

## Physical states in $d = 3, N = 2$ supergravity

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To clarify some issues raised by D'Eath's recent proposal for the physical states of  $N = 1$  supergravity in four dimensions, we study pure (topological)  $N = 2$  supergravity in three dimensions, which is formally very similar, but much easier to solve. The wave functionals solving the quantum constraints can be understood in terms of arbitrary functions on the space of moduli and supermoduli, which is not Hausdorff. We discuss the implications for the wave functionals and show that these are not amenable to expansions in fermionic coordinates, but can serve as lowest-order solutions to the quantum constraints in an expansion in  $\hbar$  in more realistic theories.

Motivated by recent work on the nature of the full non-perturbative wave functional of  $d = 4, N = 1$  supergravity [1,2] we study pure  $d = 3, N = 2$  supergravity in this letter. In contrast to supergravity in four dimensions, this theory possesses only topological degrees of freedom, so the problems associated with propagating degrees of freedom, such as operator product singularities and ordering ambiguities, which plague the canonical formulation of supergravity in four dimensions, are absent. Moreover, this model can be solved exactly at the quantum level, just like 3D gravity [3,4]. For these reasons, and because the status of perturbative renormalizability versus non-renormalizability of the  $d = 4, N = 1$  is theory still pending, it appears to be well suited to settle some of the issues raised in [1].

The reduced wave functionals of  $N = 2$  supergravity are given by arbitrary functions depending on the moduli and supermoduli of flat  $SL(2, \mathbb{R})$  connections on a connected spatial surface of arbitrary genus. However, the unreduced wave functionals still depend on an infinite number of coordinates in the full configuration space. From their structure it is immediately clear that they are not amenable to the kind of expansion in terms of the fermionic coordinates on

which the arguments of [1] are based. Furthermore, wave functionals of this type also arise in four dimensions in the limit  $\hbar \rightarrow 0$ , where they can be used as input for a consistent perturbative approach. We note that pure  $d = 3, N = 1$  supergravity was already discussed in [5]. Here we prefer to consider  $N = 2$  supergravity instead, mainly because the two Majorana gravitinos can be combined into one Dirac vector spinor and therefore admit a simpler representation of the fermionic quantum operators. As a consequence, the associated quantum constraints are formally very similar to those of  $d = 4, N = 1$  supergravity as written down in [1], apart from the fact that we work in the so-called connection representation, whereas [1] is based on the more familiar metric formulation. Moreover the  $N = 2$  theory arises naturally in the reduction of  $d = 4, N = 1$  supergravity to three space-time dimensions, which makes it an obvious starting point for this study.

We use first-order formalism for the connection field, so the basic fields are the dreibein  $e_\mu^a$ , the spin connection  $A_\mu^a$ , and the two-component complex gravitino field  $\psi_\mu$ , which transforms as a spinor under  $SL(2, \mathbb{R})$ , corresponding to the spinor representation of the Lorentz group  $SO(1,2)$ . The Lorentz-covariant

derivatives and curvatures are

$$\begin{aligned} D_\mu e_\nu^a &= \partial_\mu e_\nu^a - \varepsilon^{abc} A_{\mu b} e_{\nu c}, \\ D_\mu \psi_\nu &= \partial_\mu \psi_\nu + \frac{1}{2} A_\mu^a \gamma_a \psi_\nu, \\ F_{\mu\nu a} &= \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} - \varepsilon_{abc} A_\mu^b A_\nu^c. \end{aligned} \quad (1)$$

With this notation the Lagrangian of  $N = 2$  supergravity is given by

$$\mathcal{L} = \frac{1}{4} \varepsilon^{\mu\nu\rho} e_\mu^a F_{\nu\rho a} + \varepsilon^{\mu\nu\rho} \bar{\psi}_\mu D_\nu \psi_\rho. \quad (2)$$

It is invariant under local Lorentz transformations

$$\begin{aligned} \delta A_\mu^a &= D_\mu \omega^a, \quad \delta \psi_\mu = -\frac{1}{2} \omega^a \gamma_a \psi_\mu, \\ \delta e_\mu^a &= \varepsilon^{abc} \omega_b e_{\mu c}, \end{aligned} \quad (3)$$

and local supersymmetry transformations with parameter  $\epsilon$

$$\begin{aligned} \delta A_\mu^a &= 0, \quad \delta \psi_\mu = D_\mu \epsilon, \\ \delta e_\mu^a &= \bar{\epsilon} \gamma^a \psi_\mu - \bar{\psi}_\mu \gamma^a \epsilon. \end{aligned} \quad (4)$$

