

On the Monodromies of $N=2$ Supersymmetric Yang-Mills Theory

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Abstract

We review the generalization of the work of Seiberg and Witten on $N=2$ supersymmetric $SU(2)$ Yang-Mills theory to $SU(n)$ gauge groups. The quantum moduli spaces of the effective low energy theory parametrize a special family of hyperelliptic genus $n-1$ Riemann surfaces. We discuss the massless spectrum and the monodromies.

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1. Introduction

In two recent papers [1,2], Seiberg and Witten have investigated $N=2$ supersymmetric $SU(2)$ gauge theories and solved for their exact nonperturbative low energy effective action. Some of their considerations have recently been extended to $SU(n)$ in [3,4]. We like to review here our work and present a global description of the monodromies of $SU(3)$, previewing some results to appear in a more complete account [5].

For arbitrary gauge group G , $N=2$ supersymmetric gauge theories without matter hypermultiplets are characterized by having flat directions for the Higgs vacuum expectation values, along which the gauge group is generically broken to the Cartan subalgebra. Thus, the effective theories contain $r = \text{rank}(G)$ abelian $N=2$ vector supermultiplets, which can be decomposed into r $N=1$ chiral multiplets A^i plus r $N=1$ $U(1)$ vector multiplets W_α^i . The $N=2$ supersymmetry implies that the effective theory up to two derivatives depends only on a single holomorphic prepotential $\mathcal{F}(A)$. More precisely, the effective lagrangian in $N=1$ superspace is

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left[\int d^4\theta \left(\sum \frac{\partial \mathcal{F}(A)}{\partial A^i} \bar{A}^i \right) + \int d^2\theta \frac{1}{2} \left(\sum \frac{\partial^2 \mathcal{F}(A)}{\partial A^i \partial A^j} W_\alpha^i W_\alpha^j \right) \right]. \quad (1)$$

The holomorphic function \mathcal{F} determines the quantum moduli space and, in particular, its metric. This space has singularities at subspaces of complex codimension one, where additional fields become massless. At these regions the effective action description breaks down. A crucial insight is that the electric and magnetic quantum numbers of the fields that become massless at a given singularity are determined by the left eigenvectors (with eigenvalues equal to +1) of the monodromy matrix associated with that singularity.

For $G = SU(2)$ considered in [1,2], besides the point at $u = \infty$ there are singularities at $u = \pm\Lambda^2$, where Λ is the dynamically generated scale of the theory, and $u = \frac{1}{2}\langle a^2 \rangle$, where $a \equiv A|_{\theta=0}$. On the other hand, $u = 0$ is not singular in the exact quantum theory, which means that, in contrast to the classical theory, no massless non-abelian gauge bosons arise here (nor at any other point in moduli space). The singularities at $u = \pm\Lambda^2$ correspond to a massless monopole and a massless dyon, respectively. The parameter region near $u = \infty$ describes the semiclassical regime and is governed by the perturbative beta function with only one-loop contributions [6,7]. It gives rise to a non-trivial monodromy as well (arising from the logarithm in the effective coupling constant), but there are no massless states associated with it.

Although in refs.[1,2] it appears as an assumption, we believe that it can be proven that only two extra singularities besides infinity can exist for $G = SU(2)$. On physical grounds this would follow from the fact that to the best of our understanding of gauge theories only one mass scale Λ is generated. Having more singularities would probably imply that there are more independent scales in the problem. What is easy to prove is that if indeed there are only two singularities at finite points, they correspond up to conjugation to monopole and dyon states with (magnetic;electric) charges $(1; -4n)$ and $(1; -2 - 4n)$ ($n \in \mathbb{Z}$). The proof follows by looking for solutions to the set of diophantine equations $M_{(g_1; q_1)} M_{(g_2; q_2)} = M_\infty$, where

$$M_{(g; q)} = \begin{pmatrix} 1 - gq & -q^2 \\ g^2 & 1 + gq \end{pmatrix} \quad (2)$$

is the monodromy matrix for a massless dyon with electric charge q and magnetic charge g and $M_\infty = \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix}$. We also have considered the situation of four singular points and could exclude solutions to the corresponding diophantine equations, $\prod_{i=1}^4 M_{(g_i; q_i)} = M_\infty$ for $g_i, q_i \leq 10$.

