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Non-parametric Mean Curvature Evolution with Boundary Conditions

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Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, with smooth boundary $\partial\Omega$ and let the minimal surface operator A be given by

$$A = -D_i(a^i(p)), \qquad a^i = p^i(1+|p|^2)^{-1/2}.$$

Then the evolution equation

$$\dot{u} + Au = 0 \qquad \text{in} \quad \Omega \times [0, T) \tag{1}$$

was studied for both Dirichlet and Neumann boundary conditions (see [2, 3, 7]). Here the surfaces $M_t = \operatorname{graph} u(\cdot, t)$ move in x^{n+1}-direction with speed given by their mean curvature $H = -Au$. Recent work on parametric surfaces moving by their mean curvature suggests that it is geometrically more natural to consider surfaces whose speed in direction of their *unit normal* is equal to the mean curvature [1, 5, 6]. In the graphical setting one is then naturally led to the evolution equation

$$\dot{u} + (1+|Du|^2)^{1/2} \cdot Au = 0 \qquad \text{in} \quad \Omega \times [0, T). \tag{2}$$

We want to show here that as in the case of closed convex surfaces (see [5]), this flow has the property to level out the curvature asymptotically. We prove in Theorem 1.1 that surfaces with vertical contact angle at the boundary asymptotically converge to a constant function. The main difficulty in the proof is a time-independent gradient bound. Such an estimate is established with the Sobolev inequality and an iteration method. In case of Dirichlet boundary conditions asymptotically $u(\cdot, t)$ approaches the solution of the minimal surface equation, provided $\partial\Omega$ has positive mean curvature. This result is obtained more easily since the same barriers as in the time-independent case can be used. It also follows from the work of Lieberman [8].

The author recently learned that Eq. (2) was used by G. Dziuk to compute minimal surfaces numerically.

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1. Surfaces with Vertical Contact Angle

In the following we assume that $\partial\Omega$ is of class $C^{2,\alpha}$ and denote by $\gamma = (\gamma_1, ..., \gamma_n)$ the outer unit normal to $\partial\Omega$. We extend γ to a uniformly Lipschitz continuous vector field in Ω which is absolutely bounded by 1.

1.1. Theorem. *Let $u_0 \in C^{2,\alpha}(\bar{\Omega})$ satisfy $a^i(Du_0)\cdot\gamma_i = 0$ on $\partial\Omega$. Then the boundary value problem*

$$\begin{aligned} \dot{u} + (1+|Du|^2)^{1/2}\, Au &= 0 \qquad \text{in} \quad \Omega\times[0,\infty) \\ a^i(Du)\cdot\gamma_i &= 0 \qquad \text{on} \quad \partial\Omega\times[0,\infty) \\ u(\cdot,0) &= u_0 \end{aligned} \tag{3}$$

has a smooth solution u and $u_t = u(\cdot,t)$ converges to a constant function as $t\to\infty$.

Since Eq. (2) is uniformly parabolic as long as $|Du|$ is bounded, the existence part of Theorem 1.1 follows if we have an a priori estimate for the C^1-norm of u_t. To accomplish this we study the function

$$v = (1+|Du|^2)^{1/2}.$$

If $a^i = a^i(Du) = v^{-1} D_i u$ and $a^{ij} = \partial a^i/\partial p_j$, then we obtain from (2)

1.2. Lemma. *We have the identities*

$$\begin{aligned} \dot{v} &= a^l D_l(vH) \\ &= D_i(a^{ij} D_j v)v + Ha^l D_l v - a^{ij}a^{kl} D_i D_k u\, D_j D_l u\cdot v. \end{aligned} \tag{4}$$

Notice that the last term on the RHS is positive; in fact, we have

$$a^{ij}a^{kl} D_i D_k u\, D_j D_l u = v^{-2}\, |\delta^2 u|^2, \tag{5}$$

where $\delta^2 u$ denotes the second tangential derivatives of u (see [4, Lemma 1.3]). We can now derive a uniform supnorm bound for u on $Q = \Omega\times[0,\infty)$.

