TARGET SPACE MODULAR INVARIANCE AND LOW-ENERGY COUPLINGS IN ORBIFOLD COMPACTIFICATIONS ☆

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We show how the action of the target space modular group restricts the parameters of the low-energy effective action of orbifold compactifications These constraints are shown to be consistent with explicit computations of string S-matrix elements

Recently restrictions imposed by modular invariance on the scalar field configurations for general (four-dimensional) $N=1$ space–time supersymmetric field theories were investigated [1]. The analysis was restricted to one chiral superfield but is clearly valid in a broader context Modular invariance severely constrains the form of the superpotential and provides a link to the theory of modular functions. The analysis was motivated by the duality symmetry in string theory [2—5] In this context the single chiral field can be viewed as the modulus which is associated to the overall scale of the internal six-dimensional manifold and modular invariance of the action reflects duality invariance of the string theory. The superpotential of the modulus can receive non-zero contributions only from non-perturbative string effects

In this letter we show that the previous analysis applies exactly when discussing modular invariance of the $N=1$ low-energy supergravity action which arises in orbifold compactifications, including moduli and charged matter fields in both the untwisted and twisted sectors It is well known [6,7] that already at string tree level, due to non-perturbative world-sheet effects, the superpotential for the charged twisted fields depends on the moduli Therefore, the question of modular invariance of the low-energy supergravity action becomes non-trivial already at lowest order in string perturbation theory

let us briefly recall the main result of ref [1] The $N=1$ supergravity action is described (setting all gauge fields to zero) by a single function [8]

$$\mathscr{G}(t,\bar{t})=K(t,\bar{t})+\log W(t)+\log \bar{W}(\bar{t}) , \qquad (1)$$

where t is the scalar component of a chiral superfield. The discrete duality transformation $R\to 1/R$ extends in $N=1$ supergravity to the modular transformations $\mathrm{PSL}(2,\mathbb{Z})$ acting on the complex modulus t as

$$t\to \frac{at-ib}{ict+d}, \quad ad-bc=1 \qquad (2)$$

It follows that the t-moduli space $\mathrm{SU}(1,1)/\mathrm{U}(1)\cong \mathrm{H}_+$ has to be restricted to the fundamental region $\mathrm{H}_+/\mathrm{PSL}(2,\mathrm{Z})$ Modular invariance of the $N=1$ supergravity action requires $\mathscr{G}(\bar{t},t)$ to be a modular invariant function This means that $K(\bar{t},t)$ must be invariant up to a Kahler transformation which has to be absorbed by the transformation of the superpotential $W(t)$ Specifically, choosing $K(\bar{t},t)=-n\log(t+\bar{t})$ which leads to the correct Kahler metric of the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ non-linear σ-model, modular invariance can be maintained if the t-dependent superpotential transforms under modular transformations (up to a t-independent phase) like a modular function of weight $-n$, i e if

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$$W(t) \rightarrow \exp[-i\alpha(a,b,c,d)](ict+d)^{-n}W(t) \quad (3)$$

As we will see in the following, this result will not be changed by the inclusion of charged/twisted fields in the orbifold models.

Let us first consider only the compactification on the two-dimensional Z_3 orbifold based on the two-torus $T_2 = R^2/\Lambda$ where Λ is the root lattice of SU(3). This amounts to fixing one of the two complex background fields, the complex structure of T_2, and to keeping the second, $\tau = it = 2B + i\sqrt{3}\,R^2$ as a free parameter. Duality, i.e. a modular transformation (eq. (2)) on t, changes the conformal dimensions of the untwisted (winding) states, keeping however the whole untwisted Hilbert space invariant. The action of modular transformations on the twisted Hilbert space was recently given in refs. [9–11]. Here, the three twist fields σ_α ($\alpha = 1, 2, 3$) transform into linear combinations under modular transformations. Specifically, the generators S and T of the modular group act on the twist fields σ_α (apart from possible τ dependent phases which we discuss later) as

$$S = -\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \rho & \rho^2 \\ 1 & \rho^2 & \rho \end{pmatrix}, \quad T = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4)$$

($\rho = \exp(2\pi i/3)$). The twist fields are invariant under the congruence subgroup $\bar{\Gamma}(3) = \{\bar{\gamma} \in SL(2, Z) \mid \bar{\gamma} \equiv (\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}) \bmod 3\}$ but transform non-trivially under the quotient $\bar{\Gamma}/\bar{\Gamma}(3) \cong SL(2, Z_3)$ which is equal to the binary extension $\bar{\mathscr{T}}$ of the tetrahedral group $\mathscr{T}$ [10]. We have to consider the binary extension since $S^2 \neq 1$ on the twist fields σ_θ.