The singularity structure and knowledge of the monodromies allow to completely determine the holomorphic prepotential \mathcal{F} . The monodromy group is $\Gamma_0(4) \subset SL(2, \mathbb{Z})$ consisting of all unimodular integral matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $b = 0 \pmod{4}$. The matrices act on the vector $(a_D; a)^t$, where a_D is the magnetic dual of a , with $a_D \equiv \frac{\partial F(a)}{\partial a}$. The quantum moduli space is the u -plane punctured at $\pm\Lambda^2$ and ∞ and can be thought of as $\mathbb{H}/\Gamma_0(4)$ (\mathbb{H} is the upper half-plane).

The basic idea [1] in solving for the effective theory is to consider a family of holomorphic curves whose monodromy group is $\Gamma_0(4)$ and which can be represented as follows:

$$y^2 = (x^2 - u)^2 - \Lambda^4. \quad (3)$$

By transforming to Weierstrass form, this curve can be shown to be equivalent to the curve given in [2], which, in contrast to the curve given in [1], is the form appropriate for generalization to $SU(n), n > 2$. The curves (3) represent a double cover of the x -plane with the four branch points at $\pm\sqrt{u \pm \Lambda^2}$, and describe a genus one Riemann surface. That is, the quantum moduli space of the $SU(2)$ super Yang-Mills theory coincides with the moduli space of a particular torus; this torus becomes singular when two branch points in (3) coincide. The derivatives of the electric and magnetic coordinates $(a_D; a)^t$ with respect to u are given by the periods of the holomorphic one-form $\frac{dx}{y}$ with respect to a symplectic homology basis. Their ratio, the modular

parameter τ , is positive definite and well-defined in the u -plane, and equals the metric of the moduli space. Integrating the periods yields $a(u), a_D(u)$ and integrating a_D finally determines the prepotential $\mathcal{F}(a)$.

We want to indicate next how these ideas of Seiberg and Witten generalize to gauge groups $G = SU(n)$. To be specific, we will mainly consider $G = SU(3)$, but the generalization to higher n is straightforward.

2. Semi-classical Regime

We will denote the gauge invariant order parameters (Casimirs) of $SU(n)$ by

$$u_k = \frac{1}{k} \text{Tr} \langle \phi^k \rangle, \quad k = 2, \dots, n \quad (4)$$

where we can always take the scalar superfield component to be $\phi = \sum_{k=1}^{n-1} a_k H_k$ with $H_k = E_{k,k} - E_{k+1,k+1}$, $(E_{k,l})_{i,j} = \delta_{ik} \delta_{jl}$. The anomaly free global \mathbb{Z}_{2n} subgroup of $U(1)_{\mathcal{R}}$ acts as $u_k \rightarrow e^{i\pi k/n} u_k$. For $SU(3)$ this means that classically $u \equiv u_2 = a_1^2 + a_2^2 - a_1 a_2$, $v \equiv u_3 = a_1 a_2 (a_1 - a_2)$ with \mathbb{Z}_6 action $u \rightarrow e^{2\pi i/3} u$, $v \rightarrow -v$. For generic eigenvalues of ϕ , the $SU(3)$ gauge symmetry is broken to $U(1) \times U(1)$, whereas if any two eigenvalues are equal, the unbroken symmetry is $SU(2) \times U(1)$. These classical symmetry properties are encoded in the following, gauge and globally \mathbb{Z}_6 invariant discriminant:

$$\Delta_0 = 4u^3 - 27v^2 = (a_1 + a_2)^2 (2a_1 - a_2)^2 (a_1 - 2a_2)^2. \quad (5)$$

The lines $\Delta_0 = 0$ in (u, v) space correspond to unbroken $SU(2) \times U(1)$ and have a cusp singularity at the origin, where the $SU(3)$ symmetry is restored. As we will see, in the full quantum theory this cusp is resolved, $\Delta_0 \rightarrow \Delta_\Lambda = 4u^3 - 27v^2 + O(\Lambda^6)$, which, in particular, prohibits a phase with massless non-abelian gluons. Other singularities will however appear, signalling the appearance of massless monopoles and dyons in the spectrum.

