime dimensions are chosen to be periodic (hence "finite in all directions" [24]). More specifically, for the 26-dimensional bosonic string there is a unique choice of maximal symmetry, namely the even selfdual Lorentzian lattice $II_{25,1}$ which indeed provides a "large" algebra — the infinite rank fake monster Lie algebra introduced by Borcherds [3]. To gain insight into the mathematical structure of these symmetry algebras it is instructive to study toy models such as the 10-dimensional bosonic string compactified with momentum lattice $II_{9,1}$.

The above, to some extent heuristic arguments were recently put on more solid ground in [28]. In this paper, it was established that the fake monster Lie algebra *is* a symmetry of string theory in the sense that every physical state leads to a symmetry of the string scattering amplitudes. In view of this result one could now pose the question to which extent the vertices are already fixed by stipulating the fake monster Lie algebra as symmetry algebra. The degree of uniqueness would then give us a clue of how small the algebra is in comparison with the universal string gauge algebra. Certainly, they cannot be the same. For on the one hand it is clear that the string vertices describe the string field theory, on the other hand we know (see [24]) that the fake monster Lie algebra does not contain all Lie algebras arising from other string backgrounds. It is worth mentioning that the calculations were carried out in the so-called group theoretical approach to string theory which seems to be a powerful formalism to analyze the issue of string symmetries.

Apart from possible relations to string theory, hyperbolic Kac Moody algebras might appear in the dimensional reduction of (extended) supergravity theories to one dimension [19]. Some evidence for this conjecture was presented in [25], where it was argued that the Matzner Misner group arising in the reduction of Einstein's theory to two dimensions can be generalized to a "Matzner Misner SL$(3,\mathbb{R})$" group providing precisely the two nodes needed to extend the Dynkin diagram to a hyperbolic one. The null Killing reduction required for this investigation has been recently studied in [20]. We also mention that these hidden symmetries may be related to S,T and U duality symmetries arising in string theories (see [18] for recent progress in this direction).

Let us begin by reviewing how one constructs a Kac Moody algebra from a given Cartan matrix $A = (a_{ij})$, where the indices i, j are assumed to take d values (so d is the rank of this algebra), and where the matrix A may also be indefinite. The basic building blocks are the Chevalley generators e_i, f_i, h_i, which are subject to the following relations

$$[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \tag{1}$$
$$[e_i, f_j] = \delta_{ij} h_i. \tag{2}$$

The elements h_i then automatically obey $[h_i, h_j] = 0$ and constitute a basis of the Cartan subalgebra of $\mathfrak{g}(A)$. The free Lie algebra associated with A is obtained by forming multiple commutators of the elements $\{e_i, f_i, h_i \mid i\}$ in all possible ways taking into account the above relations. To obtain the Kac Moody algebra $\mathfrak{g}(A)$ itself we must still divide out by the Serre relations

$$(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0, \quad (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0. \tag{3}$$

It is a standard result [21] that this algebra can be written as a direct sum

$$\mathfrak{g}(A) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- . \tag{4}$$

The subalgebras $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are defined to consist of all linear combinations of multiple commutators of the form $[f_{i_1}, [f_{i_2}, \ldots [f_{i_{n-1}}, f_{i_n}] \ldots]]$ and $[e_{i_1}, [e_{i_2}, \ldots [e_{i_{n-1}}, e_{i_n}] \ldots]]$, respectively, modulo the multilinear Serre relations (3). Since the subalgebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are conjugate to each other, it is in practice sufficient to analyze either of them. In this way one gets for positive definite or positive semi-definite A just the finite or affine Kac Moody algebras, respectively (for the affine algebras, one still has to add an outer derivation to $\mathfrak{h}$ due to the degeneracy of the Cartan matrix). When A is indefinite, on the other hand, the number of multiple commutators "explodes", and no manageable way to handle them is known that would be analogous to the realization of affine Kac Moody algebras in terms of current algebras. More specifically, the characteristic feature of indefinite Kac Moody algebras is the exponential growth in the number of Lie algebra elements associated with a root $\boldsymbol{\Lambda}$ as $\boldsymbol{\Lambda}^2 \to -\infty$. Thus for a given number of Chevalley generators $e_{i_1}, \ldots, e_{i_n}$, the number of inequivalent ways of arranging them into non-vanishing multiple commutators increases exponentially with $-\boldsymbol{\Lambda}^2$, where $\boldsymbol{\Lambda} = \mathbf{r}_{i_1} + \ldots + \mathbf{r}_{i_n}$ and $\mathbf{r}_i$ are the simple roots associated with e_i. This problem does not occur for either finite or affine Kac Moody algebras, for which $\boldsymbol{\Lambda}^2 = 2$ or 0 are the only possibilities. The problem here is not so much the enormous number of commutators, but rather the fact that all those multiple commutators which contain the Serre relation somewhere inside, or can be brought to such a form by use of the Jacobi identities, must be identified and discarded. Even the more modest question as to how many elements there are for a given root has defied all attempts at a general solution so far. For a limited number of cases, one knows explicit multiplicity formulas counting the dimensions of the root spaces, but the complete root multiplicities are not known for a single Kac Moody algebra of indefinite type (root multiplicities can be determined in principle from the Peterson recursion formula [23], but this formula quickly becomes too unwieldy for practical use).