1.3. Lemma. *If $u = u(x,t)$ is a solution of (3), then*

$$\sup_Q |u| = \sup_\Omega |u_0|.$$

Proof. Let $k = \sup_\Omega |u_0|$ and $u_k = \max(u-k, 0)$. Then multiply the evolution equation for u with u_k and integrate. Then we obtain from Lemma 1.2 with $A(k) = \{x\in\Omega \mid u(x) > k\}$

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_k^2 v\,dx &= \int_{A(k)} u_k^2 a^l D_l(vH)\,dx + 2\int_{A(k)} u_k v^2 H\,dx\\ &= -\int_{A(k)} u_k^2 H^2 v\,dx - 2\int_{A(k)} u_k\,|Du|^2\,H\,dx\\ &\quad + 2\int_{A(k)} u_k v^2 H\,dx\\ &= -\int_{A(k)} u_k^2 H^2 v\,dx + 2\int_{A(k)} u_k\,D_l(a^l)\,dx\\ &= -\int_{A(k)} u_k^2 H^2 v\,dx - 2\int_{A(k)} v^{-1}\,|Du|^2\,dx \leqslant 0. \end{aligned}\tag{6}$$

Since $a^l\cdot\gamma_l = 0$ on $\partial\Omega$, no boundary integrals occurred. We conclude that $u(x,t)\leqslant k$ on Q and a similar calculation shows that $-u(x,t)\leqslant k$.

To obtain a gradient estimate, we denote by $\delta=(\delta^1,...,\delta^{n+1})$ the tangential derivatives on $M_t = \text{graph}\, u_t$ such that

$$\delta^i g = D^i g - \nu_i\cdot\sum_{k=1}^{n+1}\nu_k\cdot D^k g,\quad i=1,...,n+1,$$

where $\nu=(\nu_1,....,\nu_{n+1})$ is the exterior unit normal to M_t, i.e.,

$$\nu = v^{-1}(-D_1 u, -D_2 u, ..., -D_n u, 1).$$

We will use some estimates which were derived in [4].

1.4. LEMMA. *There is a constant c_1 depending only on $\partial\Omega$ such that for any positive function $\eta\in W^{1,\infty}(\Omega)$*

$$\int_{\partial\Omega} v\eta\,dH_{n-1}\leqslant c_1\int_{M_t}|\delta\eta| + (|H|+1)\eta\,dH_n.$$

1.5. LEMMA. *There is a constant c_2 depending only on $\partial\Omega$ such that on $\partial\Omega$ the estimate*

$$|\gamma_i\cdot a^{ij} D_j v|\leqslant c_2$$

is valid as long as $a^i(Du)\gamma_i = 0$ holds on $\partial\Omega$.

We will also need a Sobolev inequality which was shown in [4] for functions with compact support and in [9] for the case of nonvanishing boundary values:

1.6. LEMMA. *For any function $g \in C^1(\bar{\Omega})$ the inequality*

$$\left(\int_M |g|^{n/(n-1)}\, dH_n\right)^{(n-1)/n} \leqslant c_3 \left\{\int_M |\delta g| + |H|\,|g|\, dH_n + \int_{\partial\Omega} |g|\, v\, dH_{n-1}\right\}$$

is valid with a constant c_3 only depending on n.

Now we want to estimate

$$w = \log v.$$

For that purpose let $w_k = \max(w-k, 0)$ and multiply the evolution equation for v in Lemma 1.2 with w_k^2. After integration we derive

$$\begin{aligned}
\frac{d}{dt}\int_\Omega w_k^2 v\, dx &= \int_\Omega w_k^2 a^l D_l(vH)\, dx \\
&\quad + 2\int_\Omega w_k \cdot \{D_i(a^{ij} D_j v)v + Ha^l D_l v \\
&\quad - a^{ij}a^{kl} D_i D_k u\, D_j D_l u\}\, dx \\
&\leqslant -\int_\Omega H^2 w_k^2 v\, dx - 2\int_\Omega w_k a^l D_l vH\, dx \\
&\quad - 2\int_\Omega a^{ij} D_i v\, D_j v w_k\, dx + \int_{\partial\Omega} a^{ij}\cdot \gamma_i D_j v w_k v\, dH_{n-1} \\
&\quad + 2\int_\Omega w_k a^l D_l vH\, dx - 2\int_\Omega a^{ij} D_i w_k\, D_j vv\, dx.
\end{aligned}$$