The prepotential \mathcal{F} in the semi-classical, perturbative regime can easily be computed with the result

$$\mathcal{F}_{\text{class}}^{\text{semi-}} = \frac{i}{4\pi} \sum_{i < j}^3 (e_i - e_j)^2 \log[(e_i - e_j)^2 / \Lambda^2]. \quad (6)$$

Here, e_i denote the roots of the equation

$$W_{A_2}(x, u, v) \equiv x^3 - ux - v = 0, \quad (7)$$

whose bifurcation set is given by the discriminant Δ_0 in (5), i.e.

$$\begin{aligned} e_1 - e_2 &= a_1 + a_2 \\ e_1 - e_3 &= 2a_1 - a_2 \\ e_2 - e_3 &= a_1 - 2a_2. \end{aligned} \quad (8)$$

The Casimirs u, v are gauge invariant and, in particular, invariant under the Weyl group \mathcal{W} of $SU(3)$. This group is generated by any two of the transformations

$$\begin{aligned} r_1 : (a_1, a_2) &\rightarrow (a_2 - a_1, a_2) \\ r_2 : (a_1, a_2) &\rightarrow (a_1, a_1 - a_2) \\ r_3 : (a_1, a_2) &\rightarrow (-a_2, -a_1). \end{aligned} \quad (9)$$

Due to the multi-valuedness of the inverse map $(u, v) \rightarrow (a_1, a_2)$, closed paths in (u, v) space will, in general, not close in (a_1, a_2) space, but will close up to Weyl transformations. Such a monodromy will be non-trivial if a given path encircles a singularity in (u, v) space — in our case, the singularities will be at “infinity” and along the lines where the discriminant vanishes.

It is indeed well-known [8] that the monodromy group of the simple singularity of type A_2 (7) is given by the Weyl group of $SU(3)$, and acts as Galois group on the e_i (and analogously for $W_{A_{n-1}}$ related to $SU(n)$). This will be the starting point for our generalization.

What we are interested in is however not just the monodromy acting on (a_1, a_2) , but the monodromy acting on $(a_{D1}, a_{D2}; a_1, a_2)^t$, where

$$a_{Di} \equiv \mathcal{F}_i = \frac{\partial}{\partial a_i} \mathcal{F}(a_1, a_2). \quad (10)$$

Performing the Weyl reflection r_1 on $(a_1, a_2)^t$, we easily find

$$\begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} + N \begin{pmatrix} 6 & -3 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \quad (11)$$

The second term, i.e., the “quantum shift”, arises from the logarithms and is not determined by the finite, “classical” Weyl transformation acting on the a_i , but rather

depends on the particular path in (u, v) space. (We considered in (11) only paths for which all three logarithms in (6) contribute with the same sign; other paths do exist where the logarithms contribute differently and yield a quantum shift matrix different from that in (11) [5]). For example, for the closed loop given by $(u(a_i(t)), v(a_i(t)))$ for $t \in [0, 1]$, where $a_1(t) = e^{i\pi t} a_1 + \frac{1}{2}(1 - e^{i\pi t}) a_2$, $a_2(t) = a_2$ we find $N = 1$. Therefore, the matrix representation of r_1 acting on $(a_{D1}, a_{D2}; a_1, a_2)^t$ is:[†]

$$r_1 = \begin{pmatrix} -1 & 0 & 6 & -3 \\ 1 & 1 & -3 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv r_1^{\text{class}} T^{-3}, \quad (12)$$

where r_1^{class} is the ‘‘classical’’ Weyl reflection (given by the block diagonal part of r_1), and T the ‘‘quantum monodromy’’

$$T = \begin{pmatrix} \mathbb{1} & C \\ 0 & \mathbb{1} \end{pmatrix}, \quad \text{where } C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad (13)$$

is the Cartan matrix of $SU(3)$. The other Weyl reflections are given analogously by $r_i = r_i^{\text{class}} T^{-3}$. The r_i are related to each other by conjugation, and, in particular, rotate into each other via the Coxeter element, $r_{\text{cox}}^{\text{class}} = r_1^{\text{class}} r_2^{\text{class}}$.

3. The Curves for $SU(n)$

Let us now generalize the $SU(2)$ curve to a sequence of curves \mathcal{C}_n whose moduli spaces are supposed to coincide with the quantum moduli space of effective low energy $N = 2$ supersymmetric $SU(n)$ Yang-Mills theories.