Our conventions are as in [5,6]. The space-time manifold is assumed to be a direct product of a connected two-dimensional manifold of genus  $n$  (with local coordinates  $\mathbf{x}$ ) and the real line. Greek indices  $\mu, \nu, \dots$  denote coordinates on the space-time manifold; with respect to the direct product structure they take the values  $t$  for the time coordinate and  $i, j, \dots$  for the local coordinates on the spatial manifold  $\mathcal{M}$ . Flat  $SO(1, 2)$  indices are designated by  $a, b, \dots = 0, 1, 2$ .

The canonical treatment of the Lagrangian (1), which is explained at length in [5,6]<sup>#1</sup>, shows that the time components  $e_t^a, A_t^a, \psi_t$  and  $\bar{\psi}_t$  become Lagrange multipliers. They generate the constraints to be given below. The components with spatial indices  $i, j, \dots$  span the phase space. After quantization one obtains the (anti)commutation relations

$$\begin{aligned} [A_i^a(\mathbf{x}), e_j^b(\mathbf{y})] &= -2i\hbar \varepsilon_{ij} \eta^{ab} \delta(\mathbf{x}, \mathbf{y}), \\ \{\psi_{i\alpha}(\mathbf{x}), \bar{\psi}_{j\beta}(\mathbf{y})\} &= -i\hbar \varepsilon_{ij} \delta_{\alpha\beta} \delta(\mathbf{x}, \mathbf{y}). \end{aligned} \quad (5)$$

(We will frequently suppress the  $SL(2, \mathbb{R})$  spinor indices  $\alpha, \beta \dots$  on  $\psi_i$  and  $\bar{\psi}_i$ .) An operator realization

<sup>#1</sup> Standard references on the canonical formulation of gravity are [7].

is easily found (this would be slightly more tricky for the  $N = 1$  theory where the gravitino is Majorana). As the basic variables, we take the connection  $A_i^a(\mathbf{x})$  and  $\psi_i(\mathbf{x})$ . Consequently, the wave functionals are of the form  $\Psi = \Psi[A_i^a, \psi_i]$ . The canonically conjugate fields  $e_i^a$  and  $\bar{\psi}_i$  are represented by the functional differential operators

$$e_i^a = -2i\hbar \varepsilon_{ij} \frac{\delta}{\delta A_{ja}}, \quad \bar{\psi}_{i\alpha} = i\hbar \varepsilon_{ij} \frac{\delta}{\delta \psi_{j\alpha}}. \quad (6)$$

Inserting these into the classical constraints we obtain the quantum constraints

$$\left( iD_i(A) \frac{\delta}{\delta A_i^a} + \frac{1}{2} \frac{\delta}{\delta \psi_i} \gamma_a \psi_i \right) \Psi = 0, \quad (7)$$

$$\varepsilon^{ij} F_{ija}(A) \Psi = 0, \quad (8)$$

$$\varepsilon^{ij} D_i(A) \psi_j \Psi = 0, \quad (9)$$

$$D_i(A) \frac{\delta}{\delta \psi_i} \Psi = 0. \quad (10)$$

Observe that there are no operator ordering ambiguities or singularities in these expressions. Solving quantum supergravity in three dimensions amounts to solving these four functional differential equations. The Lorentz constraint (7) implies that  $\Psi$  is invariant under local Lorentz transformations and is thus trivially solved. The constraints (8) and (9) tell us that  $\Psi[A, \psi]$  has support on flat  $SL(2, \mathbb{R})$  connections  $A_i^a$ , and on gravitino fields  $\psi_i$  whose Rarita-Schwinger field strength vanishes. The former constraint is just the Wheeler-DeWitt equation, which is implied by the constraints (9) and (10). This last constraint requires  $\Psi$  to be invariant under the supersymmetry transformations  $\delta \psi_i = D_i \epsilon$ .

Unlike the constraints given in [1] for supergravity in four dimensions, our constraints do not depend on  $\hbar$ , which reflects the topological nature of the theory. In both cases the supersymmetry constraints are of first order and homogeneous in the fermionic operators. Consequently, the fermionic constraints can be studied separately on wave functionals  $\Psi$  with definite fermion number. Application of the constraints to the zero-fermion sector of supergravity in four dimensions has lead to conflicting conclusions [1,2].

