A Periodic Analog of the Schwarzschild Solution

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ABSTRACT. We construct a new exact solution of Einstein's equations in vacuo in terms of Weyl canonical coordinates. This solution may be interpreted as a black hole in a space-time which is periodic in one direction and which behaves asymptotically like the Kasner solution with Kasner index equal to $4ML^{-1}$, where L is the period and M is the mass of the black hole. Outside the horizon, the solution is free of singularities and approaches the Schwarzschild solution as $L \to \infty$.

In this article, we present a new exact solution of Einstein's equations in vacuo (for a comprehensive review of exact solutions, see [1]). This solution constitutes a periodic analog of the Schwarzschild solution [2], and is likewise free of singularities. Asymptotically, it behaves like the Kasner solution. Before describing the new solution, we would like to remind the reader of the link between meromorphic functions on the Riemann sphere $\mathbf{C}P^1$ and on the torus $T = \mathbf{C}/\{L_1, L_2\}$ (L_1 and L_2 are the two periods on the lattice defining T). Given some meromorphic function $f_0(\xi)$, $\xi \in \mathbf{C}$, we can consider the formal expression

$$f(\xi) = \sum_{m,n=-\infty}^{\infty} \Big\{ f_0(\xi + mL_1 + nL_2) + a_{mn} \Big\} \tag{1}$$

Provided we can choose the constants a_{mn} in such a way that this series converges, the function $f(\xi)$ is meromorphic on the torus T with the same number and positions of poles as the function $f_0(\xi)$ on $\mathbf{C}P^1$. In this case, $f(\xi)$ is a doubly periodic analog of the original function f_0. The simplest example of a function $f_0(\xi)$ admitting an appropriate choice of convergence generating constants a_{mn} is $f_0(\xi) = \xi^{-2}$ with $a_{mn} = -(mL_1 + nL_2)^{-2}$ for $(m, n) \neq (0, 0)$ and $a_{00} = 0$, leading to the well-known Weierstrass $\wp$-function. (For $f_0(\xi) = \xi^{-1}$, on the other hand, no choice of a_{mn} will render the series (1) convergent since this would imply the existence of a meromorphic function on the torus having only one pole in contradiction to the Riemann-Roch theorem.)

The same procedure works, of course, for any *real* harmonic function $\omega(\xi, \bar{\xi})$ which can be represented as $\omega = \mathrm{Re} f(\xi)$ for some (locally) holomorphic function f. For $\omega(\xi, \bar{\xi})$ we have the Laplace equation

$$\omega_{\xi\bar{\xi}} = 0 \tag{2}$$

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Real doubly periodic solutions of (2) may be constructed starting from an arbitrary solution $\omega_0(\xi,\bar{\xi})$ of (2) if we can ensure the convergence of the series

$$\sum_{m,n=-\infty}^{\infty}\Big\{\omega_0(\xi+mL_1+nL_2)+b_{mn}\Big\} \tag{3}$$

by appropriate choice of the real constants b_{mn}. We emphasize that this construction rests on two features of the Laplace equation (2), namely (i) its linearity and (ii) on its invariance with respect to arbitrary translations $\xi\to\xi+L$, $L\in\mathbf{C}$.

Now consider the Euler-Darboux equation

$$\omega_{\xi\bar{\xi}}-\frac{\omega_\xi-\omega_{\bar{\xi}}}{2(\xi-\bar{\xi})}=0\,,\quad \omega(\xi,\bar{\xi})\in\mathbf{R} \tag{4}$$

or, in terms of real coordinates (x,ρ) (where $\xi=x+i\rho$),

$$\omega_{xx}+\frac{1}{\rho}\omega_\rho+\omega_{\rho\rho}=0$$

This equation satisfies the same properties as (2) with one important modification: it is invariant with respect to *real* translations $\xi\to\xi+L$ only (i.e. $L\in\mathbf{R}$). Thus there are no doubly periodic solutions of (4), but we can still try to apply a similar scheme to obtain solutions of (4) which are periodic in the x-direction. Namely, let $\omega_0(x,\rho)$ be some solution of (4). In analogy with (3) we consider the expression

$$\omega(x,\rho)=\sum_{n=-\infty}^{\infty}\Big\{\omega_0(x+nL,\rho)+a_n\Big\} \tag{5}$$

where a_n are constants to be chosen in such a way that the series (5) becomes convergent. It is important that these coefficients do not depend on (x,ρ) since otherwise the sum could not possibly satisfy the original equation (4). If convergence can be achieved, the function (5) describes a solution of (4) with period L.