We first list the requirements we impose on the curves \mathcal{C}_n : (i) We seek surfaces with $2(n-1)^2$ periods (corresponding to $(\partial_{u_j} a_{Di}; \partial_{u_j} a_i)$), whose period matrices $\Omega_{ij} = \frac{\partial a_{Di}}{\partial a_j}$ are positive definite. (ii) For $\Lambda \rightarrow 0$ the classical situation must be recovered. That is, the discriminant of \mathcal{C}_n should have, for $\Lambda = 0$, a factor of Δ_0 (cf. eq.(5)). This means that for $\Lambda = 0$ the curves should have the form $y^m = \mathcal{C}_n(x) \equiv W_{A_{n-1}} \times (\dots)$ for some m . (iii) The curves must behave properly under the cyclic global transformations acting on the Casimirs u_k ; in other words, there should be a natural dependence on Casimirs, for all groups. Finally, from [2] we know that Λ should appear in $\mathcal{C}(x)$ with

[†] In [3] we represented these matrices for rescaled a_i . However, the present normalization is more appropriate if one wants the other monodromies to be given by integral matrices.

a power that corresponds to the charge violation of the one-instanton process, which is $2n$ for $SU(n)$.

Taking these requirements together leads us to consider the following genus $g = n - 1$ hyperelliptic curves for $SU(n)$:

$$y^2 = \mathcal{C}_n(x) \equiv \left(W_{A_{n-1}}(x, u_i) \right)^2 - \Lambda^{2n}, \quad (14)$$

where

$$W_{A_{n-1}}(x, u_i) = x^n - \sum_{i=2}^n u_i x^{n-i}. \quad (15)$$

are the A -type simple singularities related to $SU(n)$. The square of W reflects having both electric and magnetic degrees of freedom. Since there is a general relationship [8] between Arnold's simple A-D-E singularities, perturbations by Casimirs, monodromy and Weyl groups, we conjecture that (14) describes surfaces for the other simply laced gauge groups G as well. One simply replaces $W_{A_{n-1}}(x, u_i)$ by the corresponding D- or E-type singularity, and Λ^{2n} by Λ^{2h} , where h is the corresponding Coxeter number.

For the following it will be useful to write

$$\begin{aligned} \mathcal{C}_n(x) &= \left(W_{A_{n-1}}(x, u_i) + \Lambda^n \right) \left(W_{A_{n-1}}(x, u_i) - \Lambda^n \right) \\ &= \prod_{i=1}^n (x - e_i^+) (x - e_i^-). \end{aligned} \quad (16)$$

Critical surfaces occur whenever two roots of $\mathcal{C}_n(x)$ coincide, that is, whenever the discriminant $\Delta_\Lambda = \prod_{i < j} (e_i^\pm - e_j^\pm)^2$ vanishes. We expect this to happen when monopoles or dyons, whose quantum numbers are determined by the corresponding monodromy matrices, become massless. For example, for $G = SU(3)$ the quantum discriminant is:

$$\Delta_\Lambda = \Lambda^{18} \Delta_\Lambda^+ \Delta_\Lambda^-, \quad \Delta_\Lambda^\pm = 4u^3 - 27(v \pm \Lambda^3)^2. \quad (17)$$

By construction, the hyperelliptic curves (14) are represented by branched covers over the x -plane. More precisely, we have n \mathbf{Z}_2 cuts, each linking a pair of roots e_i^+ and e_i^- , $i = 1, \dots, n$. As an example, we present the picture for $G = SU(3)$ in Fig.1. In the classical theory, where $\Lambda \rightarrow 0$, the branch cuts shrink to n doubly degenerate points: $e_i^- \rightarrow e_i^+ \equiv e_i$. These points, given for $SU(3)$ in eq. (8), correspond to the weights of the n -dimensional fundamental representation (the picture represents a deformed projection of the weights onto the unique Coxeter eigenspace with \mathbf{Z}_n

action). This means that the branched x -plane transforms naturally under the finite “classical” Weyl group that permutes the points. This finite Weyl group is just the usual monodromy group of the A_{n-1} singularity alluded to earlier. In the quantum theory, where the degenerate points are resolved into branch cuts, there are in addition possibilities for “quantum monodromy”, which involves braiding of the cuts.