We are here mainly interested in *hyperbolic* Kac Moody algebras which are based on indefinite A, but obey the additional requirement that the removal of any point from the Dynkin diagram leaves a Kac Moody algebra which is either of affine or finite type. One can then show that the maximal rank is ten and that the associated root lattice must be Minkowskian, i.e. with metric signature $(+\ldots+|-)$. There are altogether three hyperbolic algebras of maximal rank. Of these, E_{10} is not only the most interesting, containing E_8 and its affine extension E_9 as subalgebras, but also distinguished by the fact that it has only one affine subalgebra that can be obtained by removing a point from the E_{10} Dynkin diagram, while the other two rank 10 algebras contain at least two regular affine subalgebras. Furthermore, the root lattice $Q(E_{10})$ coincides with the (unique) 10-dimensional even unimodular Lorentzian lattice $II_{9,1}$ [5], whereas the root lattices of the other two maximal rank hyperbolic algebras are not self-dual. For the further discussion it is useful to introduce the notion of *level*. Denoting the "over-extended root", which turns an affine into a hyperbolic Kac Moody algebra, by $\mathbf{r}_{-1}$ (e.g. for E_{10}, this is the left-most point in the Dynkin diagram, see below), one defines the level $\ell \in \mathbb{Z}$ of a root such that positive ℓ counts the number of e_{-1} generators in $[e_{i_1}, [e_{i_2}, \ldots [e_{i_{n-1}}, e_{i_n}] \ldots]]$ (similarly, if ℓ is negative, $-\ell$ counts the number of f_{-1} generators in $[f_{i_1}, [f_{i_2}, \ldots [f_{i_{n-1}}, f_{i_n}] \ldots]]$). In terms of the corresponding root $\boldsymbol{\Lambda} = \mathbf{r}_{i_1} + \ldots + \mathbf{r}_{i_n}$, ℓ is defined by

$$\ell := -\boldsymbol{\Lambda} \cdot \boldsymbol{\delta}, \tag{5}$$

where $\boldsymbol{\delta}$ is the null vector of the affine subalgebra obtained by deleting the over-extended node from the Dynkin diagram (in principle one could also use the null vector of other regular affine subalgebras to define a level which would be different from the above; however, then not all of the results to be stated below remain valid, e.g. the level-one states would no longer form the basic representation of this affine subalgebra). The level derives its importance from the fact that it grades the hyperbolic Kac Moody algebra with respect to the affine subalgebra [7]. Consequently, the subspaces belonging to a fixed level can be decomposed into irreducible representations of the affine subalgebra, the level being equal to the eigenvalue of the central term on this representation (the full hyperbolic algebra contains representations of *all* integer levels). Multiplicities are known for levels $\ell = 0$ (corresponding to the affine subalgebra) and $\ell = 1$ (corresponding to the so-called basic representation). More precisely, we have [21]

$$\text{mult}(\boldsymbol{\Lambda}) = p_{d-2}(1 - \tfrac{1}{2}\boldsymbol{\Lambda}^2), \tag{6}$$

where d is the rank of the algebra and the function $p_{d-2}(n)$ counts the partitions of $n \in \mathbb{N}$ into "parts" of $d-2$ "colours". For $\ell = 2$, one knows general multiplicity formulas in some cases, and in particular for E_{10}. Beyond $\ell = 2$, no general formula seems to be known.

Now it is well known that, at least in principle, the string vertex operator construction can provide a more concrete realization of an indefinite Kac Moody algebra. To exploit it one interprets the root lattice as the momentum lattice of a fully compactified string, and tries to understand the multiple commutators in terms of string vertex operators associated with the excited string states. The real roots then correspond to spacelike (tachyon) momenta and the imaginary roots to either lightlike or timelike momenta. For the simple roots $\mathbf{r}_i$, all of which obey $\mathbf{r}_i{}^2 = 2$ (and hence are real), we have the following correspondence between Chevalley generators and tachyon and photon states:

$$e_i \mapsto |\mathbf{r}_i\rangle, \tag{7}$$

$$f_i \mapsto -|-\mathbf{r}_i\rangle, \tag{8}$$

$$h_i \mapsto \mathbf{r}_i(-1)|\mathbf{0}\rangle. \tag{9}$$

Here we use the shorthand notation $\mathbf{r}_i(-1) \equiv \mathbf{r}_i \cdot \alpha_{-1}$ (where α^{μ}_{-1} is just the lowest string oscillator); furthermore, we define the states in such a way that cocycle factors have been absorbed and do not appear explicitly. The commutator between any two physical states φ and ψ is quite generally defined by the formula (cf. [1])

$$[\psi, \varphi] := \mathrm{Res}_z\,(\mathcal{V}(\psi, z)\varphi)\,, \tag{10}$$

where $\mathcal{V}(\psi, z)$ is the string vertex operator associated with the state ψ (the residue formula here is completely equivalent to the contour integrals employed down in [14]). An important consequence of this formula is that the physical string states always form a Lie algebra (not to be confused with the Kac Moody algebra, see remarks below).

Apart from yielding a concrete realization of an abstract algebraic structure, the string vertex operators construction has the advantage that the Serre relations (3) are built in from the outset: in this context they simply state that the string has no excited states "below" the tachyons. For finite and affine Kac Moody algebras, no other states beside tachyons and photons occur, whereas for indefinite A, excited string states of arbitrarily high levels must be taken into account. These will appear with certain polarizations, whose number increases rapidly with the mass level of the string state. Thus, in more physical parlance, the multiplicity of a root is equal to the number of linearly independent polarizations of the corresponding string state that can be reached by multiple commutators. This "easy" realization of the Kac Moody algebra might suggest that the problem of classifying the Lie algebra elements is essentially solved by the string vertex operator construction. Unfortunately, this is by no means the case because the problem of accounting for the Serre relations is now replaced by another one: not all physical states can be obtained in terms of multiple commutators. Denoting the Lie algebra of *all* physical states by $\mathfrak{g}_\Lambda$, we rather have the *proper* inclusion

