BRIEF REPORTS

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Gauge independence in Hadamard renormalization

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We examine the dependence of the Hadamard-renormalized stress-energy tensor on the gauge-fixing parameter for electromagnetism and one-loop gravity. Simple arguments show that the stress-energy tensor is independent of the parameter.

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I. INTRODUCTION

In an interesting paper, Nielsen and van Nieuwenhuizen [1] pointed out that when one introduces a parameter α^{-1} in front of the standard term used to break the gauge invariance of the electromagnetic action,

$$\frac{1}{\alpha}S_{\rm GB}=-\frac{1}{2\alpha}\int d^4x\sqrt{g}\,(A_a{}^{;a})^2 \ ,$$

the ghost action also acquires an α dependence. This parameter dependence may formally be removed by a rescaling of the antighost field and so has generally been ignored by previous authors, including ourselves for the analogous case of linearized gravity [2]. However, Nielsen and van Nieuwenhuizen argue that this formal argument is invalidated by regularization and that a nontrival α dependence of the ghost effective action can occur. In particular, they show that the electromagnetic trace anomaly is not α dependent, as suggested by Endo [3]; Endo's α-dependent terms are canceled by opposite terms which arise from a careful treatment of the ghost contribution to the trace anomaly.

In this Brief Report we investigate the effect of the observation of Nielsen and van Nieuwenhuizen on our Hadamard-renormalization scheme for electromagnetism [4] and linearized gravity [2] and make some general statements about their work. For simplicity we shall carry out the detailed part of our argument for the Maxwell field, but the extension to the gravitational case is straightforward.

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II. ELECTROMAGNETISM

In terms of the vector potential A_a, the Maxwell action is

$$S_M=-\tfrac{1}{4}\int d^4x\sqrt{g}\,F_{ab}F^{ab} \ , \tag{2.1}$$

where the field strength $F_{ab}\equiv 2\nabla_{[a}A_{b]}$. The field strength is invariant under the gauge transformation $A_a \to A_a+\nabla_a\Lambda$ for an arbitrary scalar field Λ. To quantize the theory, one must introduce a gauge-breaking term into the action; the standardly used term is

$$S_{\rm GB}=-\tfrac{1}{2}\int d^4x\sqrt{g}\,(A_a{}^{;a})^2 \ . \tag{2.2}$$

We consider the effects of using $\alpha^{-1}S_{\rm GB}$ as the gauge-breaking term, introducing the positive real parameter α. Following Nielsen and van Nieuwenhuizen, the ghost action needed to compensate for this choice of gauge-breaking term is

$$S_{\rm gh}=\alpha^{-1/2}\int d^4x\sqrt{g}\,\bar\eta\Box\eta \ , \tag{2.3}$$

where η and $\bar\eta$ are the complex anticommuting scalar ghost and antighost fields, respectively. For the purposes of quantization, the total action is then $S_M+\alpha^{-1}S_{\rm GB}+S_{\rm gh}$. The action gives the following equation of motion for A_a:

$$[g_a{}^b\Box-R_a{}^b+(\alpha^{-1}-1)\nabla_a\nabla^b]A_b=0 \ . \tag{2.4}$$

The remaining equations of motion are $\Box\eta=0$ and $\Box\bar\eta=0$ for the ghost and antighost fields.

The Feynman two-point functions for the vector potential and ghost field are defined as expectation values via a path integral. The two-point function of the ghost field satisfies

$$\Box i\langle T\bar\eta(x)\eta(x')\rangle=-\alpha^{1/2}\delta(x,x') \ , \tag{2.5}$$

where T denotes time ordering. The field rescaling $\bar{c}=\alpha^{-1/2}\bar{\eta}$, $c=\eta$, enables the action to be rewritten as
The field rescaling $\bar{c}=\alpha^{-1/2}\bar{\eta}$, $c=\eta$, enables the action to be rewritten as

$$S_{\rm gh}=\int d^4x\sqrt{g}\,\bar{c}\Box c \; . \tag{2.6}$$

We denote the Feynman two-point functions by

$$G^{\alpha}_{ab'}(x,x')=i\langle TA_a(x)A_{b'}(x')\rangle_{\alpha} \; , \tag{2.7}$$

for the vector potential, and

$$\tilde{G}(x,x')=i\langle T\bar{c}(x)c(x')\rangle \; , \tag{2.8}$$

for the rescaled ghost field. The rescaled ghost two-point function now satisfies