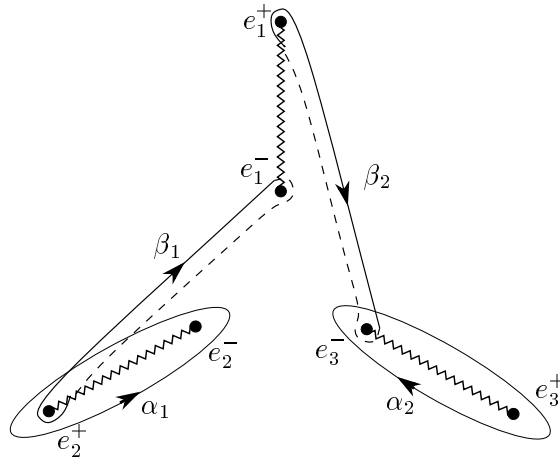


Fig.1: Branched x -plane with cuts linking pairs of roots of $C_3 = 0$. We depicted a choice of basis for the homology cycles that is adapted to the cyclic \mathbb{Z}_3 symmetry generated by the classical Coxeter element. Note that the locations of the cuts are \mathbb{Z}_3 -symmetric only for $u = 0$.

4. Monodromies for $G=SU(3)$

The semi-classical monodromy discussed so far was determined from properties relying on asymptotic freedom of the $SU(3)$ theory. We now turn to the monodromies M_λ around singularities related to massless dyons of charges $\lambda \equiv (\vec{g}, \vec{q})^t \equiv (g_1, g_2; q_1, q_2)^t$ (the charge vectors will be given in a basis generated by the two fundamental weights of $SU(3)$). Analogous to [1], the charge vectors are left eigenvectors of the corresponding monodromy matrices with eigenvalue equal to one, i.e. $\lambda^t M_\lambda = \lambda^t$. Obviously, the monodromies of any two dyons whose charge vectors are related via $\lambda_2 = N^t \lambda_1$, $N \in Sp(4, \mathbb{Z})$, are conjugated: $M_{\lambda_2} = N^{-1} M_{\lambda_1} N$. If we require the monodromies to depend only on the charges of the corresponding states, the matrices are determined to take the following form, which is the generalization of (2):

$$M_{(\vec{g}; \vec{q})} = \begin{pmatrix} \mathbb{1} - \vec{q} \otimes \vec{g} & -\vec{q} \otimes \vec{q} \\ \vec{g} \otimes \vec{g} & \mathbb{1} + \vec{g} \otimes \vec{q} \end{pmatrix} \quad (18)$$

This form of the matrix is obviously the same for all $SU(n)$. Here we have fixed some freedom with hindsight to our results below. Note that for purely magnetically charged monopoles, where $q_i = 0$, one has: $a_{Di} \rightarrow a_{Di}$, and this reflects the fact that the a_{Di} are good local coordinates in the vicinity of the singular loci where the monopoles become massless.

We now indicate how to determine the monodromies M_λ of the curves \mathcal{C}_n . Fixing a base point u_0 in the moduli space \mathcal{M}_n of \mathcal{C}_n , there is a homomorphism of the fundamental group $\pi_1(\mathcal{M}_n, p)$ into $Sp(2n-2; \mathbb{Z})$ whose image is the monodromy group. Note that the singular locus of \mathcal{C}_n , $\Delta_\Lambda = 0$, does itself have singularities. For $n = 3$ there are cusps at $u = 0, v = \pm\Lambda^3$ ($4\delta u^3 - 27\delta v^2 = 0$) and nodes at $u^3 = \frac{27}{4}\Lambda^6, v = 0$ ($\rho\delta u^2 - \delta v^2 = 0, \rho^3 = \Lambda^6/4$), as discussed in [4]. We have depicted in Fig.2 the moduli space for real v .

The monodromy matrices reflect the action of braiding and permuting the cuts on the vector $(a_{Di}; a_i)^t$. This action is expressed in terms of the action on the homology cycles via

$$a_{Di} = \oint_{\beta_i} \lambda, \quad a_i = \oint_{\alpha_i} \lambda, \quad (19)$$

where α_i, β_j is some symplectic basis of $H_1(\mathcal{C}_n; \mathbb{Z})$: $\langle \alpha_i, \beta_j \rangle = -\langle \beta_j, \alpha_i \rangle = \delta_{ij}$, $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0$, $i, j = 1, \dots, g$ and λ is some suitable chosen meromorphic differential. From the theory of Riemann surfaces it is clear that the monodromy group is contained in $Sp(2g, \mathbb{Z}) = Sp(2n-2, \mathbb{Z})$. For $G = SU(3)$, we have depicted our choice of homology basis in Fig.1.