We now discuss the solutions of the quantum constraints, following the analysis given in [5]. We start

with (8) and (9), since this is where most of the subtleties reside. For this purpose, we find it convenient to employ differential forms  $A$  and  $\psi$  on  $\mathcal{M}$  defined by

$$A := \frac{1}{2} A_i^a \gamma_a dx^i, \quad \psi := \psi_i dx^i. \quad (11)$$

Consequently, we must solve the conditions  $F(A) = 0$  and  $D(A)\psi := (d + A)\psi = 0$ . Clearly, these equations tell us that  $A$  and  $\psi$  are pure gauge locally, hence the absence of propagating degrees of freedom. Globally, this need not be true in general, and the leftover degrees of freedom are called “moduli” (for  $A$ ) and “supermoduli” (for  $\psi$ ). Locally  $A$  and  $\psi$  are thus expressed by

$$A = g^{-1} dg, \quad \psi = g^{-1} d\phi, \quad (12)$$

where  $g$  is an element of  $SL(2, \mathbb{R})$  and  $\phi$  a fermionic spinor function. Unlike  $A$  and  $\psi$ ,  $g$  and  $\phi$  are not necessarily globally defined (they are single-valued on the covering manifold, however). Moreover they are only defined up to multiplication of  $g$  by a constant group element  $h_0$  and shifts of  $\phi$  by a constant spinor  $\epsilon_0$ . Hence  $g, \phi$  and  $g', \phi'$  related by

$$\begin{aligned} g'(\mathbf{x}) &= h_0^{-1} g(\mathbf{x}), \\ \phi'(\mathbf{x}) &= h_0^{-1} (\phi(\mathbf{x}) - \epsilon_0), \end{aligned} \quad (13)$$

are equivalent. We note also that under local Lorentz and supersymmetry transformations  $g$  and  $\phi$  transform as

$$\begin{aligned} g(\mathbf{x}) &\longrightarrow g(\mathbf{x}) h(\mathbf{x}), \\ \phi(\mathbf{x}) &\longrightarrow \phi(\mathbf{x}) + g(\mathbf{x}) \epsilon(\mathbf{x}), \end{aligned} \quad (14)$$

where  $h(\mathbf{x})$  and  $\epsilon(\mathbf{x})$  are single-valued on  $\mathcal{M}$ .

A representation of  $g$  and  $\phi$  can be constructed as follows. Pick an arbitrary point  $\mathbf{x}_0 \in \mathcal{M}$  and let  $\Gamma$  denote the first fundamental group of  $\mathcal{M}$  with base point  $\mathbf{x}_0$ . Given arbitrary field configurations  $A$  and  $\psi$  on  $\mathcal{M}$  with vanishing field strengths  $F(A)$  and  $D(A)\psi$ , we define  $g(\mathbf{x})$  and  $\phi(\mathbf{x})$  by

$$g(\mathbf{x}) := \mathcal{P} \exp \int_{\mathbf{x}_0}^{\mathbf{x}} A, \quad \phi(\mathbf{x}) := \int_{\mathbf{x}_0}^{\mathbf{x}} g \psi. \quad (15)$$

These expressions depend on the base point  $\mathbf{x}_0$ , but are insensitive to continuous deformations of the path connecting  $\mathbf{x}_0$  to  $\mathbf{x}$ . Therefore they are affected by local Lorentz and supersymmetry transformations at the base point, which induce the transformations (13) with  $h_0 \equiv h(\mathbf{x}_0)$  and  $\epsilon_0 \equiv \epsilon(\mathbf{x}_0)$ . Likewise, changing the basepoint and thus the path connecting it to  $\mathbf{x}$ , changes  $g$  and  $\phi$  in accord with (13).