Here we used that $a^i\gamma_i = 0$ on $\partial\Omega$ and (5). Notice that in a convex domain the boundary integral has the right sign and can be neglected (cf. [12]). Now observe that

$$a^{ij} D_i g\, D_j g = v^{-1}\, |\delta g|^2.$$

Then we derive from Lemma 1.5 that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega w_k^2 v\, dx &\leqslant -2\int_{A(k)} |\delta w|^2\, v\, dx - \int_{A(k)} H^2 w_k^2 v\, dx \\
&\quad + c_2 \int_{\partial\Omega} w_k v\, dH_{n-1},
\end{aligned}$$

where $A(k)$ is the set $\{(x, u(x)) \in M \mid w(x) > k\}$. Since on M $dH^n = v\,dx$, we conclude from Lemma 1.4

$$\begin{aligned}\frac{d}{dt}\int_M w_k^2\,dH_n &\leqslant -2\int_{A(k)} |\delta w|^2\,dH_n - \int_{A(k)} H^2 w_k^2\,dH_n\\ &\quad + c_2 c_1 \int_{A(k)} |\delta w_k| + (|H|+1)\,w_k\,dH_n\\ &\leqslant -\int_{A(k)} |\delta w|^2\,dH_n - \frac{1}{2}\int_{A(k)} H^2 w_k^2\,dH_n\\ &\quad + c\,|A(k)| + c\int_{A(k)} w_k^2\,dH_n. \end{aligned} \tag{7}$$

Combining now Lemmas 1.4 and 1.6 we have

$$\left(\int_M |g|^{n/(n-1)}\,dH_n\right)^{(n-1)/n} \leqslant c\left(\int_M |\delta g| + (|H|+1)\,|g|\,dH_n\right)$$

for some constant depending on c_1 and c_3. The Hölder inequality then implies that

$$\left(\int_M |g|^{2q}\,dH_n\right)^{1/q} \leqslant c\int_M |\delta g|^2 + (|H|^2+1)\,|g|^2\,dH_n, \tag{8}$$

where

$$q = \begin{cases} n/(n-2), & n > 2,\\ < \infty, & n = 2.\end{cases}$$

Now let

$$\|A(k)\|_T = \int_0^T |A(k)|\,d\tau = \int_0^T \int_{A(k)} dH_n\,d\tau.$$

Then for each $T < \infty$ we derive from (7) and (8)

$$\begin{aligned}\sup_{[0,T]} \int_{A(k)} w_k^2\,dH_n + c\int_0^T \left(\int_{A(k)} w_k^{2q}\,dH_n\right)^{1/q} dt\\ \leqslant c\,\|A(k)\|_T + c\int_0^T \int_{A(k)} w_k^2\,dH_n\,dt\end{aligned}$$

provided $k \geq k_0 = \sup_\Omega (w|_{t=0})$. From interpolation inequalities for L^p-spaces we have

$$\left(\int_{A(k)} w_k^{2q_0}\, dH_n\right)^{1/q_0} \leq \left(\int_{A(k)} w_k^{2q}\, dH_n\right)^{a/q} \left(\int_{A(k)} w_k^2\, dH_n\right)^{1-a},$$

$$\frac{1}{q_0} = \frac{a}{q} + (1-a),$$

where $a = 1/q_0$ such that $1 < q_0 < q$. So we derive

$$\left(\int_0^T \int_{A(k)} w_k^{2q_0}\, dH_n\, dt\right)^{1/q_0}$$

$$\leq c\, \|A(k)\|_T + c\left(\int_0^T \int_{A(k)} w_k^{2q_0}\, dH_n\, dt\right)^{1/q_0} \|A(k)\|_T^{1-1/q_0}. \tag{9}$$