$$\mathfrak{g}(A) \subset \mathfrak{g}_\Lambda\,. \tag{11}$$

In other words, the Lie algebra of physical states is well understood in physical terms, but the actual Kac Moody algebra $\mathfrak{g}(A)$ is only a subalgebra thereof, and all the complications now reside in the way in which $\mathfrak{g}(A)$ is embedded in the bigger, but simpler algebra $\mathfrak{g}_\Lambda$. In particular, there are "missing states", i.e. physical states that cannot be represented as multiple commutators of the Chevalley generators. A possible explanation for this phenomenon is the following. For continuous momenta it is well known that one can generate any physical state by multiple scattering of tachyons (multiple scattering is the equivalent of taking multiple commutators), so there can be no "missing states". This is no longer the case for the compactified string: when the tachyon momenta are on the lattice, certain "intermediate states" are no longer allowed, and therefore not all physical states are accessible in this way any more. The construction given in [12] is an attempt to make this intuitive argument more precise. As a further consequence of (11), the multiplicities are *not* given by the numbers of excited states of the string (which are of course well known), but only bounded above by them:

$$\mathrm{mult}\boldsymbol{\Lambda} \leq p_{d-1}(1 - \tfrac{1}{2}\boldsymbol{\Lambda}^2) - p_{d-1}(-\tfrac{1}{2}\boldsymbol{\Lambda}^2). \tag{12}$$

Only for Euclidean lattices the two Lie algebras coincide, and we have equality in (11).

To summarize: the root system of the Kac Moody algebra $\mathfrak{g}(A)$ is well understood though its root multiplicities are not completely known for a single example; whereas the Lie algebra of physical string states $\mathfrak{g}_\Lambda$ has a much simpler structure (although it will not be easy to define a root system associated with it). Thus a complete understanding of (11) requires a "mechanism" which tells us how $\mathfrak{g}(A)$ has to be filled up with physical states to reach the complete Lie algebra of physical states.

We mention that recently Borcherds [2] has introduced a new type of generalized Kac Moody algebras by admitting "imaginary simple" roots (i.e. obeying $\mathbf{r}_i{}^2 \leq 0$). In the present interpretion this means that one adds "by hand" those physical states not obtainable as multiple commutators; the corresponding momenta will then be counted as new simple roots, whose multiplicity is simply given by the number of associated independent (missing) polarizations. However, this program has so far only been carried to completion for the 26-dimensional bosonic string, where special properties such as the no-ghost theorem play a crucial role. In this example, all missing states are under control (though not explicitly known): one has to adjoin a certain (infinite) set of photonic states as new Lie algebra generators to the ordinary Kac Moody generators in order to get a complete set of generators for the Lie algebra of physical states. The resulting algebra constitutes an example of a generalized Kac Moody algebra and has been dubbed fake monster Lie algebra (for historical reasons). The imaginary simple roots corresponding to the extra generators are just the positive integer multiples of the (lightlike) Weyl vector for the lattice $II_{25,1}$, and their multiplicities are equal to the number of photon states (i.e. = 24). On the other hand, for algebras such as E_{10}, not much is gained by this change of perspective, because supplying the missing generators "by hand" presupposes knowledge of what the missing Lie algebra elements are (not to mention the potential arbitrariness as to the number of ways in which this can be consistently done). So the problem identifying the elements belonging to $\mathfrak{g}_\Lambda$ and not to $\mathfrak{g}(A)$ in (11) remains.

In [12] it is proposed to understand Kac Moody algebras of hyperbolic type, and in particular the maximally extended hyperbolic algebra E_{10}, from a more "physical" (i.e. pedestrian) point of view by focussing on the "missing states" rather than on the Serre relations. For this purpose, we make use of a lattice version of the DDF construction, which provides the most direct and explicit solution of the physical state constraints in string theory [6]. The physical states, which by definition are annihilated by the Virasoro constraints, are simply obtained in this scheme by acting on a tachyonic groundstate with the DDF operators, which commute with all Virasoro generators and form a spectrum generating algebra. Our key observation is that for Kac Moody algebras of "subcritical" rank (i.e. $d < 26$), there appear *longitudinal* string states and vertex operators beyond level one, whose significance in this context has so far not been recognized to the best of our knowledge. The appearance of longitudinal states is already obvious from the known multiplicity formulas for level $\ell = 2$: for sufficiently large (negative) $\boldsymbol{\Lambda}^2$, one can check that there are roots $\boldsymbol{\Lambda}$ such that mult$(\boldsymbol{\Lambda}) > p_{d-2}(1 - \frac{1}{2}\boldsymbol{\Lambda}^2)$ (however, there may also exist higher level roots whose multiplicity is *less* than the number of transversal states). This clearly implies that whereas transversal states are sufficient to characterize the elements of an affine Kac Moody algebra (see below for further explanations of this point), they are no longer sufficient for indefinite Kac Moody algebras.

The necessary adaptation of the DDF construction involves a discretization of the string vertex operator formalism. As is well known [14], the allowed momenta of the string excitations must be elements of the weight lattice of the corresponding (affine or indefinite) Kac Moody algebra. For the definition of DDF operators one must choose a special Lorentz frame, in terms of which one can distinguish transversal and longitudinal DDF operators. On the lattice it is no longer possible to find a frame such that the relevant DDF vectors (see below for details) are still on the lattice, and we therefore are forced to make use of a rational extension of the (self-dual) root lattice as an auxiliary device. This is a curious feature of our construction, not encountered in previous studies. Despite the fact that our DDF vectors are not on the lattice, we employ them in our analysis because we can still use the associated (transversal and longitudinal) DDF operators to construct a complete basis for any root space of the Lie algebra of physical states $\mathfrak{g}_{II_{9,1}}$, of which the corresponding root space of the Kac Moody algebra $\mathfrak{g}(A)$ is then a proper subspace.