For  $\gamma \in \Gamma$  and  $f(\mathbf{x})$  an arbitrary and not necessarily single-valued function on  $\mathcal{M}$ , we denote by  $f(\mathbf{x}_0 + \gamma)$  the value of  $f$  obtained by starting at  $\mathbf{x}_0$  and letting  $\mathbf{x}$  traverse the loop  $\gamma$  once. Assuming single-valuedness for  $A$  and  $\psi$  implies that the effect of traversing the loop may lead to a deficit of the form (13). The bosonic and fermionic holonomies  $g_\gamma$  and  $\phi_\gamma$  parametrize this deficit and are thus defined by

$$\begin{aligned} g(\mathbf{x}_0 + \gamma) &= g_\gamma g(\mathbf{x}_0), \\ \phi(\mathbf{x}_0 + \gamma) &= \phi_\gamma + g_\gamma \phi(\mathbf{x}_0). \end{aligned} \quad (16)$$

Under local Lorentz and supersymmetry transformations  $g_\gamma$  and  $\phi_\gamma$  are invariant. As  $g$  and  $\phi$  are only defined up to the transformations (13), the bosonic and fermionic holonomies related by

$$\begin{aligned} g'_\gamma &= h_0^{-1} g_\gamma h_0, \\ \phi'_\gamma &= h_0^{-1} (\phi_\gamma + (g_\gamma - 1)\epsilon_0) \end{aligned} \quad (17)$$

should be identified. Therefore the wave functionals depend only on the conjugacy classes with respect to (17). (This can for instance be ensured by choosing a reduced wave functional that is invariant with respect to the transformations (17).)

The space of moduli and supermoduli on a spatial (Riemann) surface of genus  $n$  is now defined as the space of holonomies (16) modulo the transformations (17) and the constraint

$$\prod_{j=1}^n \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1} = 1, \quad (18)$$

defining the first fundamental group on a Riemann surface of genus  $n$ , where  $\alpha_j, \beta_j$  ( $j = 1, \dots, n$ ) constitute the usual basis of homology cycles on the Riemann surface [8]. More rigorously, the space of (super)moduli consists of the set of conjugacy classes of group homomorphisms from the fundamental group

into the gauge group (the combined group of local Lorentz and supersymmetry transformations). This homomorphism is defined by (15). Condition (18) thus translates into similar constraints on the corresponding quantities  $(g_{\alpha_j}, \phi_{\alpha_j})$  and  $(g_{\beta_j}, \phi_{\beta_j})$ , because  $g(\mathbf{x})$  and  $\phi(\mathbf{x})$  must be single-valued around the contractible loop defined by (18). Here one must use the composition law induced by the homomorphism,

$$g_{\gamma_1 \circ \gamma_2} = g_{\gamma_1} g_{\gamma_2}, \quad \phi_{\gamma_1 \circ \gamma_2} = \phi_{\gamma_2} + g_{\gamma_2} \phi_{\gamma_1}, \quad (19)$$

where we used that  $\phi$  vanishes at the base point.

Detailed discussions of the bosonic moduli space of flat  $SL(2, \mathbb{R})$  connections may be found in [4] and [9]. However, the first reference discusses only elliptic conjugacy classes of  $SL(2, \mathbb{R})$ , while the second deals only with hyperbolic conjugacy classes, which are shown to be directly related to Teichmüller space<sup>#2</sup>. On the other hand, it could be plausibly argued that any discussion of 3D quantum supergravity should take into account *all* of (super)moduli space (see also [10]). An unexpected property of the bosonic moduli space is that it is not a Hausdorff space in general [11]. This feature is usually related to the fact that moduli space is defined as the quotient of two *infinite* dimensional spaces, and has been known to mathematicians for a long time [12]; we will explain it in terms of an elementary example at the end of this paper when discussing the torus. It also has implications for the fermionic moduli and for the wave functionals. In particular, the presence of extra fermionic moduli seems to be correlated with the lack of the Hausdorff property. To explain this point, let us count the number of fermionic moduli. There are  $2n$   $\phi_\gamma$ 's, each with two spinor components, hence altogether  $4n$  fermionic holonomies. Since  $\phi(\mathbf{x})$  must be single-valued when transported around the contractible loop (18), only  $4n - 2$  of them are independent (see the discussion above). Moreover, for generic bosonic holonomies, we can use (17) to gauge away two more fermionic holonomies, so in general there will be  $4n - 4$  fermionic moduli. For non-generic  $g_\gamma \in SL(2, \mathbb{R})$ , the matrices  $(g_\gamma - 1)$  may