$$\Box\tilde{G}(x,x')=-\delta(x,x') \; . \tag{2.9}$$

The other important aspect for us is the Ward identity, which may be derived from the Becchi-Rouet-Stora (BRS) invariance of the full action under the perturbations $\delta A_a=\nabla_a c\,\delta\zeta$, $\delta\bar{c}=\alpha^{-1/2}A_a{}^{;a}\delta\zeta$, $\delta c=0$. This invariance implies the Ward identity

$$G^{\alpha}_{ab'}(x,x')^{;a}+\alpha\tilde{G}(x,x')_{;b'}=0 \; . \tag{2.10}$$

A similar equation can be obtained for the Ward identity obeyed by the two-point function of η.

Here we face the major difference between Hadamard renormalization and those renormalization methods which use what might be called analytic methods of regularization. The Hadamard method deals directly with the two-point functions of the theory and uses Wald's axioms to determine $\langle T_{\mu}{}^{\nu}\rangle_R$ as uniquely as possible. In this case regularization is achieved by point separation in the two-point functions; geometrical singular terms are then removed before letting the two points come together. The formal equivalence of this procedure to a renormalization of coupling constants in a generalized Einstein action was discussed by Christensen [5]. As can be seen above, at the level of two-point functions, there are no subtleties involved in rescaling fields. It is more convenient to work with $\bar{c}$ and c because $\tilde{G}(x,x')$ and $G^1_{ab'}$ have the standard Hadamard singularity structure. This is the procedure that was adopted in Ref. [2].

We shall discuss analytic methods in the next section, but here it is worth pursuing Hadamard renormalization with arbitrary α a stage further, to demonstrate explicitly the α independence obtained using the renormalization prescription of Ref. [2] as applied to electromagnetism.

The two-point functions $\tilde{G}(x,x')$ and $G^1_{ab'}(x,x')$ have the Hadamard form [4]

$$G^1_{ab'}(x,x')=\frac{i}{8\pi^2}\left\{\frac{\Delta^{1/2}g_{ab'}}{\sigma}+V_{ab'}\ln\sigma+W^1_{ab'}\right\} , \tag{2.11}$$

$$\tilde{G}(x,x')=\frac{i}{8\pi^2}\left\{\frac{\Delta^{1/2}}{\sigma}+\tilde{V}\ln\sigma+\tilde{W}\right\} , \tag{2.12}$$

where the singular parts are purely geometrical and all state dependence is contained in $W^1_{ab'}$ and $\tilde{W}$. $G^{\alpha}_{ab'}$ may be obtained from $G^1_{ab'}$ and $\tilde{G}$ with the relation [2]

$$G^{\alpha}_{ab'}(x,x')=G^1_{ab'}(x,x')+(\alpha-1)\nabla_a\nabla_{b'}\int d^4x''\sqrt{g''}\tilde{G}(x,x'')\tilde{G}(x'',x'). \tag{2.13}$$

As shown in Ref. [2], the convolution integral may be rewritten as

$$\int d^4x''\sqrt{g''}\tilde{G}(x,x'')\tilde{G}(x'',x')=-\left[\frac{\partial}{\partial m^2}\tilde{G}(x,x';m^2)\right]_{m^2=0} , \tag{2.14}$$

from which it follows that it may be written as

$$-\frac{i}{8\pi^2}\left\{\left[\frac{1}{2}\Delta^{1/2}+\sum_{n=1}^{\infty}\frac{1}{2n}\tilde{V}_n\sigma^n\right]\ln\sigma+\dot{W}\right\} , \tag{2.15}$$

where

$$\dot{W}(x,x')\equiv\left[\frac{\partial}{\partial m^2}\tilde{W}(x,x';m^2)\right]_{m^2=0} .$$

Expression (2.15) splits the convolution into a geometrical singular part and a state-dependent regular part. We shall define the regular part of $-8i\pi^2G^{\alpha}_{ab'}$ to be

$$W^{\alpha}_{ab'}\equiv W^1_{ab'}-(\alpha-1)\nabla_a\nabla_{b'}\dot{W} \; . \tag{2.16}$$