In order to exploit these observations for the construction of new solutions of Einstein's equations, we start from a stationary axisymmetric space-time with the metric

$$ds^2=f^{-1}[e^{2k}(dx^2+d\rho^2)+\rho^2d\phi^2]+f(dt+Ad\phi)^2 \tag{6}$$

where (x,ρ) are Weyl canonical coordinates (here $\rho\geq 0$ is the radial coordinate). The functions $f(x,\rho)$, $k(x,\rho)$ and $A(x,\rho)$ are related to the complex-valued Ernst potential $\mathcal{E}(x,\rho)$ [3] as follows:

$$f=\mathrm{Re}\mathcal{E}\qquad A_\xi=2\rho\frac{(\mathcal{E}-\bar{\mathcal{E}})_\xi}{(\mathcal{E}+\bar{\mathcal{E}})^2}\qquad k_\xi=2i\rho\frac{\mathcal{E}_\xi\bar{\mathcal{E}}_\xi}{(\mathcal{E}+\bar{\mathcal{E}})^2} \tag{7}$$

where again $\xi=x+i\rho$. In terms of $\mathcal{E}(x,\rho)$, the Einstein equations may be written in the form

$$(\mathcal{E}+\bar{\mathcal{E}})(\mathcal{E}_{xx}+\frac{1}{\rho}\mathcal{E}_\rho+\mathcal{E}_{\rho\rho})=2(\mathcal{E}_x^2+\mathcal{E}_\rho^2) \tag{8}$$

If $\mathcal{E}\in\mathbf{R}$, the coefficient A in (6) vanishes, and the metric (6) becomes static, i.e. describes a space-time without rotation. In this case, the Ernst equation (8) can be linearized by defining

$$\omega=\log\mathcal{E}$$

and is thereby reduced to the Euler-Darboux equation (4). The metric (6) becomes

$$ds^2=e^{-\omega}[e^{2k}(dx^2+d\rho^2)+\rho^2d\phi^2]+e^\omega dt^2 \tag{9}$$

where the conformal factor is determined from the first order equation

$$k_\xi = \frac{i\rho}{2}(\omega_\xi)^2$$

or, equivalently,

$$k_\rho = \frac{\rho}{4}(\omega_\rho^2 - \omega_x^2) \qquad k_x = \frac{\rho}{2}\omega_x\omega_\rho \tag{10}$$

We can now construct x-periodic analogs of known static solutions by means of the procedure outlined above. Note that in order to obtain a truly periodic metric (9) we must also verify the periodicity of the function $k(x,\rho)$ defined by (10).

The following theorem shows that our method of construction is really quite general.

Theorem 1 *Let $\omega_0(x,\rho)$ be any solution of the Euler-Darboux equation corresponding to an asymptotically flat metric (9), i.e.*

$$\omega_0(x,\rho) = \frac{\beta}{r} + O(r^{-2}) \qquad \text{as} \qquad r \to \infty \tag{11}$$

where $r = \sqrt{x^2+\rho^2}$; $M = -\frac{1}{2}\beta$ is the mass. Let

$$a_n = -\frac{\beta}{L|n|} \quad , \quad n \neq 0 \quad , \quad a_0 = 0 \tag{12}$$

Then series (5) is convergent for all (x,ρ) except the points (x_0+nL,ρ_0), where the function $\omega_0(x,\rho)$ is singular ($n \in \mathbf{Z}$), and defines a periodic function with period L.

Proof: For large n we have

$$\omega_0(x+nL,\rho) + a_n = \beta\Big(\frac{1}{\sqrt{(x+nL)^2+\rho^2}} - \frac{1}{L|n|}\Big) + O(\frac{1}{n^2}) = O(\frac{1}{n^2})$$

by (11), and therefore the series (5) converges if $(x+nL,\rho)$ does not coincide with a singular point of $\omega_0(x,\rho)$ for any n. $\Box$