The holomorphic differentials on a genus $g = n-1$ hyperelliptic curve are $\omega_{n-i} = x^i \frac{dx}{y}$, $i = 0, \dots, g-1$. The $g \times 2g$ period matrix is $(A, B) = A(\mathbf{1}, \Omega)$ with $A_{ij} = \int_{\alpha_j} \omega_i$ and $B_{ij} = \int_{\beta_j} \omega_i$, which are related to (a_{Di}, a_i) as follows:

$$A_{ij} = \frac{\partial a^j(u)}{\partial u_i}, \quad B_{ij} = \frac{\partial a_D^j(u)}{\partial u_i}. \quad (20)$$

Note that due to (20), the second Riemann bilinear relation, $\text{Im } \Omega > 0$, ensures the positivity of the metric

$$(ds)^2 = \text{Im} \frac{\partial \mathcal{F}}{\partial a_i \partial a_j} da_i d\bar{a}_j = \text{Im} \sum_{i=1}^g da_D^i d\bar{a}_i.$$

This generalizes the situation at genus one [1]. Note also that (20) represents a non-trivial integrability condition for the periods, $\partial_i A_{jk} = \partial_j A_{ik}$, $\partial_i B_{jk} = \partial_j B_{ik}$,

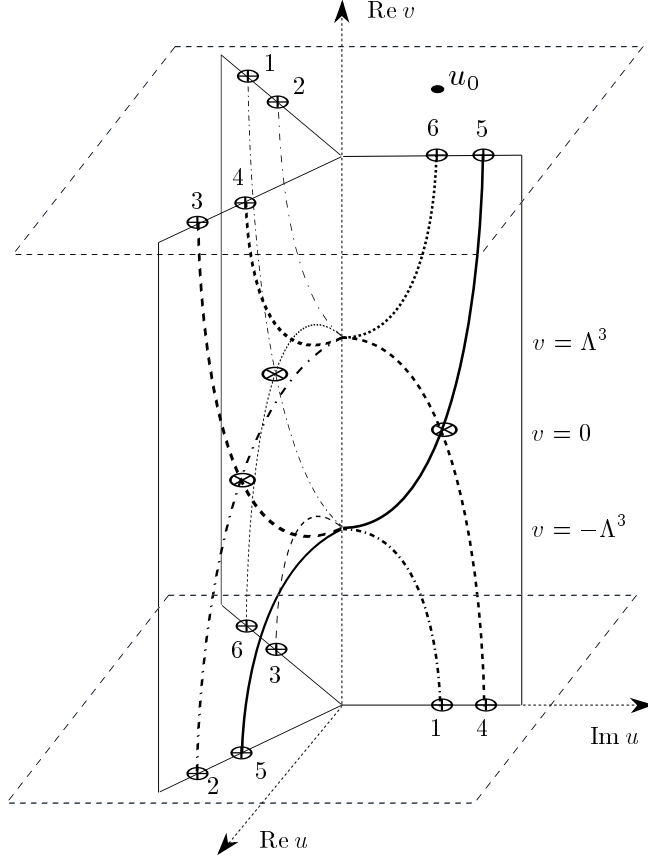


Fig.2: $SU(3)$ quantum moduli space for real v . The six lines are the singular loci where $\Delta_\Lambda = 0$ and where certain dyons become massless. At each of the three crossing points at $v = 0$, two mutually local dyons become simultaneously massless, and the theory is semi-classical in the corresponding dual variables. The various markings of the lines indicate how the association with particular monodromy matrices changes when moving through the cusps.

which holds for our parametrization of the curve (14). The periods, which can be obtained as solutions of the Picard-Fuchs equations [5], can thus be integrated to yield $a_{D_i}(u, v)$, $a_i(u, v)$ and by further integration to yield $\mathcal{F}(u, v)$. Alternatively one can choose the meromorphic differential λ in (19) s.t. $\omega_i \sim \partial_i \lambda$ up to exact forms, e.g. $\lambda \sim (3x^3 - ux) \frac{dx}{y}$ as in [4], and try to integrate (19).