This discussion generalizes in a straightforward manner to the compactification on the six-dimensional Z_3 orbifold with gauge group $E_6 \times SU(3)$ [12]. (At special values of the background parameters there is an additional $U(1)^6$ gauge symmetry which we will not discuss in the following.) There are nine complex moduli $T_{i\bar{j}}$ ($i, \bar{j}) = 1, 2, 3$) in the untwisted sector. (The moduli are of course neutral under $E_6 \times SU(3)$, however not under the gauge symmetries which might be present at special values of the background parameters or, in other words, of the vacuum expectation values of the untwisted moduli.) In addition there are 27 moduli in the twisted sector which are singlets under the E_6 gauge symmetry. In fact, the twisted fields are not moduli of the orbifold compactification since their vacuum expectation values describe the blowing up of the orbifold singularities. All moduli are of the (1, 1) type and therefore correspond to changes of the Kahler structure.

The full modular group acts as a discrete coordinate transformation on moduli space which is parametrized by all moduli, untwisted and twisted. Here we will limit ourselves to that part of the modular group which acts on the untwisted moduli only. It means that we consider only those duality transformations which do not involve the blowing up of the orbifold singularities, i.e. we discuss the theory for small values of the twisted moduli. The moduli space of the untwisted moduli is locally given by the coset $SU(3,3)/[SU(3) \times SU(3) \times U(1)]$ [13]. This space can be, for example, obtained by performing a Z_3-invariant truncation of the manifold $SO(6,6)/[SO(6) \times SO(6)]$ which parametrizes the G_{ij}, B_{ij} moduli space of a six-dimensional torus compactification. The action of the generalized duality transformations on the torus moduli G_{ij}, B_{ij} is given by the discrete group $SO(6,6,Z)$ [3,4]. The maximal discrete subgroup of $SO(6,6,Z)$ which is also a subgroup of $SU(3,3)$, thus leaving the untwisted spectrum invariant, is obviously given by $SU(3,3,Z)$. It is therefore natural to conjecture that $SU(3,3,Z)$ is the complete duality group of the untwisted moduli.

Consider now the simplified case where all untwisted moduli are set to zero except for those three whose real part parametrizes the overall size of the three SU(3) tori, i.e. the three internal dilatons whose imaginary parts are the corresponding axions.

$$T_{i\bar{j}} = t_i \delta_{i\bar{j}}, \quad t_i = \sqrt{3}\,R_i^2 - 2iB_i, \quad i = 1, 2, 3 \quad (5)$$

We therefore consider from now on the modular transformations $SU(1,1,Z)^3 \subset SU(3,3,Z)$.

A similar structure arises also in other (2, 2) orbifold theories [14,15]. The possible gauge groups are (apart from the case already considered) $E_6 \times SU(2) \times U(1)$ and $E_6 \times U(1)^2$ corresponding to 5 and 3 untwisted (1, 1) moduli, parametrizing $[SU(1,1)/U(1)] \times SO(2,4)/[SO(2) \times SO(4)]$ and $[SU(1,1)/U(1)]^3$, respectively. The duality transformations on the untwisted moduli are then given by $SU(1,1,Z) \times SO(2,4,Z)$ and $SU(1,1,Z)^3$ respectively. Thus, if we keep only three moduli t_i, duality transformations are always given by SU(1, 1, Z)^3

Let us now introduce also the charged chiral multiplets and formulate the $N=1$ modular invariant supergravity action for moduli and charged fields. The t_i are accompanied by their three superpartners A_i ($i=1, 2, 3$) under the $n=2$ superconformal algebra. They transform in the **27**'s of E_6. Throughout we suppress E_6 and SU(3) indices. The 27 twisted moduli and twisted **27**'s will be denoted by C_d and A_d ($d=(\alpha\beta\gamma)=1, ..., 27$). At the special values of the moduli where the gauge group is enhanced there are extra massless matter fields. We will set them to zero.