To proceed further, we need to estimate $\|A(k)\|_T$. We use estimate (6) with $k = 0$ to obtain

$$\frac{d}{dt}\int_\Omega u^2 v\, dx \leq -\int_\Omega H^2 u^2 v\, dx - 2\int_\Omega v^{-1}\, |Du|^2\, dx.$$

It follows that

$$\int_0^T \int_\Omega v^{-1}\, |Du|^2\, dx\, dt \leq \int_\Omega u^2 v\, dx|_{t=0} = c_4 \tag{10}$$

uniformly in T. Now notice that on $A(k)$

$$v^{-1}\, |Du|^2 = v - v^{-1} \geq v(1 - e^{-k}) = \alpha(k) v.$$

Then for arbitrary but fixed $k_0 > 0$ we have $\alpha(k_0) > 0$ and therefore

$$\|A(k)\|_T \leq \int_0^T \int_{A(k)} v\, dx\, dt$$

$$\leq \alpha^{-1}(k_0) \int_0^T \int_{A(k)} v^{-1}\, |Du|^2\, dx\, dt \leq \alpha^{-1}(k_0)\, c_4 \tag{11}$$

for all $k \geq k_0 > 0$. Similarly, we can compute from Lemmata 1.2 and 1.4 that

$$\frac{d}{dt}\int_{A(k)} u^2 w_k^2 v\, dx$$

$$\leq -2\int_{A(k)} v^{-1}\, |Du|^2\, w_k^2\, dx + c_5 \int_{A(k)} v\, dx$$

with a constant c_5 depending on $\sup |u|$, c_1, and c_2. Thus we have from (11) for $k \geqslant k_0 > 0$

$$\int_0^T \int_{A(k)} w_k^2 \, dH_n \, dt \leqslant \alpha^{-1}(k_0) \int_0^T \int_{A(k)} v^{-1} |Du|^2 w_k^2 \, dx \, dt$$
$$\leqslant \alpha^{-1}(k_0) c_6.$$

In particular we get

$$\|A(k)\|_T \leqslant (k-k_0)^{-2} \int_0^T \int_{A(k)} w_{k_0}^2 \, dH_n \, dt$$
$$\leqslant (k-k_0)^{-2} \alpha^{-1}(k_0) c_6.$$

Choosing then $k_1 \geqslant k_0$ sufficiently large, we get from (9) that

$$\left(\int_0^T \int_{A(k)} w_k^{2q_0} \, dH_n \, dt \right)^{1/q_0} \leqslant c \, \|A(k)\|_T, \qquad \forall k \geqslant k_1.$$

Therefore by Hölders inequality

$$|h-k|^2 \, \|A(h)\|_T \leqslant c_7 \, \|A(k)\|_T^{2-1/q_0} = c_7 \, \|A(k)\|_T^{\gamma}, \quad \gamma > 1$$

for all $h > k \geqslant k_1$. The constant c_7 depends only on c_1, c_2, and $\sup |u|$, but not on T. By a well-known result (see, e.g., [11, Lemma 4.1]) we conclude that on $\Omega \times [0, T]$

$$w \leqslant k_1 + d, \qquad d^2 = c_7 2^{2\gamma/(\gamma+1)} \, \|A(k_1)\|_T^{\gamma-1}.$$

Together with (11) this completes the proof of the gradient estimate.

1.6. Proposition. *If u is a smooth solution of* (3) *in $\Omega \times [0, T]$ then*

$$\sup_{\Omega \times [0,T]} [|u(x,t)| + |Du(x,t)|] \leqslant c_8$$

with a constant c_8 depending only on n, u_0, and $\partial\Omega$.