not be invertible. From (17), it is evident that we can still remove two fermionic degrees of freedom as long as there remains at least one homology cycle  $\gamma$ , for which  $(g_\gamma - 1)$  is invertible. Otherwise, we cannot use (17) to gauge away fermionic holonomies, and at such non-generic points, there will be more fermionic moduli. This means that the "superspace" spanned by the bosonic and fermionic moduli is not a direct product space, but rather more like a sheaf! The existence of such singular points is a feature which is entirely due to the non-compactness of  $SL(2, \mathbb{R})$ , since invertibility may only fail for parabolic conjugacy classes. Since all matrices  $(g_\gamma - 1)$  must be non-invertible for extra fermionic moduli to exist, the singular points form a set of very low dimension and therefore become "less and less important" with increasing genus  $n$ . Nevertheless, we still face the obvious question how to define wave functionals on such a space. If we insist on continuity, these must be constant along those bosonic moduli for which the Hausdorff property breaks down; this is also the point of view adopted in [11]. For the same reason, they could not depend then on the extra fermionic moduli, either. One can also avoid the singular points altogether by restricting the space of bosonic moduli to the hyperbolic sector from the outset as proposed in [9]. In any case, different prescriptions can be expected to lead to inequivalent theories of quantum (super)gravity.

The physics content of the theory is completely encapsulated in the wave functions  $f(g_{\alpha_j}, g_{\beta_j}, \phi_{\alpha_j}, \phi_{\beta_j})$  depending on the conjugacy classes of the fermionic and bosonic holonomies as defined by (17) and the constraint (18). Since these moduli and supermoduli form a finite dimensional space at each genus, all further manipulations are in principle well-defined. So, we can now define a scalar product by means of a suitable measure and evaluate the observables introduced in [5] on the states. We note, however, that it is a priori not clear what measure to choose due to the non-compactness of the group  $SL(2, \mathbb{R})$  and how to define the class of admissible functions  $f$  (see e.g. [10]).

We can equivalently describe the solutions in terms of wave functionals on the full configuration space spanned by  $A_i^a(\mathbf{x})$  and  $\psi_i(\mathbf{x})$ . The resulting expression is somewhat cumbersome and reads

<sup>#2</sup> It would be interesting to see whether the space of supermoduli as defined above can be similarly related to the super-Teichmüller space of a Riemann surface if one restricts the bosonic conjugacy classes to the hyperbolic sector. The dimensions of these spaces are the same.

$$\begin{aligned} \Psi[A_i^a, \psi_i] = & \int' \prod_{j=1}^n \left\{ dg_{\alpha_j} d\phi_{\alpha_j} \right. \\ & \times \delta\left(g_{\alpha_j} - \mathcal{P} \exp \oint_{\alpha_j} A\right) \delta\left(\phi_{\alpha_j} - \oint_{\alpha_j} g \psi\right) \left. \right\} \\ & \times \prod_{j'=1}^n \left\{ dg_{\beta_{j'}} d\phi_{\beta_{j'}} \delta\left(g_{\beta_{j'}} - \mathcal{P} \exp \oint_{\beta_{j'}} A\right) \right. \\ & \times \delta\left(\phi_{\beta_{j'}} - \oint_{\beta_{j'}} g \psi\right) \left. \right\} f(g_{\alpha_j}, g_{\beta_{j'}}, \phi_{\alpha_j}, \phi_{\beta_{j'}}) \\ & \times \prod_x \delta(F(A(x))) \delta(\varepsilon^{ij} D_i(A) \psi_j(x)), \end{aligned} \quad (20)$$

where  $dg$  denotes the Haar measure on  $SL(2, \mathbb{R})$  and  $d\phi$  are Grassmann integrals; the prime attached to the (finite-dimensional) integral is to indicate that it is to be performed only over those holonomies satisfying the constraints following from (18). The delta functionals in the last line enforce the quantum constraints (8) and (9). The fermionic delta functional is defined as the infinite product over its arguments in the usual way. Evidently, this makes an expansion of the wave functional in terms of fermionic coordinates starting “at the bottom” or “at the top” of the Dirac sea rather pointless, because the sea is neither empty nor completely filled. Applying the last constraint (10) to the above wave functional we get two contributions. On the fermionic delta functional (last line of (20)), it produces a curl  $\varepsilon^{ij} D_i D_j$ , which vanishes by  $F(A) = 0$ , while on the delta functions (second line of (20)), it gives zero because the moduli are locally supersymmetric. Similarly, (20) is annihilated by (7) because the delta functionals are, and the remaining part of the integral involves only manifestly Lorentz invariant quantities. Thus, the wave functional (20) satisfies all constraints.