We know from ref. [16] that the non-vanishing components of the Kähler metric, correct to lowest order in the twisted moduli and the charged fields, are

$$\mathcal{G}_{t_i\bar{t}_j}=\frac{\delta_{ij}}{(t_i+\bar{t}_i)^2}, \quad \mathcal{G}_{A_i\bar{A}_j}=\frac{\delta_{ij}}{t_i+\bar{t}_i},$$
$$\mathcal{G}_{C_d\bar{C}_e}=\frac{\delta_{d\bar{e}}}{\prod_{i=1}^{3}(t_i+\bar{t}_i)}, \quad \mathcal{G}_{A_d\bar{A}_e}=\frac{\delta_{d\bar{e}}}{\prod_{i=1}^{3}(t_i+\bar{t}_i)^{2/3}} \tag{6}$$

They can be derived from the Kähler potential

$$K=-\ln\left(\prod_{i=1}^{3}(t_i+\bar{t}_i-A_i\bar{A}_i)-C_d\bar{C}_{\bar{d}} -A_d\bar{A}_{\bar{d}}\prod_{i=1}^{3}(t_i+\bar{t}_i)^{1/3}\right) \tag{7}$$

From the known transformation law of $t_i+\bar{t}_i$ under SL(2, Z) transformations with parameters a_i, b_i, c_i, $d_i\in$ Z, $a_id_i-b_ic_i=1$,

$$t_i+\bar{t}_i\to\frac{t_i+\bar{t}_i}{|ic_it_i+d_i|^2}, \tag{8}$$

we find through the restriction that the Kähler potential has to be invariant up to a Kähler transformation, the following transformation properties of the twisted moduli and the charged fields:

$$A_i\to\frac{A_i}{ic_it_i+d_i},$$
$$C_d\to\frac{M_{de}C_e}{\prod_{i=1}^{3}(ic_it_i+d_i)},$$
$$A_d\to\frac{M_{de}A_e}{\prod_{i=1}^{3}(ic_it_i+d_i)^{2/3}}. \tag{9}$$

Here $M_{de}=M_{\alpha\delta}M_{\beta\epsilon}M_{\gamma\eta}$ and $M_{\alpha\delta}$ describes the non-trivial action of the modular group on the twisted ground states σ_δ and can be composed of a finite product of S and T, eq. (4). Invariance of the supergravity action, in particular the gravitino mass term $\exp(\mathcal{G}/2)$, requires that the superpotential W transforms as (up to a field-independent phase)

$$W\to\frac{W}{\prod_{i=1}^{3}(ic_it_i+d_i)} \tag{10}$$

So far we obtained the transformation properties eqs. (9) only from the requirement that the Kähler potential is modular invariant up to a Kähler transformation. Now we want to verify eqs. (9) by considering expressions for the string scattering amplitudes that enter the low energy supergravity lagrangian. Here we are in particular interested in the Yukawa couplings. Through supersymmetry they are related to the scalar self-interactions.

From $N=1$ supersymmetry we know the general structure of the low energy lagrangian. In particular the Yukawa couplings take the form

$$\exp(\mathcal{G}/2)\bar{\chi}^I_{\mathrm{R}}(\mathcal{G}_{IJ}-\mathcal{G}^{KL^*}\mathcal{G}_{IJL^*}\mathcal{G}_K+\mathcal{G}_I\mathcal{G}_J)\chi^J_{\mathrm{R}}+\mathrm{h.c.}, \tag{11}$$

with capital indices $I=\{i, d\}$. χ_{R} are the fermionic superpartners of the charged matter scalars. Since we are only considering massless fields we know that the superpotential is, to lowest order in the charged fields,

$$W=W_{def}(t_i, C_d)A^dA^eA^f+W_{ijk}(t_i, C_d)A^iA^jA^k, \tag{12}$$

and to this order the Yukawa couplings reduce to

$$\left(\prod_{i=1}^{3}R_i\right)^{-1}\bar{\chi}^I_{\mathrm{R}}W_{IJK}\chi^J_{\mathrm{R}}A^K \tag{13}$$