As it turns out, longitudinal states are absent only for levels zero and one; this accounts for the comparative simplicity of the corresponding multiplicity formulas.

A central role in the DDF construction is played by the tachyon momentum $\mathbf{a}$ (i.e. $\mathbf{a}^2 = 2$) of the ground state $|\mathbf{a}\rangle$, on which the physical states are built by means of DDF operators, and a null vector $\mathbf{k}$, subject to the condition $\mathbf{k}\cdot\mathbf{a} = 1$. For continuous momenta $\mathbf{a}$, we can always find suitable $\mathbf{k} = \mathbf{k}(\mathbf{a})$; moreover, we can rotate these vectors into a convenient frame by means of a Lorentz transformation [26]. On the lattice, however, the full Lorentz invariance is broken to a discrete subgroup (containing the Weyl group generated by the fundamental Weyl reflections), and for generic roots Λ, the associated DDF vectors $\mathbf{a}$ and $\mathbf{k}$ will *not* be elements of the root lattice in general. We can, however, still rotate the vectors $\mathbf{a} - n\mathbf{k}$ into the *fundamental Weyl chamber* for n sufficiently large. The lightlike momentum $\mathbf{k}$ is then proportional to the null-root δ characterizing the affine subalgebra; the latter is always in the fundamental Weyl chamber. Now we invoke the (obvious) fact that any root Λ in the fundamental Weyl chamber can be represented in the form

$$\Lambda = \ell\mathbf{r}_{-1} + M\delta + \mathbf{b}, \tag{13}$$

where ℓ is the level of Λ and $\mathbf{b}$ an element of the E_8-root lattice $Q(E_8)$ ($\mathbf{b}$ need not be positive by itself as only $M\delta + \mathbf{b}$ must be positive). We then define the **DDF decomposition** of Λ by

$$\Lambda = \mathbf{a} - n\mathbf{k}(\mathbf{a}), \tag{14}$$

where

$$\mathbf{k}(\mathbf{a}) := -\tfrac{1}{\ell}\delta \tag{15}$$

and

$$n = 1 - \tfrac{1}{2}\Lambda^2 = 1 + (M - \ell)\ell - \tfrac{1}{2}\mathbf{b}^2. \tag{16}$$

By construction, $\mathbf{a}$ obeys $\mathbf{a}^2 = 2$ and is therefore associated with a tachyon state, and n is the number of steps required to reach the root Λ by starting from $\mathbf{a}$ and decreasing the momentum by $\mathbf{k}$ at each step (n is non-negative because $\Lambda^2 \leq 2$; note also that $\mathbf{k}$ is always a *negative* root, so Λ is positive for all n). Obviously, for $\ell > 1$, neither $\mathbf{k}$ nor $\mathbf{a}$ belong to the lattice in general. As a consequence, the intermediate DDF states associated with momenta $\mathbf{a} - m\mathbf{k}$ not on the lattice will not correspond to elements of the algebra, but they are nonetheless indispensable for the construction of a complete basis for any given root space. On the other hand, states associated with the root Λ do belong to the algebra of physical states, and the DDF decomposition enables us to write down all possible polarization states associated with the root Λ in terms of transversal and longitudinal DDF states; the totality of these states constitutes the complete set of elements in the root space $\mathfrak{g}^{(\Lambda)}_{II_{9,1}}$, whose dimension equals $p_{d-1}(1 - \tfrac{1}{2}\Lambda^2) - p_{d-1}(-\tfrac{1}{2}\Lambda^2)$. Explicitly, given a tachyon momentum $\mathbf{a}$, the physical states are

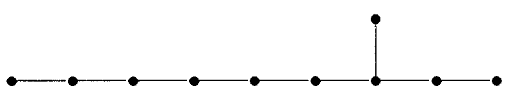
$$A^{i_1}_{-m_1}(\mathbf{a}) \ldots A^{i_M}_{-m_M}(\mathbf{a})\mathfrak{L}_{-n_1}(\mathbf{a}) \ldots \mathfrak{L}_{-n_N}(\mathbf{a})|\mathbf{a}\rangle, \tag{17}$$

explicitly indicating the dependence of the DDF operators and their polarizations on the tachyon momentum $\mathbf{a}$ and the associated lightlike vector $\mathbf{k}(\mathbf{a}) = -\tfrac{1}{\ell}\delta$, and assuming $n_i \geq 2$ to exclude null states. The transversal DDF operators associated with a tachyonic momentum $\mathbf{a}$ and its light-like momentum $\mathbf{k}$ are defined by the well known formula

$$A^i_{-m} = \mathrm{Res}_z \left[\mathcal{V}(\boldsymbol{\xi}_i(-1)| - m\mathbf{k}\rangle, z)\right], \tag{18}$$

i.e. is given by the contour integral of the vertex operator corresponding to the photon state $\boldsymbol{\xi}_i{\cdot}\boldsymbol{\alpha}_{-1}|\frac{m}{l}\boldsymbol{\delta}\rangle$. The transversal DDF operators always form a $(d-2)$-fold Heisenberg algebra. Note, however, that we have to deal with a whole plethora of transversal Heisenberg algebras, namely one for each admissible pair $(\mathbf{a}, \mathbf{k})$; this is in contrast to the single set of primitive oscillators, $\{\alpha^\mu_m | 1 \le \mu \le d, m \in \mathbb{Z}\}$, which makes up the full Fock space when acting on the groundstates. The longitudinal DDF operators are given by more complicated (but also standard) expressions; they involve a logarithmic contribution (cf. [4]). Again, for each admissible pair $(\mathbf{a}, \mathbf{k})$, we end up with a different set of operators. An important technical point is that the longitudinal vertex operators cannot be associated with definite states, as their action cannot be defined on all of Fock space, including the vacuum state $|\mathbf{0}\rangle$. Put differently, they do not correspond to summable operators in the sense of [10]; in this respect, vertex algebras encompassing longitudinal states transcend the definition given in [1], [10].