Obviously, the above expression is highly formal and at best of heuristic value; rigorous manipulations can only be performed in terms of the functions  $f$ . With this in mind, we now return to the constraints of  $d = 4, N = 1$  supergravity as written down in [1]. They read

$$\begin{aligned} & \left( \varepsilon^{ijk} e_{AA'i}(x) D_j \psi_k^A(x) - \frac{1}{2} \hbar \kappa^2 \psi_i^A(x) \frac{\delta}{\delta e_i^{AA'}(x)} \right) \\ & \times \Psi[e, \psi] = 0, \end{aligned} \quad (21)$$

$$\begin{aligned} & \left( D_i \frac{\delta}{\delta \psi_i^A(x)} + \frac{1}{2} \hbar \kappa^2 \frac{\delta}{\delta e_i^{AA'}(x)} D_{ji}^{BA'} \frac{\delta}{\delta \psi_j^B(x)} \right) \\ & \times \Psi[e, \psi] = 0, \end{aligned} \quad (21 \text{ cont'd})$$

where the notation has been appropriately changed to four dimensions (with two-component spinor notation). It is not relevant for our arguments that [1] uses the dreibein (or metric) representation rather than the connection representation that we have been using up to this point. Observe that these constraints depend on  $\hbar$ , which reflects the presence of propagating degrees of freedom in this theory. Obviously, these constraints are much harder to solve, because both the dreibein and its conjugate momentum appear in (21), so the constraint operator is not just a multiplication operator as in the 3D theory. The dependence of (21) on  $\hbar$  suggests an expansion of  $\Psi$  in terms of  $\hbar$ . In analogy with (20) the first of the constraints (21) is schematically solved by

$$\begin{aligned} \Psi[e, \psi] = & \prod_x \delta(R^{(3)}(e(x))) \\ & \times \delta(D(\omega(e(x))) \psi(x)) F(\text{“moduli”}) \\ & + O(\hbar) \end{aligned} \quad (22)$$

to lowest order in  $\hbar$ , where by “moduli” we mean all the bosonic and fermionic degrees of freedom contained in the dreibein and gravitino which are not killed by the first constraint (there are now infinitely many such moduli, because we have propagating modes). Note that the argument of the delta functionals are just the  $O(\hbar^0)$  parts of the Wheeler–DeWitt operator and the first supersymmetry constraint in (21), respectively. To lowest order, the second constraint then annihilates the wave functional (22) as well, because application of the  $O(\hbar^0)$  term produces a commutator of two covariant derivatives, which after a little calculation is found to be proportional to the curvature scalar, and thus vanishes on (22) <sup>#3</sup>. Apart from the highly singular nature of the  $O(\hbar^0)$  term, we again see that an expansion

<sup>#3</sup> For this argument we have to make use of the formula

$$\begin{aligned} R_{ijkl} = & g_{ik} R_{jl} - g_{jk} R_{il} - g_{il} R_{jk} + g_{jl} R_{ik} \\ & + \frac{1}{2} R(g_{ik} g_{jl} - g_{jk} g_{il}), \end{aligned}$$

which is valid in three dimensions only.

in the fermionic coordinates does not seem feasible. Just as in (20), the fermionic occupation is such that  $\Psi$  starts “in the middle” of the Dirac sea at infinite distance from both its top and bottom, but there will now be further terms at higher orders in  $\hbar$ . With (22) as lowest-order solution we can thus apply perturbation theory in  $\hbar$ , at least in principle. In practice, matters become quite complicated and we have not attempted to carry these considerations any further. We suspect that the short-distance singularities, which were so conspicuously absent from [1], will reappear in higher orders.

We conclude this paper by briefly discussing the space of moduli and supermoduli for the torus and explicitly demonstrating that this space is not Hausdorff. There are two non-trivial homology cycles  $\alpha$  and  $\beta$ . The condition (18) and the corresponding condition for the fermionic holonomies (cf. (19)) reduce to

$$g_\alpha g_\beta g_\alpha^{-1} g_\beta^{-1} = 1, \\ (g_\alpha - 1)\phi_\beta = (g_\beta - 1)\phi_\alpha. \tag{23}$$

Two sets of holonomies are then equivalent if they are related as in (17). From the first condition in (23), we infer that  $g_\alpha$  and  $g_\beta$  must commute, and hence belong to the same conjugacy class of  $SL(2, \mathbb{R})$ . There are four special points corresponding to the matrices  $g_\alpha, g_\beta = \pm 1$ . Only for  $g_\alpha = g_\beta = 1$  there are fermionic moduli. In principle there are four such moduli (corresponding to the four constant  $\psi_i$  coordinates), but by requiring equivalence with respect to  $SL(2, \mathbb{R})$  we are left with only two. Apart from these four points there are three types of conjugacy classes which we now discuss in turn.