So starting from an arbitrary static asymptotically flat solution we can construct its x-periodic analog. In the remainder we will, however, restrict attention to the Schwarzschild solution, which is characterized by the Ernst potential

$$\omega_0 = \log\mathcal{E}_0 \qquad \mathcal{E}_0(x,\rho) = \frac{\sqrt{(x-M)^2+\rho^2}+\sqrt{(x+M)^2+\rho^2}-2M}{\sqrt{(x-M)^2+\rho^2}+\sqrt{(x+M)^2+\rho^2}+2M} \tag{13}$$

where $M \in \mathbf{R}$ is an arbitrary positive constant (the mass of the black hole). Here, all square roots are taken to be positive; this means that we do not consider (13) inside the event horizon, which coincides with the segment $\rho = 0$, $x \in [-M, M]$. The coefficient β in (11) is therefore equal to $-2M$. Thus the periodic analog of the Schwarzschild solution (13) has the following form:

$$\mathcal{E}(x,\rho) = \mathcal{E}_0(x,\rho)\prod_{n=1}^{\infty}\mathcal{E}_0(x+nL,\rho)\mathcal{E}_0(x-nL,\rho)\exp\Big(\frac{4M}{nL}\Big) \tag{14}$$

Obviously,

$$\mathcal{E}(x+L,\rho) = \mathcal{E}(x,\rho)$$

In the sequel we will assume $\frac{1}{2}L > M$ (for $L \leq 2M$, the interpretation of the solution is not clear as the horizon overlaps with itself, and the Ernst potential vanishes on the symmetry axis). Convergence of the infinite product (14) is equivalent to convergence of the series (5) and thus guaranteed by Theorem 1 (for non-negative values of the square roots in (13)). Consequently, the solution (14) is a periodic function on the upper half plane $\rho \geq 0$ with "fundamental region" $\mathcal{F}$ defined by $\rho \geq 0$, $-\frac{1}{2}L \leq x \leq \frac{1}{2}L$.

Theorem 2 *The function $\mathcal{E}(x,\rho)$ defined by (14) is smooth everywhere on $\mathcal{F}$ away from the points $x = \pm M$, $\rho = 0$. Moreover, it is non-zero everywhere except on the horizon (i.e. $\rho = 0$, $|x| \leq M$).*

Proof: These properties are a consequence of analogous properties of the Schwarzschild solution and the convergence of (5). □

As mentioned above we must also check the periodicity of $k(x,\rho)$.

Theorem 3 *Let $L > 2M$. Then the function $k(x,\rho)$ corresponding to the Ernst potential (10) is periodic in x with period L, i.e.*

$$k(x+L,\rho) = k(x,\rho)$$

Proof: It is sufficient to show that

$$\int_{-L/2}^{L/2} k_x dx = 0$$

where the derivative k_x is to be evaluated by means of (10). Consider the integral

$$\int_l \Big\{ \frac{\rho}{4}(\omega_\rho^2 - \omega_x^2) d\rho + \frac{\rho}{2}\omega_x \omega_\rho dx \Big\} \tag{15}$$

where the closed contour l is chosen according to Fig.1. This integral vanishes because the function ω obeys (4) and is smooth everywhere inside of l. The integrals along the edges $[(-\frac{1}{2}L,0),\ (-\frac{1}{2}L,\rho)]$ and $[(\frac{1}{2}L,\rho),\ (\frac{1}{2}L,0)]$ cancel due to the periodicity of $\omega(x,\rho)$. Owing to the presence of the factor ρ in (15) the contribution of the interval $[(\frac{1}{2}L,0),\ (-\frac{1}{2}L,0)]$ reduces to a sum of contributions of two small rectangular paths around the points $x = -M$ and M (cf. Fig.1.), where the derivatives ω_x and ω_ρ become singular. These contributions cancel by virtue of the symmetry

$$\omega(-x,\rho) = \omega(x,\rho)$$

inherited by solution $\mathcal{E}$ from $\mathcal{E}_0$. So the integral along the contour $[(-\frac{1}{2}L,\rho),\ (\frac{1}{2}L,\rho)]$ also vanishes and we get $k(-\frac{1}{2}L,\rho) = k(\frac{1}{2}L,\rho)$. □

Hence the metric (9) corresponding to the periodic solution (14) is also periodic. Furthermore, we have

Theorem 4 *The asymptotic behavior of the Ernst potential (14) is given by*

$$\mathcal{E} = C\rho^{4M/L}\Big(1 + o(1)\Big) \quad \text{as} \quad \rho \to \infty \tag{16}$$

where C is some constant.