Now standard results imply that (3) has a smooth solution on $\Omega \times [0, \infty)$ for arbitrary $u_0 \in C^{2,\alpha}(\bar{\Omega})$. It remains to show that u approaches a constant function as $t \to \infty$. To show this observe that

$$\frac{d}{dt} \int_\Omega v \, dx = -\int_\Omega H^2 v \, dx,$$

such that

$$\int_0^\infty \int_\Omega H^2 v \, dx \, dt \leqslant \int_\Omega v \, dx|_{t=0} = c.$$

Thus we obtain from (10) and the uniform gradient bound that

$$\int_0^T \int_\Omega |\dot{u}|^2 \, dx \, dt + \int_0^T \int_\Omega |Du|^2 \, dx \, dt \leqslant c. \tag{12}$$

In view of Proposition 1.6 our evolution equation is uniformly parabolic, implying uniform estimates on all higher derivatives of u. Thus (12) shows that u converges uniformly to a constant function, completing the proof of Theorem 1.1.

2. Dirichlet Boundary Conditions

The case of Dirichlet boundary conditions was studied by Lieberman in [8] for general quasilinear parabolic equations and his structure conditions cover our equation. For the convenience of the reader we include a short proof based on Lemma 1.2.

Assuming that φ is a function in $C^{2,\alpha}(\bar{\Omega})$, we have

2.1. Theorem. *Let $u_0 \in C^{2,\alpha}(\bar{\Omega})$ satisfy $u_0 = \varphi$ on $\partial\Omega$. If $\partial\Omega$ has nonnegative mean curvature, the boundary value problem*

$$\begin{aligned} \dot{u} + (1 + |Du|^2)^{1/2} Au &= 0 \qquad &&\text{in} \quad \Omega \times [0, \infty) \\ u &= \varphi &&\text{on} \quad \partial\Omega \\ u(\cdot, 0) &= u_0 && \end{aligned} \tag{12}$$

has a smooth solution and $u_t = u(\cdot, t)$ converges to the solution of the minimal surface equation with boundary data φ as $t \to \infty$.

Proof. Again we need uniform a priori estimates for $\sup_\Omega |u|$ and $\sup_\Omega |Du|$. From the parabolic maximum principle or an argument as in Lemma 1.3 we get immediately that

$$\sup_{\Omega \times [0, T]} |u| = \sup_\Omega |u_0|.$$

Furthermore, since $\partial\Omega$ has nonnegative mean curvature, it is well known (see, e.g., [10]) that one can construct barriers δ^+ and δ^- with

$$\begin{aligned} A\,\delta^+ &\geqslant 0, \qquad & \delta^+|_{\partial\Omega} &= \varphi \\ A\,\delta^- &\leqslant 0, \qquad & \delta^-|_{\partial\Omega} &= \varphi. \end{aligned}$$

It is easy to see that one can also achieve $\delta^- \leqslant u_0 \leqslant \delta^+$.

In view of the parabolic maximum principle we have then $\delta^- \leqslant u_t \leqslant \delta^+$ for all times. Thus there is a constant c_8 depending only on φ, u_0, and $\partial\Omega$ such that

$$|Du| \leqslant c_8 \qquad \text{uniformly on } \partial\Omega \times [0, T].$$

Applying then the parabolic maximum principle to the evolution equation for v in Lemma 1.2 we conclude that $\sup_{\Omega \times [0,T]} |Du|$ can be bounded uniformly in time by a constant depending on c_8 and $\sup_\Omega |Du_0|$. As in the proof of Theorem 1.1 the uniform a priori estimate on $\|u\|_{C^1}$ ensures the existence of a solution to (12) for all times $0 < t < \infty$. Moreover, the gradient estimate ensures that the evolution equation is uniformly parabolic, so all higher derivatives of u are bounded as well. Now we compute

$$\frac{d}{dt} \int_\Omega v\, dx = -\int_\Omega H^2 v\, dx$$

since $H = \dot{u} v^{-1} = 0$ on $\partial\Omega$. Therefore

$$\int_0^\infty \int_\Omega H^2 v\, dx\, dt \leqslant \int_\Omega v\, dx|_{t=0}.$$

Since v is already bounded, we conclude that both $\sup_\Omega |\dot{u}|$ and $\sup_\Omega |H|$ converge to zero uniformly as $t \to \infty$. This completes the proof of Theorem 2.1.

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