We quickly summarize some pertinent results about E_{10}. It is defined from its Cartan matrix in terms of multiple commutators of the Chevalley generators as described above. The root lattice $Q(E_{10})$ of E_{10} is the unique even self-dual $II_{9,1}$ in ten dimensions, which can be defined as the set of points $\mathbf{x} = (x_1, \ldots, x_9|x_0)$ in 10-dimensional Minkowski space for which the x_μ's are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$ and which have integer inner product with the vector $\mathbf{l} = (\frac{1}{2}, \ldots, \frac{1}{2} \mid \frac{1}{2})$, all norms and inner products being evaluated in the Minkowskian metric $\mathbf{x}^2 = x_1^2 + \ldots + x_9^2 - x_0^2$ (cf. [27]). According to [5], a set of positive norm simple roots for $II_{9,1}$ is given by the ten vectors $\mathbf{r}_{-1}, \mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_8$ in $II_{9,1}$ for which $\mathbf{r}_i^2 = 2$ and $\mathbf{r}_i \cdot \boldsymbol{\rho} = -1$ where the Weyl vector is $\boldsymbol{\rho} = (0, 1, 2, \ldots, 8|38)$ with $\boldsymbol{\rho}^2 = -1240$. The corresponding Coxeter-Dynkin diagram looks as follows



(19)

and is associated with the Cartan matrix:

$$A \equiv (a_{ij}) = (\mathbf{r}_i \cdot \mathbf{r}_j) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

Let us first describe the E_9 subalgebra in terms of physical states. The Cartan subalgebra of E_{10} (and also of E_9) is spanned by the states

$$\boldsymbol{\delta}(-1)|\mathbf{0}\rangle \quad =: \quad K, \tag{20}$$

$$(\mathbf{r}_{-1} + \boldsymbol{\delta})(-1)|\mathbf{0}\rangle \quad =: \quad d, \tag{21}$$

$$\boldsymbol{\xi}_i(-1)|\mathbf{0}\rangle \qquad \text{for } i = 1, \ldots, 8\,, \tag{22}$$

where K represents the central element, d is the derivation of E_9, and $\{\boldsymbol{\xi}_i(-1)|\mathbf{0}\rangle \mid i = 1, \ldots, 8\}$ span the Cartan subalgebra of E_8. This is the standard "light-cone" basis of

$\mathfrak{h}(E_9)$ in the sense that K and d are lightlike. The allowed (positive and negative) roots are all $\mathbf{r} \in II_{9,1}$ obeying $\mathbf{r}^2 = 2$ and $\mathbf{r}\cdot\boldsymbol{\delta} = 0$ (hence having no $\mathbf{r}_{-1}$ component), and $m\boldsymbol{\delta}$ for $m \in \mathbb{Z}^{\times}$. These correspond to the tachyonic states $|\mathbf{r}\rangle$ and the photonic states $\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle$ (where $\boldsymbol{\xi}_i\cdot\boldsymbol{\delta} = \boldsymbol{\xi}_i\cdot\mathbf{r}_{-1} = 0$) with multiplicities 1 and 8, respectively. The following commutation relations are already enough for a complete characterization of E_9

$$\begin{aligned}
\big[\boldsymbol{\eta}(-1)|\mathbf{0}\rangle, \boldsymbol{\zeta}(-1)|\mathbf{0}\rangle\big] &= 0, && (23)\\
\big[\boldsymbol{\eta}(-1)|\mathbf{0}\rangle, \boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle\big] &= m(\boldsymbol{\eta}\cdot\boldsymbol{\delta})\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle, && (24)\\
\big[\boldsymbol{\eta}(-1)|\mathbf{0}\rangle, |\mathbf{r}\rangle\big] &= (\boldsymbol{\eta}\cdot\mathbf{r})|\mathbf{r}\rangle, && (25)\\
\big[\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle, \boldsymbol{\xi}_j(-1)|n\boldsymbol{\delta}\rangle\big] &= m\delta_{m+n,0}(\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j)\boldsymbol{\delta}(-1)|\mathbf{0}\rangle. && (26)\\
\big[\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle, |\mathbf{r}\rangle\big] &= (\boldsymbol{\xi}_i\cdot\mathbf{r})|\mathbf{r}+m\boldsymbol{\delta}\rangle, && (27)\\
\big[|\mathbf{r}\rangle, |\mathbf{s}\rangle\big] &= \begin{cases} 0 & \text{if } \mathbf{r}\cdot\mathbf{s} \geq 0, \\ \epsilon(\mathbf{r},\mathbf{s})|\mathbf{r}+\mathbf{s}\rangle & \text{if } \mathbf{r}\cdot\mathbf{s} = -1, \\ -\mathbf{r}(-1)|m\boldsymbol{\delta}\rangle & \text{if } \mathbf{r}+\mathbf{s} = m\boldsymbol{\delta}, \end{cases} && (28)
\end{aligned}$$

for $\boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathfrak{h}(E_9)$ and E_9 roots $\mathbf{r}, \mathbf{s}$. In the last formula ϵ denotes the cocycle factor. The bra and ket notation used here may appear a bit unusual (and is not really necessary at this point), but will prove invaluable as one goes to higher level elements of the hyperbolic algebra. The multiplicities of the corresponding Lie algebra elements can be read off directly, and are given by 1 and 8, respectively, for the tachyonic and lightlike roots in accord with the formula (6) above.