Proof: The function $\omega = \log \mathcal{E}$ is defined by

$$\omega(x,\rho) = \omega_0(x,\rho) + \sum_{n=1}^{\infty} \Big[\omega_0(x+nL,\rho) + \omega_0(x-nL,\rho) + \frac{4M}{nL}\Big]$$

Substituting the explicit expression for $\omega_0(x,\rho)$ (13) and differentiating with respect to ρ^2, we obtain

$$\frac{\partial \omega}{\partial(\rho^2)}(x,\rho) = \sum_{n=-\infty}^{\infty} \frac{2M[s_1(n)+s_2(n)]}{[s_1(n)+s_2(n)+2M][s_1(n)+s_2(n)-2M]s_1(n)s_2(n)}$$

where $s_1(n) = \sqrt{(x+nL+M)^2+\rho^2}$ and $s_2(n) = \sqrt{(x+nL-M)^2+\rho^2}$. The leading term in this series for large ρ can be estimated by approximating the sum by an integral; it is given by

$$\sum_{n=-\infty}^{\infty} \frac{M}{((x+nL)^2+\rho^2)^{3/2}} = \frac{2M}{L}\frac{1}{\rho^2}\Big(1+o(1)\Big)$$

Thus,

$$\omega = \frac{2M}{L}\log\rho^2 + O(1) \qquad \text{as} \qquad \rho \to \infty$$

and $\mathcal{E} = C\rho^{4M/L}(1+o(1))$ for some constant C. $\Box$

Hence, as $\rho \to \infty$, the metric (9) tends to the Kasner solution

$$ds^2 = \tilde{C}\rho^{\frac{\alpha^2}{2}-\alpha}(dx^2 + d\rho^2) + C^{-1}\rho^{2-\alpha}d\phi^2 - C\rho^{\alpha}dt^2 \tag{17}$$

where $\tilde{C}$ is another constant of integration and the Kasner parameter α is related to the period L by $\alpha = 4ML^{-1}$, so that $0 \le \alpha < 2$ with our assumption on the range of M.

The new solution has a compact event horizon coinciding with the segment $\rho = 0\,,\ -M \le x \le M$. Outside the horizon it is everywhere non-singular, including the segment of the symmetry axis outside the horizon. Using the standard product representation for the Γ-function, we find

$$\mathcal{E}(x, \rho = 0) = \exp\Big(\frac{4\gamma M}{L}\Big)\,\frac{\Gamma\Big(\frac{|x|+M}{L}\Big)\Gamma\Big(1-\frac{|x|-M}{L}\Big)}{\Gamma\Big(\frac{|x|-M}{L}\Big)\Gamma\Big(1-\frac{|x|+M}{L}\Big)} \tag{18}$$

for $M \le |x| \le \frac{1}{2}L$ (γ is the Euler-Mascheroni constant), and $\mathcal{E} \equiv 0$ for $\rho = 0$ and $|x| \le M$. As a consequence of the reflection and translation symmetry, the free integration constant in equation (10) may be chosen in such a manner that conical singularities on the part of the symmetry axis outside the horizon are avoided (this requirement fixes the constant $\tilde{C}$ in (17)). In the limit $L \to \infty$ the solution obviously tends (pointwise) to the ordinary Schwarzschild solution. The leading term in the asymptotic expansion then approaches the flat metric, as it should be. Alternatively, one can regard the new solution as describing an infinite chain of black holes spaced at a distance L. At first sight, it seems remarkable that this configuration does not require conical singularities on the axis between adjacent black holes for stability. Rather, it appears to be stabilized by its symmetry under reflections and translations and the presence of infinitely many black holes "on each side". However, this also indicates instability under non-periodic perturbations, which makes this interpretation somewhat less attractive.

In summary, our new solution can be interpreted as a natural analog of the Schwarzschild solution in an x-periodic Kasner universe. The existence of this solution does not in any way contradict well-known theorems on the uniqueness of black hole solutions [4] as these theorems only apply to asymptotically flat space-times. Note that the product (14) is also convergent inside the event horizon, where ρ becomes imaginary if all square roots in (13) are taken to be positive. On the second sheet of the ordinary Schwarzschild solution, where the square roots become negative and the mass is also negative, the product (14) diverges, i.e. $\mathcal{E} \to \infty$, so the second sheet is, in fact, absent. It would be desirable to study the behaviour of our solution on and inside the event horizon and to compare it with the ordinary Schwarzschild solution there. Furthermore, the Kasner solution (17) represents only the main term in the asymptotic expansion of our solution. Clearly, it would be interesting to identify the next order term and to define some analog of mass in the periodic case. Finally, the construction of periodic analogs of the Kerr solution [5] will be more complicated as the Ernst equation can no longer be linearized in that case. In addition, one must presumably take into account infinite-soliton Ernst potentials.

References

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Figure 1: Integration contour used in the proof of Theorem 3.