Now, if ν denotes the vanishing cycle for the braiding of the branch points induced by a loop γ in moduli space, the action on the homology cycle $\delta \in H_1(\mathcal{C}_n; \mathbf{Z})$ can be

simply obtained via the Picard-Lefschetz formula [8]

$$S_\nu \delta = \delta + \langle \delta, \nu \rangle \nu$$

We find that when decomposing the given vanishing cycle as $\nu = \sum_{i=1,2} (q_i \alpha_i + g_i \beta_i)$, the action on the homology basis $(\beta_1, \beta_2, \alpha_1, \alpha_2)^t$ is given by a monodromy matrix M_λ precisely of the form (18), with labels just given by the expansion coefficients: $M_\lambda \equiv M_{(g_1, g_2; q_1, q_2)}$. That is, we can read off the electric and magnetic quantum numbers of a given massless dyon just by looking at the picture of the corresponding vanishing cycle !

More specifically, we studied the monodromies of $G = SU(3)$ by fixing a base point u_0 and by carefully tracing the effects of loops in moduli space on the motions of the branch points in the x -plane. With reference to results by Zariski and van Kampen (cf. [9] and references therein) it suffices to study loops in a generic complex line through the base point. Across cusps and nodes the monodromies are related through the “van Kampen relations”. In Fig.3 the six marked points are the intersections of the complex u -plane at $\text{Re}(v) = \text{const} > \Lambda^3$, $\text{Im}(v) = 0$ with the singular set $\Delta_\Lambda = 0$. The monodromies around the six loops γ_i can be characterized by the corresponding vanishing cycles, which are depicted in Fig.4. According to what we said above, the quantum numbers of the dyons that become massless at the encircled singular lines in moduli space can be directly obtained from Fig.4, by comparing the vanishing cycles with the basis cycles in Fig.1. The corresponding monodromy matrices (18) then turn out to be:

$$\begin{aligned} M_{\lambda_1} &= M_{(1,1;-1,0)}, & M_{\lambda_2} &= M_{(1,1;0,1)}, \\ M_{\lambda_3} &= M_{(1,0;-1,1)}, & M_{\lambda_4} &= M_{(1,0;1,0)}, \\ M_{\lambda_5} &= M_{(0,1;0,-1)}, & M_{\lambda_6} &= M_{(0,1;-1,1)}. \end{aligned} \tag{21}$$

We remark that the charge vectors of each pair of lines which cross in the nodes at $u = 0$ satisfy $\lambda_i^t \Sigma \lambda_j = 0$, where $\Sigma = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$ is the symplectic metric [4]. This is a necessary condition for two types of dyons to condense simultaneously [10]. (It follows from (18) that if two charge vectors are symplectically orthogonal, the corresponding monodromy matrices commute.) More precisely, the charge labels $[\lambda_i, \lambda_j]$ of the condensing pairs are $\{[\lambda_1, \lambda_6], [\lambda_2, \lambda_3], [\lambda_4, \lambda_5]\}$. Note that by appropriately changing the homology basis, one can always conjugate the charge vectors of any given one such pair of dyons to purely magnetic charges, ie., to $(1, 0; 0, 0)$ and $(0, 1; 0, 0)$. At the nodes both dual $U(1)$'s are weakly coupled and one can verify that the monodromies

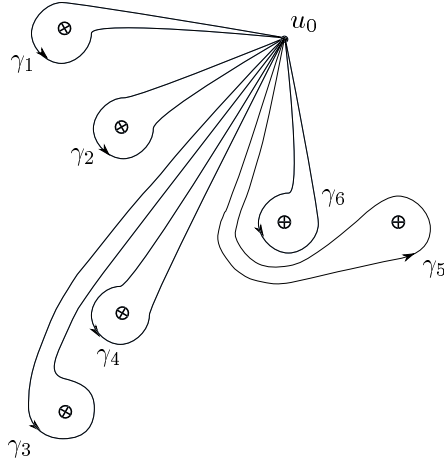


Fig.3: Loops γ_i in the u -plane at $\text{Re}(v) = \text{const} > \Lambda^3$, $\text{Im}(v) = 0$ (cf., Fig.2). u_0 denotes the base point. A loop encircling all the points yields essentially the monodromy matrix r_1 given in (12).

are consistent with the beta functions of the effective dual theory that contains two monopole hypermultiplets.