The level-one elements exhibit already a slightly more involved structure. Inspection of the inverse Cartan matrix shows that the only such roots in the fundamental Weyl chamber $\mathcal{C}$ are of the form (for $k_{-1} \in \mathbb{N}$)

$$\boldsymbol{\Lambda} = \mathbf{r}_{-1} + (2 + k_{-1})\boldsymbol{\delta}, \tag{29}$$

corresponding to the DDF decomposition (14) with $\mathbf{a} = \mathbf{r}_{-1}$, $\mathbf{k} = -\boldsymbol{\delta}$ and $n = 2 + k_{-1}$. Since all these vectors are elements of the lattice, we can straightforwardly apply the DDF construction to obtain the physical states

$$A^{i_1}_{-m_1} \cdots A^{i_N}_{-m_N}|\mathbf{r}_{-1}\rangle, \tag{30}$$

where $m_1 + \ldots + m_N = 2 + k_{-1}$ and with the polarization vectors chosen such that $\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j = \delta_{ij}$ and $\boldsymbol{\xi}_i\cdot\boldsymbol{\delta} = \boldsymbol{\xi}_i\cdot\mathbf{r}_{-1} = 0$ for $i, j = 1, \ldots, 8$. In terms of multiple commutators, these states correspond to

$$\big[\boldsymbol{\xi}_{i_1}(-1)|m_1\boldsymbol{\delta}\rangle, \big[\ldots, \big[\boldsymbol{\xi}_{i_N}(-1)|m_N\boldsymbol{\delta}\rangle, |\mathbf{r}_{-1}\rangle\big]\ldots\big]\big] \qquad \in E_{10}^{(\boldsymbol{\Lambda})}. \tag{31}$$

All relevant level-one states can now be obtained by acting with the E_9 Weyl group on these states and polarizations. Denoting the rotated DDF operators by $A^{\mathfrak{w}(i)}_{-m} \equiv A^{\mathfrak{w}(\boldsymbol{\xi}_i)}_{-m}$, we obtain the new states

$$A^{\mathfrak{w}(i_1)}_{-m_1} \cdots A^{\mathfrak{w}(i_N)}_{-m_N}|\mathfrak{w}(\mathbf{r}_{-1})\rangle \tag{32}$$

in this fashion. The so-called basic representation is spanned by all elements of the form (32); the highest weight vector of the representation is easily seen to be $|\mathbf{r}_{-1}\rangle$.

At level two, a general multiplicity formula was derived in [22]; it reads

$$\operatorname{mult}(\boldsymbol{\Lambda}) = \xi(3 - \tfrac{1}{2}\boldsymbol{\Lambda}^2), \tag{33}$$

where

$$\sum_{n\geq 0} \xi(n) q^n = \frac{1}{\phi(q)^8}\left[1 - \frac{\phi(q^2)}{\phi(q^4)}\right], \tag{34}$$

and $\phi(q)$ is the Euler function. The special example we have investigated in [12], is the level-two root $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_7$, dual to the simple root $\mathbf{r}_7$, which has $\boldsymbol{\Lambda}_7^2 = -4$ and is explicitly given by

$$\boldsymbol{\Lambda}_7 = \left[\begin{matrix} & & & & & & 7 & & \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 9 & 4 \end{matrix}\right] = (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,|\,2), \tag{35}$$

where the first notation with square brackets refers to the simple roots in the above Dynkin diagram. We can now invoke the results of [7] which tell us that level ℓ states can be obtained as $(\ell - 1)$-fold commutators of level-one states, for which we can use the representation (32). Our analysis reveals that the following states form a complete basis of the root space $E_{10}^{(\boldsymbol{\Lambda}_7)}$ (no summation convention!):

$$\begin{aligned}
&A^i_{-2}A^j_{-1}\,|\mathbf{a}\rangle && \text{for } i, j \text{ arbitrary},\\
&A^i_{-1}A^j_{-1}A^k_{-1}\,|\mathbf{a}\rangle && \text{for } i \neq j \neq k \neq i,\\
&\{A^i_{-3} - A^i_{-1}A^j_{-1}A^j_{-1}\}|\mathbf{a}\rangle && \text{for } i \neq j,\\
&\{5A^i_{-3} + A^i_{-1}A^i_{-1}A^i_{-1}\}|\mathbf{a}\rangle && \text{for } i \text{ arbitrary},\\
&\{A^i_{-3} - A^i_{-1}\mathcal{L}_{-2}\}|\mathbf{a}\rangle && \text{for } i \text{ arbitrary}.
\end{aligned} \tag{36}$$

Altogether, we get $64 + 2\cdot 56 + 2\cdot 8 = 192$ states in agreement with the formula (33) predicting $\xi(3) = 192$ [22]. Despite the fact that this number coincides with the number of transversal states, our result explicitly shows the appearance of longitudinal as well as the disappearance of some transversal states. The above states form irreducible representations of the little group; in particular, the longitudinal DDF operator is inert under the little Weyl group. To appreciate the simplicity of this result readers need only contemplate the problem of classifying the states in terms of 75-fold multiple commutators of the Chevalley generators for this example.