We now examine the dependence of the stress-energy tensor on α. The classical stress tensors of the theory are given by

$$T^{\mu\nu}_M=\frac{2}{\sqrt{g}}\frac{\delta S_M}{\delta g_{\mu\nu}}=F^{\mu}{}_aF^{\nu a}-\tfrac{1}{4}g^{\mu\nu}F_{ab}F^{ab} \; ,$$

$$T^{\mu\nu}_{\rm GB}=\frac{2}{\sqrt{g}}\frac{\delta S_{\rm GB}}{\delta g_{\mu\nu}}=-2A_a{}^{;a(\mu}A^{\nu)}+\tfrac{1}{2}g^{\mu\nu}(A_a{}^{;a})^2+g^{\mu\nu}A_a{}^{;ab}A_b \; ,$$

$$T^{\mu\nu}_{\rm gh}=\frac{2}{\sqrt{g}}\frac{\delta S_{\rm gh}}{\delta g_{\mu\nu}}=-2\bar{c}^{;(\mu}c^{;\nu)}+g^{\mu\nu}\bar{c}_{;a}c^{;a} \; .$$

The Hadamard-renormalization prescription [4] involves viewing these expression as coincidence limits of differential operators acting on the appropriate state-dependent part (W) of the appropriate two-point function, forming

$$T^{\mu\nu}_{\rm ren}=(T^{\mu\nu}_M+\alpha^{-1}T^{\mu\nu}_{\rm GB})[W^{\alpha}_{ab'}]+T^{\mu\nu}_{\rm gh}[\tilde{W}]+\text{geometrical terms} \; , \tag{2.17}$$

with the geometrical terms chosen to ensure conservation: $\nabla_{\mu}T^{\mu\nu}_{\rm ren}=0$. By "geometrical terms" we mean any polynomial expression of dimensions length^{-4} formed from the metric, the Riemann tensor, and its covariant derivatives.

Now, formally, on using the Ward identity and ghost wave equation, we may show that

$$\alpha^{-1}T^{\mu\nu}_{\rm GB}[G^{\alpha}_{ab'}]+T^{\mu\nu}_{\rm gh}[\tilde{G}]=0 \; . \tag{2.18}$$

It follows that

$$\alpha^{-1}T^{\mu\nu}_{\rm GB}[W^{\alpha}_{ab'}]+T^{\mu\nu}_{\rm gh}[\tilde{W}]\overset{*}{=}0 \; , \tag{2.19}$$

where $\overset{*}{=}$ denotes equality up to geometrical terms. This is a consequence of the equations

$$\Box \tilde{W} \overset{*}{=} 0 \ , \quad W^{\alpha}_{ab'} + \alpha \tilde{W}_{;b'} \overset{*}{=} 0 \ , \tag{2.20}$$

which follow from the ghost wave equation and Ward identity [4]. Since the total geometrical additions are determined by conservation, we may proceed as if

$$\alpha^{-1} T^{\mu\nu}_{\rm GB}[W^{\alpha}_{ab'}] + T^{\mu\nu}_{\rm gh}[\tilde{W}] = 0 \ . \tag{2.21}$$

Thus we are left to consider simply

$$\begin{aligned} T^{\mu\nu}_{\rm ren} &= T^{\mu\nu}_{M}[W^{\alpha}_{ab'}] \\ &= T^{\mu\nu}_{M}[W^{1}_{ab'}] - (\alpha-1) T^{\mu\nu}_{M}[\nabla_a \nabla_{b'} \dot{W}] \ . \end{aligned} \tag{2.22}$$

However, the second term on the right-hand side vanishes automatically by gauge invariance, and so we are left simply with the tensor $\tau^{\mu\nu} \equiv T^{\mu\nu}_{M}[W^{1}_{ab'}]$ considered in Ref. [4], where the case $\alpha=1$ was studied in detail. As the appropriate geometrical addition is determined by conservation, it follows that the renormalized stress tensor is independent of α.

III. ANALYTIC METHODS OF REGULARIZATION

Another way to demonstrate the α independence of the effective action is through the use of ζ-function regularization. The quantum part of the renormalized effective action for electromagnetism may be expressed as

$$\begin{aligned} S_{\rm effective} = &-\tfrac{1}{2} \ln \det[-g_a{}^b \Box + R_a{}^b + (1-\alpha^{-1}) \nabla_a \nabla^b] \\ &+ \ln \det(-\alpha^{-1/2} \Box) \ , \end{aligned} \tag{3.1}$$

where the first operator acts on real four-vectors and the second operator acts on real scalar functions. The first term arises from the Maxwell and gauge-breaking terms $S_M + \alpha^{-1} S_{\rm GB}$ in the action, and the second term arises from the ghost part of the action $S_{\rm gh}$.