– If both  $g_\alpha$  and  $g_\beta$  are elliptic (corresponding to mutually commuting timelike generators), they can always be brought into form of an  $SO(2)$  transformation,

$$g_\gamma = \begin{pmatrix} \cos \theta_\gamma & \sin \theta_\gamma \\ -\sin \theta_\gamma & \cos \theta_\gamma \end{pmatrix}, \quad \phi_\gamma = 0, \tag{24}$$

for  $\gamma = \alpha, \beta$ . As  $(g_\gamma - 1)$  is always invertible in this case, it is easy to see that all fermionic holonomies can be gauged away and there are thus no supermoduli. From the representation (24) it is evident that this part of the bosonic moduli space is just a torus with the four points corresponding to  $g_\alpha, g_\beta = \pm 1$  cut out.

– If both  $g_\alpha$  and  $g_\beta$  are hyperbolic (corresponding to mutually commuting spacelike generators), the standard representative is given by

$$g_\alpha = \pm \begin{pmatrix} e^{r \cos(\theta/2)} & 0 \\ 0 & e^{-r \cos(\theta/2)} \end{pmatrix}, \\ g_\beta = \pm \begin{pmatrix} e^{r \sin(\theta/2)} & 0 \\ 0 & e^{-r \sin(\theta/2)} \end{pmatrix}, \\ \phi_\gamma = 0, \tag{25}$$

for  $r > 0$  and  $0 \leq \theta < 2\pi$ . The absence of fermionic moduli is again straightforward to prove. This part of moduli space consists of four copies of the plane with the origin cut out.

– If both  $g_\alpha$  and  $g_\beta$  are parabolic (corresponding to mutually commuting lightlike generators), we get

$$g_\alpha = \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad g_\beta = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \tag{26}$$

where we can scale  $a$  and  $b$  by an  $SL(2, \mathbb{R})$  transformation to obey the restriction  $a^2 + b^2 = 1$ . This part of the bosonic moduli space is thus homeomorphic to four copies of the circle  $S^1$  (with no points cut out). In contrast to the cases discussed before, however, the matrices  $(g_\gamma - 1)$  fail to be invertible if both signs are  $+$ , and we then have extra fermionic moduli.

To see that this is not a Hausdorff space consider for instance a sequence of elliptic elements in  $SL(2, \mathbb{R})$  approaching a point on the lightlike boundary (i.e., on  $S^1$  according to the above analysis). By conjugating these matrices into the  $SO(2)$  subgroup as we did above, we obtain a sequence of points in  $SO(2)$  converging to  $\pm 1$ . However, although they converge to different limit points on the lightlike boundary, these two sequences are identified in the moduli space. Phrased otherwise, this means that for any two distinct points in the parabolic sector, we cannot find open neighbourhoods that separate them! The situation is similar for sequences approaching the boundary in the hyperbolic sector, although there are now *three* limit points, two on  $S^1$  (at opposite points) and  $\pm 1$ . Thus, the total moduli space can be modelled as follows: take a torus with four holes and four copies of the plane with a hole cut out at the origin. Glue them together in such a way that each plane gets attached to one of the holes in the torus and that each boundary on the torus winds twice around the border of the hole in

the plane to which it is attached. These boundaries have the topology of a circle and parametrize the four parabolic conjugacy classes. To each of them we put an extra point and define the open neighborhoods of these points to be all open sets containing the whole circle. There are no fermionic moduli except along one of the circles.

Finally, we note that the difficulties encountered here disappear altogether in the so-called mini-superspace approximation, where from the outset one deals only with a finite number of degrees of freedom. In the light of the results obtained it appears that this approximation cannot truly capture the remarkable features of quantum gravity and supergravity.

*Note added.* It has been demonstrated in [13] that there are no purely bosonic states in  $d = 4, N = 1$  supergravity and additional arguments are given that there are no states of any fixed but finite fermion number.

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