Having a complete description of the root space $E_{10}^{(\boldsymbol{\Lambda}_7)}$, we can now in principle explore root spaces associated with other level-two roots of the form $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_7 + n\boldsymbol{\delta}$ (i.e. the **root string** associated with $\boldsymbol{\Lambda}_7$) by commuting the states (36) with the E_9 elements $\boldsymbol{\xi}_i(-1)|m\boldsymbol{\delta}\rangle$. From (18) it is evident that all states obtained by acting with a product $A^{i_1}_{-2m_1}\ldots A^{i_M}_{-2m_M}$ on any of the states (36) belong to the root space of $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_7 + (m_1 + \ldots m_M)\boldsymbol{\delta}$ (note that each operator $A^i_{-2m}(\mathbf{a})$ shifts the momentum by $m\boldsymbol{\delta}$!). However, it is also clear that we cannot obtain all root space elements in this way. For this, it is necessary to calculate DDF commutators between appropriate elements of the form (32). An alternative, more elucidating way might be to consider the action of the Sugawara generators defined by

$$\mathcal{L}^{\text{Sug.}}_m := \frac{1}{2(\ell + h^\vee)}\left\{\sum_{n\in\mathbb{Z}}\sum_{i=1}^{8} :A^i_n A^i_{m-n}: + \sum_{\mathbf{s}\in\Delta^{\text{real}}(E_9)} {}^\times_\times \operatorname{ad}_{|\mathbf{s}\rangle} \operatorname{ad}_{|-\mathbf{s}-m\boldsymbol{\delta}\rangle} {}^\times_\times\right\} \tag{37}$$

on the states (36); here, $h^\vee = 30$ is the dual Coxeter number of E_8, the level is $\ell = 2$, and the normal-ordering of the operators $\mathrm{ad}_{|\mathbf{r}\rangle}$ is chosen as

$$ {}^\times_\times \mathrm{ad}_{|\mathbf{s}+m\boldsymbol{\delta}\rangle}\, \mathrm{ad}_{|\mathbf{t}+n\boldsymbol{\delta}\rangle} {}^\times_\times := \begin{cases} \mathrm{ad}_{|\mathbf{s}+m\boldsymbol{\delta}\rangle}\, \mathrm{ad}_{|\mathbf{t}+n\boldsymbol{\delta}\rangle} & \text{if } m \geq n, \\ \mathrm{ad}_{|\mathbf{t}+n\boldsymbol{\delta}\rangle}\, \mathrm{ad}_{|\mathbf{s}+m\boldsymbol{\delta}\rangle} & \text{if } m < n, \end{cases} \tag{38} $$

for E_8 roots $\mathbf{s}, \mathbf{t}$ and $m, n \in \mathbb{Z}$. We get

$$ \mathcal{L}_m^{\mathrm{Sug.}} |\mathbf{a}\rangle = 0 \tag{39} $$

for $m \geq 1$. Furthermore, when evaluating $\mathcal{L}_0^{\mathrm{Sug.}}$ on the ground state $|\mathbf{a}\rangle$, we find $A_0^8|\mathbf{a}\rangle = -2|\mathbf{a}\rangle$, and obtain

$$ \mathcal{L}_0^{\mathrm{Sug.}} |\mathbf{a}\rangle = \tfrac{1}{16} |\mathbf{a}\rangle, \tag{40} $$

showing that the state $|\mathbf{a}\rangle$ is a highest weight vector of weight $h = \frac{1}{16}$ for the level-two Sugawara generators. In view of the results of [22], we therefore expect these states to belong to the irreducible Virasoro module with $c = \frac{1}{2}$ and $h = \frac{1}{16}$. The problem that remains is to relate the Sugawara generators to the longitudinal DDF operators. If this can be done, a completely explicit description of *all* level-two root spaces is within reach.

Let us finally return to the issue of missing states in more detail. Comparing (6) with (12), it becomes obvious that tachyonic and photonic physical states are necessarily transversal, so that

$$ \mathfrak{g}_{II_{9,1}}^{(\boldsymbol{\Lambda})} \equiv E_{10}^{(\boldsymbol{\Lambda})} \quad \text{for } \boldsymbol{\Lambda}^2 \geq 0 \tag{41} $$

(of course, for $\boldsymbol{\Lambda}^2 > 2$, both spaces are empty). This means that there are no missing states for $\boldsymbol{\Lambda}^2 \geq 0$. But already for the massive spin 2 states, we encounter one longitudinal physical state that surely does not belong to the Kac Moody algebra E_{10}. It is clear that there is only one weight in $\mathcal{C}$ of norm -2, namely the fundamental weight $\boldsymbol{\Lambda}_0 = \mathbf{r}_{-1} + 2\boldsymbol{\delta}$. Since the latter is a level-one element, which we know to occur in E_{10} just with transversal polarizations, we infer that the longitudinal state $\mathfrak{L}_{-2}|\mathbf{r}_{-1}\rangle$ is not contained in the root space $E_{10}^{(\boldsymbol{\Lambda}_0)}$ and thus represents a missing state, so

$$ \mathfrak{g}_{II_{9,1}}^{(\boldsymbol{\Lambda}_0)} \equiv E_{10}^{(\boldsymbol{\Lambda}_0)} + \mathbb{R} \cdot \mathfrak{L}_{-2}|\mathbf{r}_{-1}\rangle . \tag{42} $$

Acting with the full Weyl group on the missing state, we obtain the associated orbit of missing states in E_{10}. Our detailed analysis of the root space for $\boldsymbol{\Lambda}_7$ now enables us to discuss the case of norm -4, for $\boldsymbol{\Lambda}_7$ is the only weight in the fundamental Weyl chamber with this property. From the multiplicity formula we learn that there have to be $201 - 192 = 9$ missing states, and in view of our DDF basis (36) we write

$$ \mathfrak{g}_{II_{9,1}}^{(\boldsymbol{\Lambda}_7)} \equiv E_{10}^{(\boldsymbol{\Lambda}_7)} + \mathrm{span}_{\mathbb{R}}\{\mathfrak{L}_{-3}|\mathbf{a}\rangle;\ A_{-3}^i|\mathbf{a}\rangle, i = 1, \ldots, 8\}, \tag{43} $$

which can be also acted on with $\mathfrak{W}(E_{10})$ to find its analogue in other chambers.