Now, because the four-dimensional vector space of potentials A_a is the direct sum of a three-dimensional vector space of transverse (divergenceless) vectors satisfying $\nabla^a A_a = 0$ and a one-dimensional space of longitudinal vectors $\nabla_a \phi$, one can reexpress the first term in the effective action as

$$\begin{aligned} &-\tfrac{1}{2} \ln \det[-g_a{}^b \Box + R_a{}^b + (1-\alpha^{-1}) \nabla_a \nabla_b] \\ &\quad = -\tfrac{1}{2} \ln \det[-g_a{}^b \Box + R_a{}^b] - \tfrac{1}{2} \ln \det[\alpha^{-1} \Box^2] \ , \end{aligned} \tag{3.2}$$

where the first operator acts on transverse vectors and the second term acts on real scalars. It is immediately apparent that this second term is of the same form as the ghost part of the effective action, but has opposite sign, and one thus obtains a cancellation of all the α dependent terms:

$$S_{\rm effective} = -\tfrac{1}{2} \ln \det[-g_a{}^b \Box + R_a{}^b] \ , \tag{3.3}$$

where the operator acts on transverse vectors.

These determinants, which are here treated in a formal fashion, can be rigorously defined in terms of generalized ζ functions. The generalized ζ function is a function of the complex variable z and is defined on some right half-z-plane by

$$\zeta(z) \equiv \sum_n \lambda_n^{-z} \ , \tag{3.4}$$

where λ_n is the spectrum of the given operator. On the remainder of the complex z plane, the generalized ζ function is defined by analytic continuation. These generalized ζ functions can be explicitly evaluated in certain special (highly symmetric) space-times [6].

The determinant of an operator Q is now formally and rigorously defined as

$$\ln \det \mu^2 Q \equiv -\zeta'(0) + \zeta(0) \ln(\mu^2) \ , \tag{3.5}$$

where μ is a renormalization mass and $\zeta(z)$ is the zeta function associated with the operator Q. Hence the "dimension of the matrix Q" is effectively given by $\zeta(0)$ and the argument above, showing that all α dependence cancels from $S_{\rm effective}$, is perfectly rigorous. We note in passing that an incorrect treatment of $S_{\rm effective}$, in which the α dependence of the ghost term is neglected, generates precisely the same ambiguity as arises from a change of renormalization mass scale $\mu^2 \to \mu'^2$ and, hence, cannot affect physically measurable quantities.

In the case of linearized quantum gravity, similar arguments hold. Yasuda has shown [7] that the one-loop part of the effective action may be expressed as

$$\begin{aligned} S_{\rm effective} = &-\tfrac{1}{2} \ln \det[\Delta_2 + R/2] \\ &+ \tfrac{1}{2} \ln \det[\Delta_1 + R/2] \ , \end{aligned} \tag{3.6}$$

where Δ_i denotes the Lichnerowicz Laplacian acting on rank-i symmetric tensors. Once again, the α dependence cancels out of the effective action, and once again, if the incorrect normalization is choosen for the ghost part of the effective action, it corresponds in a simple way to a change of the renormalization mass scale. One can also repeat the earlier arguments of Sec. II, using the results for linearized gravity given in Ref. [2], and reach conclusions analagous to those for the electromagnetic field.

These changes in the renormalization mass scale do affect the renormalized stress-energy tensor, in the sense that they affect the coefficients of the "ambiguous" or "geometrical" terms which appear in it. The significance of this ambiguity depends upon one's point of view. Generally, workers in quantum field theory in curved spacetime regard it as a genuine ambiguity within this model; this is certainly the view taken in Hadamard renormalization. The ambiguity is absorbed in the coefficients appearing in the generalized Einstein-Hilbert action. In this case the observation of Nielsen and van Nieuwenhuizen though true is ignorable. The observation of Nielsen and van Nieuwenhuizen is, however, important to those who take a more ambitious stance: first, to those who view quantum field theory in curved spacetime as an approximate theory complete in all respects except for a knowledge of the connection between the renormalization mass scale and the Planck mass [5]; second, to those who trust that (super) symmetry can lead to full anomaly cancellation.

We hope in any case that the approach to the problem from an alternative viewpoint will be helpful.

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