To go from eq. (11) to eq. (13) we have made a non-holomorphic field redefinition

$$\chi^I_{\mathrm{R}}\to(W/\bar{W})^{1/4}\chi^I_{\mathrm{R}} \tag{14}$$

Here we are only interested in the dependence of the Yukawa couplings on the untwisted moduli T_i. They have been calculated in string theory in refs. [6, 7]. It is important to remember that in these papers the fields were represented by vertex operators which create normalized states with canonical kinetic energy terms Fields with this normalization will be denoted by primes In this primed basis the Kahler structure is not manifest, which results e g. in the form of the non-holomorphicity of the Yukawa couplings or moduli-dependent phases in the modular transformations of the twisted fields

Consider first the untwisted charged field A_i The string theory Yukawa couplings for these fields are constants and non-vanishing only for $i \neq j \neq k$, the A_i' do not transform under modular transformations Furthermore, taking into account the Kähler metric for A_i in eq (6) we derive that the fields A_i' in the string basis are related to the supergravity basis by

$$A_i = A_i' R_i \exp(i\phi_i) , \tag{15}$$

where the phase factor has to transform under modular transformations as

$$\exp(i\phi_i) \to \left(\frac{-ic_i\bar{t}_i + d_i}{ic_i t_i + d_i}\right)^{1/2} \exp(i\phi_i) \tag{16}$$

Using this one shows the first equation in (9). We note that $\prod_{i=1}^3 \exp(i\phi_i)$ transforms like $(W/\bar{W})^{1/2}$, i.e $\prod_{i=1}^3 \exp(i\phi_i) = (W/\bar{W})^{1/2} \exp(i\alpha)$ where α is a modular invariant phase With this, one shows that W_{ijk} is a constant Actually, the supergravity action containing only the untwisted fields t_i and A_i is invariant not only under the discrete modular transformations eqs (1) and (9) with integer coefficients, but under the full continuous non-compact SU(1, 1)

Let us now turn to the twisted fields Since they are generated by the twist field vertex operators $\sigma_d' = \sigma_\alpha \sigma_\beta \sigma_\gamma$ one has to determine the already mentioned phase in the modular transformation rules of these fields For the S transformation this phase was given in ref [11] and it is straightforward to obtain the result for general modular transformations

$$\sigma_d' \to \prod_{i=1}^3 \left(\frac{ic_i t_i + d_i}{-ic_i\bar{t}_i + d_i}\right)^{-1/6} M_{de}\sigma_e' , \tag{17}$$

where M_{de} are the t_i independent matrices described above Note that this is a non-holomorphic transformation in the string basis

Using the expressions for the Kahler metrics in eq (6) we find the following relation between the supergravity and the string basis for the twisted fields

$$C_d = C_d'\left(\frac{W}{\bar{W}}\right)^{1/3} \prod_{i=1}^3 R_i, \quad A_d = A_d'\left(\frac{W}{\bar{W}}\right)^{1/6} \prod_{i=1}^3 R_i^{2/3} \tag{18}$$

These relations are up to a SL(2, Z)^3 invariant phase which cannot be determined without the knowledge of higher order interactions calculated in the string basis Using these transformation rules and also eq. (17) leads immediately to the last two equations in (9). The Yukawa couplings eq. (11) become to lowest order in the fields

$$\left(\prod_{i=1}^3 R_i\right)\bar{\chi}_{dR}' W_{def}(t_i)\chi_{eR}' A_f'$$

$$= \left(\prod_{i=1}^3 R_i\right)^{-1} \left(\frac{W}{\bar{W}}\right)^{-1/2} \bar{\chi}_{dR} W_{def}(t_i)\chi_{eR} A_f \tag{19}$$

Above transformation rules imply, comparing with eq. (8), that $W_{def}(t_i)$ has to transform under modular transformations as follows

$$W_{def}(t_i) \to \left(\prod_{i=1}^3 (ict_i + d_i)\right)$$

$$\times M_{d\bar{d}'}^{-3} M_{e\bar{e}'}^{-3} M_{f\bar{f}'}^{-3} W_{d'e'f'}(t_i) \tag{20}$$