One type of monodromy at infinity is given by a loop encircling all the singular points in Fig.3, that is, by the product of the matrices in (21). It turns out to be precisely the monodromy (12) deduced from the semi-classical effective action (6), up to a change of basis:

$$M_\infty \equiv \prod_{i=1}^6 M_{\lambda_i} = S^{-1} (r_1)^{-1} S,$$

with $S = \begin{pmatrix} \mathbf{1} & s \\ 0 & \mathbf{1} \end{pmatrix}$, $s \equiv \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$. (In this basis, the other matrices r_2, r_3 can be obtained as well, by starting from a different base point.) We take this as a non-trivial physical consistency check.

One can also consider monodromies for paths lying in other complex lines in the moduli space, like in the line $u = \text{const}$. as discussed in [4]. This plane is however not in generic position but is special, in that it does not cut through all six, but only through four singular lines. (The corresponding monodromy matrices form a subset of the matrices in (21)). The total monodromy obtained by encircling the four points in the v -plane is different from the monodromy in the u -plane discussed above, however it is closely related to it: its classical (block-diagonal) part is just given by the Coxeter element $r_{\text{cox}}^{\text{class}} \equiv r_1^{\text{class}} r_2^{\text{class}}$ of the Weyl group of $SU(3)$. The Coxeter element is of

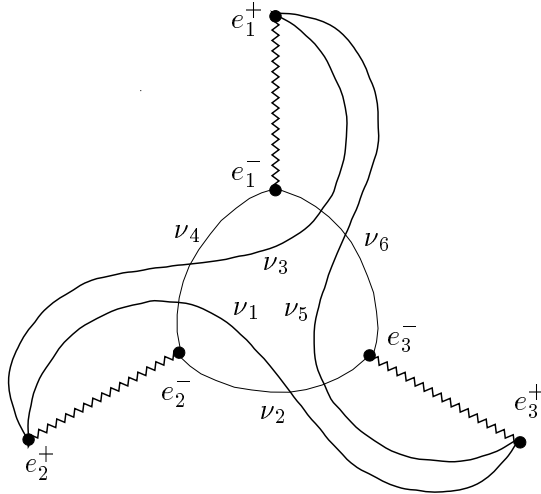


Fig.4: Vanishing cycles ν_i associated with the loops γ_i in Fig.3. The dyon charges can be directly inferred from this picture, by expanding the cycles in terms of the homology basis given in Fig.1 (e.g. $\nu_6 = \beta_2 - \alpha_1 + \alpha_2 \Rightarrow \lambda_6 = (0, 1; -1, 1)$). We have depicted here only the paths on the upper sheet, and not the return paths on the lower sheet.

order three, and its \mathbb{Z}_3 action corresponds to the cyclic rotation of the cuts in Fig.1. The complete monodromy in the v -plane, which includes the quantum shift, is the same as the semi-classical monodromy obtained in [4], up to change of basis. It can be obtained from the effective action (6) as well, by choosing an appropriate path.

This Coxeter monodromy is closely related to a well-known fact in singularity theory [8]. Here one considers perturbations $W = W_0(u_k = 0) + \Lambda$, where the parameter circles around the origin: $\Lambda = e^{2\pi it}$, $t \in [0, 1]$. This induces what is called, ironically, “classical monodromy”, and for the A-D-E simple singularities, this classical monodromy is precisely given by the Coxeter element of the corresponding A-D-E type Weyl group. For our curves (14), such loops in the Λ -plane indeed reproduce the above-mentioned Coxeter monodromy in the v -plane, if we set $u, v = 0$. On the other hand, if $|\Lambda| < |v|$, these loops do not induce the cyclic rotation of the cuts in

Fig.1, but only induce simultaneous braiding of the cuts. This corresponds to pure “quantum monodromy”, given by the shift matrix T in (13) [3].

Summarizing, the classical piece of the total monodromy in the u -plane is given by any one of the Weyl group generators, say r_1^{class} , whereas for the v -plane one obtains the Coxeter element, $r_1^{\text{class}}r_2^{\text{class}}$. We expect for $SU(n)$ that when looping around infinity in a parameter plane u_k related to a Casimir of degree k , the classical part of the monodromy is given by a Weyl group conjugacy class of the corresponding order. In particular, encircling the top Casimir parameter plane u_n will induce the Coxeter monodromy, which has order n .

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