The above formulas naturally suggest two ways of how to proceed. If we are primarily interested in E_{10}, we can try to systematize the way of splitting of $\mathfrak{g}_{II_{9,1}}$ into E_{10} states and missing states. In other words, we are seeking a mechanism which satisfactorily answers the following question: How do the missing states decouple from the E_{10} states? That this idea is not far-fetched shows the example of the 26-dimensional bosonic string. There we separate the longitudinal physical states from the transversal ones by introducing a positive semidefintite contravariant bilinear form which renders the former states to be null physical states. If one prefers the modern cohomological treatment then the decoupling is furnished by the nilpotent BRST operator and its cohomology. Thus we may rephrase the above question as: *Is there a cohomology describing how E_{10} is embedded in the Lie algebra of physical states, $\mathfrak{g}_{II_{9,1}}$?*

The other point of view, as advocated by Borcherds, involves a generalization of the framework of Kac Moody algebras. We know how a large part of $\mathfrak{g}_{II_{9,1}}$, namely the E_{10} part, can be formulated in terms of generators and relations. The idea then is to extend this approach to the whole Lie algebra. We would have to find an additional set of Chevalley generators which, when adjoined to the generators for E_{10}, produce all physical states as multiple commutators. For example, we certainly have to add $\mathfrak{L}_{-2}|\mathbf{r}_{-1}\rangle$ as such a new generator. This amounts to saying that Λ_0 constitutes an imaginary simple root with multiplicity one. Kac Moody algebras allowing for imaginary ($\equiv$ nonpositive norm) simple roots were invented by Borcherds [2]. So far, the introduction of this new generator seems to be very natural and appealing, but the second step of the procedure is subtle and becomes cumbersome when repeatedly done. In order to decide which missing states for the case of Λ_7 have to be chosen as new generators, we need to take into account the previous additional generator $\mathfrak{L}_{-2}|\mathbf{r}_{-1}\rangle$. Thus we ought to calculate the commutator $[|\mathbf{s}\rangle, \mathfrak{L}_{-2}|\mathbf{r}\rangle]$ (where $\mathbf{s}+\mathbf{r}+2\delta = \Lambda_7$) and express it in terms of the DDF basis for $\mathfrak{g}_{II_{9,1}}^{(\Lambda_7)}$ to see which missing states now do appear. We have not completed this calculation yet, for we are mainly interested in E_{10} itself and hence focus on the first approach. Alternatively, it is also possible to determine recursively the imaginary simple roots by anticipating $\mathfrak{g}_{II_{9,1}}$ as a Borcherds algebra and then plug its well-known root multiplicities, $p_9(1-\frac{1}{2}\mathbf{r}^2) - p_9(-\frac{1}{2}\mathbf{r}^2)$, into the Weyl–Kac–Borcherds denominator formula [2].

So, what have we learnt from our analysis of the root space $E_{10}^{(\Lambda_7)}$ and how may it be relevant for other hyperbolic Kac Moody algebras? Our approach suggests that root spaces of E_{10} and other algebras of that type carry an additional structure related to polarization; this differs from the conventional point of view that a root space is essentially, up to its dimension, a black box. The DDF framework, as developed here, provides adequate tools for the analysis of the complicated structure of hyperbolic algebras.

In particular, we now have a deeper understanding why Frenkel's conjecture [8] is wrong. Inspired by the example of the 26-dimensional bosonic string and the results about the canonical hyperbolic extension of $\mathfrak{su}(2)$ [7], he conjectured that for every hyperbolic algebra $\mathfrak{g}$ of rank d one has, for any root $\mathbf{r}$, $\dim \mathfrak{g}^{(\mathbf{r})} \le p_{d-2}(1-\frac{1}{2}\mathbf{r}^2)$ as an upper bound. This conjecture was disproved in [22] by establishing the level-two multiplicity formula for E_{10} as a counterexample. We argue that the 26-dimensional bosonic string represents a rather untypical example, because there the longitudinal states span the radical of the contravariant bilinear form which is divided out. Hence only transversal states survive and we end up indeed with the exact multiplicity formula $p_{24}(1-\frac{1}{2}\mathbf{r}^2)$. In the generic case, on the other hand, the longitudinal states do appear as Lie algebra elements. In terms of the DDF realization the following picture emerges. At level-zero and level-one we naturally obtain all transversal states giving the affine subalgebra and its basic representation, respectively. By commuting transversal level-one states, which is necessary for generating higher level elements, we cannot escape from producing longitudinal states, too. Hence there is no reason to expect a connection between higher level root multiplicities and the formula $p_{d-2}(1-\frac{1}{2}\mathbf{r}^2)$, which just counts the number of transversal states. Of course, we start off from the transversal level-one states, but the more commutators we take between them the more subtle the entanglement of longitudinal and transversal states becomes.

For example, look at the canonical hyperbolic extension of $\mathfrak{su}(2)$ whose level-two root multiplicities coincide with the number of transversal states, $p_1(1 - \frac{1}{2}\mathbf{r}^2)$, up to $\mathbf{r}^2 \leq -36$ (see [21, Table H_3]) and then drop below this bound. We conjecture that, when we perform the DDF construction for this example, we shall see at level-two from the very beginning longitudinal states to appear and transversal states to be missed, even though the multiplicity superficially suggests the existence of transversal states alone. For higher levels we predict an increasing mixing of longitudinal and transversal states which manifests itself in an increasing deviation of the multiplicities from the number of transversal states. Thus the DDF analysis of a single level-two root space of E_{10} allows us to make some reasonable predictions for some features occurring in other hyperbolic algebras of that type.

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