This is indeed the behaviour under modular transformations of the string tree level Yukawa couplings for the Z_3 orbifold [11,6]

$$W_{def}(t_i) \sim \prod_{i=1}^3 \eta^2(it_i)\chi_{\alpha_i}(it_i) \tag{21}$$

χ_α are the three level one characters of SU(3) and η the Dedekind function The t independent phase in the transformation of η can be absorbed in the transformation law of the twist fields The indices on the left- and right-hand sides of eq (21) are related as follows If the three twisted **27**'s all sit, with respect to the ith torus, at the same fixed point, $\alpha_i = 0$ and χ_0 is the character of the root conjugacy class of SU(3) If they all sit at different fixed points we have either $\alpha_i = 1$ or $\bar{1}$, depending on the order χ_1 and $\chi_{\bar{1}}$ are the characters of the two fundamental weight conjugacy classes of SU(3) They are equal, i e. $\chi_1 = \chi_{\bar{1}}$ For all other combinations the space-group selection rules are not satisfied and the Yukawa couplings involving twist fields vanish in the orbifold limit

It is now also straightforward to verify that the E_6 and SU(3) D-terms are invariant under modular transformations

Let us summarize our results The inclusion of the twisted fields, which arise in the orbifold compactification scheme, in the string-induced $N=1$ supergravity action gives a concrete example of a modular invariant supersymmetric field theory with non-trivial superpotential. The Kahler potential eq. (7) reproduces the kinetic energy terms of the untwisted and twisted moduli and charged matter fields to lowest order as discussed and is modular invariant up to Kahler transformations In addition we have shown that the known expressions for the Yukawa couplings among the twisted fields satisfy the requirement of modular invariance of the supergravity effective action at lowest order in the twisted fields On the other hand, setting the twisted fields to zero and keeping only the untwisted moduli and charged fields, the realization of the modular invariance of the supergravity action is trivial in the sense that no modular functions are required to build the superpotential of the untwisted fields, to cubic order it is independent of the moduli t_i. This situation is similar to the case where one considers the action for the t-field alone with vanishing t-field superpotential. Then the supergravity action is modular invariant due to its geometrical interpretation as an $SU(1, 1)/U(1)$ nonlinear σ-model It is the introduction of the superpotential for the twisted fields (as well as for the t-field) which links the supergravity action to the theory of modular functions In fact, the lagrangian in the untwisted sector, including the Yukawa couplings, is invariant under the full $SU(1, 1)$ non-compact group. (This is a no-scale supergravity model [17]) Note that one of the $SU(1, 1)$ transformations (namely $t \to t - ib$) is the Peccei–Quinn symmetry associated to the internal axion $B \sim \mathrm{Im}\, t$ The Yukawa couplings in the twisted sector, which arise from non-perturbative world sheet effects (instantons) break the Peccei–Quinn symmetry, but not completely in the sense that a discrete shift of B is still allowed, analogously to shifts in θ-terms in self-dual gauge models [4] This residual discrete Peccei–Quinn symmetry is part of the modular invariance of the non-perturbative part of the effective lagrangian for the twisted states We have thus seen that while the untwisted Yukawa couplings are independent of the size R of the internal manifold and are thus the same as computed in field theory [18,19], the twisted Yukawa couplings are zero in σ-model perturbation theory and entirely due to instanton effects

One has also to stress that the homogeneous transformation behaviour of the Kähler potential eq (7) under modular transformations is only valid up to quadratic order in the twisted moduli, i.e up to that order in which the Kahler metric of the twisted fields is independent of these fields For smooth Calabi–Yau manifolds which arise by blowing up the orbifold singularities, higher order terms in K will be relevant, and the duality transformations will not take the simple form as displayed in eq (9).

Finally we want to comment on the relation of this work to the duality symmetries in $N=2$ Landau–Ginzburg models [10,11] which can be used to describe the string compactification on these orbifold spaces The parameters both in the Landau–Ginzburg and space–time superpotentials are essentially given by string amplitudes in the twisted sectors. The modular weights and phases of the twist fields in eqs (9) and (17) are irrelevant in the Landau–Ginzburg approach since there the superpotential is only defined up to an overall factor Therefore, the modular parameter $a(\tau)$ which appears in the Landau–Ginzburg superpotential (which can be written as the ratio of two twisted Yukawa couplings) must be a $\Gamma(3)$ invariant modular function On the other hand, the space–time superpotential eq (12) is not invariant under $\Gamma(3)$ transformations but acquires a non-trivial weight factor.

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