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Immersed Hypersurfaces with Constant Weingarten Curvature

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In a recent paper Korevaar [5] used the Alexandrov reflection principle to show that closed embedded hypersurfaces in $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$ or the upper hemisphere of $\mathbb{S}^{n+1}$ are umbilic spheres provided a certain function f of the principal curvatures $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ is constant. He only had to assume that f is positive on the positive cone $\mathscr{C}=\{\lambda|\lambda_i>0\ \forall i\}$ and that f is elliptic on the component Γ of $\{\lambda\,|\,f(\lambda)>0\}$ which contains $\mathscr{C}$. Here f is said to be elliptic if $\partial f/\partial\lambda_i>0$ for all i, $1\leqq i\leqq n$. This generalization of earlier sphere theorems (see [9] and [10] for references) cannot be extended to hypersurface immersions in view of recent counterexamples, [11].

However, assuming additional curvature conditions Walter derived in [10] global results for hypersurface immersions in a space $N^{n+1}(c)$ of constant curvature c, which have a constant higher mean curvature function H_r. Here H_r is the r-th symmetric function of the principal curvatures. It was shown that such hypersurfaces are of constant mean curvature H_1, provided they have non-negative sectional curvature *and* non-negative principal curvatures. As a consequence they have to be isoparametric with at most two distinct principal curvatures and can therefore be completely classified.

Here we show that it is not necessary to assume all principal curvatures to be non-negative. Moreover we extend Walter's result to general symmetric functions $f=f(\lambda)$. Let M^n be a smooth, connected and compact manifold without boundary. Then we have the following result.

1. Theorem. *Let $F:M^n\to N^{n+1}(c)$ be a smooth isometric hypersurface immersion of M^n into a Riemannian manifold of constant curvature c, such that the sectional curvature of M^n is non-negative. Assume that $f=f(\lambda)$ is a smooth symmetric function of $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ satisfying the following conditions*

i) *$f(\lambda)>0$ whenever $\lambda\in\mathscr{C}$.*

ii) *On the component $\Gamma(f)$ of $\{\lambda\,|\,f(\lambda)>0\}$ containing $\mathscr{C}$, f is elliptic (i.e. $\partial f/\partial\lambda_i>0\ \forall i$) and concave (i.e. $[\partial^2 f]\leqq 0$).*

If $f(\lambda)\equiv\text{const}>0$ for each principal curvature vector $\lambda=\lambda(p)$, $p\in M^n$, and if $\lambda(p_0)\in\Gamma(f)$ for some point $p_0\in M^n$, then the mean curvature H_1 of M^n is constant and hence M^n is isoparametric with at most two distinct principal curvatures.

Remarks. i) We only have to show that the mean curvature H_1 is constant: Nomizu and Smyth established in [8] that then $F(M^n)$ has parallel second fundamental form. This in turn implies that F is isoparametric with at most two distinct principal curvatures by a result of Lawson, [6].

ii) The condition that $\lambda(p_0)\in\Gamma(f)$ for at least one point $p_0\in M$ is automatically satisfied if $F(M^n)$ is contained in $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$ or in the upper hemisphere of $\mathbb{S}^{n+1}$. More generally it is sufficient that $F(M^n)$ lies in the domain of a strictly convex function, compare [10, Remark 5B.].

iii) It is shown in [4] that $f=(H_r)^{1/r}$ satisfies our conditions for all r, $1\leqq r\leqq n$. Other examples including the harmonic means functions $f(\lambda)=(\lambda_1^{-1}+\lambda_2^{-1}+\dots\lambda_n^{-1})^{-1}$ can be found in [7, Chap. 2].

For the proof of Theorem 1 we need an inequality for concave symmetric functions in the plane.

2. Lemma. *Let $f=f(x,y)$ be a symmetric function on $\mathbb{R}^2$ which is concave on an open convex and symmetric subset G. Then the inequality*

$$\left(\frac{\partial f}{\partial x}(x_0,y_0)-\frac{\partial f}{\partial y}(x_0,y_0)\right)(x_0-y_0)\leqq 0$$

holds for every $(x_0,y_0)\in G$.

Proof. Consider the straight line $\gamma(t)$ orthogonal to the diagonal $\{x=y\}$ through (x_0,y_0), parametrized by

$$\gamma(t)=\tfrac{1}{2}(x_0+y_0-2t,\,x_0+y_0+2t).$$

Since f is concave and symmetric in x and y, $f|_\gamma$ has a maximum at $t=0$, is nonincreasing in t for $t\geqq 0$ and nondecreasing in t for $t\leqq 0$. As

$$\frac{d}{dt}f(\gamma)(t)=\frac{\partial f}{\partial y}(\gamma(t))-\frac{\partial f}{\partial x}(\gamma(t)),$$

this implies the desired inequality.

Proof of Theorem 1. Since F is a smooth immersion and f is a smooth symmetric function of the λ_i's, f as a function of the principal curvature vector $\lambda(p)$, $p\in M^n$, is a smooth function on M^n. Let ∇ denote covariant differentiation on $F(M^n)$ and let f'_{ij} be the derivative of f when considered as a function of the second fundamental form $A=\{h_{ij}\}$. Since $f(h_{ij})$ is constant on M^n by assumption, the Laplace-Beltrami operator Δ applied to f yields zero. Computing in a local orthonormal frame and summing over repeated indices we obtain

$$\begin{aligned}0=\Delta f=\nabla_m\nabla_m f&=\nabla_m(f'_{ij}\nabla_m h_{ij})\\&=f'_{ij}\Delta h_{ij}+f''_{ijkl}\nabla_m h_{ij}\nabla_m h_{kl}.\end{aligned}$$

Now observe that as in [10, identity 3.16]

$$\Delta h_{kl}=\nabla_k\nabla_l H_1+R_{ikim}h_{ml}+R_{iklm}h_{mi},$$

where R_{ijkl} is the curvature tensor on $F(M^n)$. Thus we obtain

$$\begin{aligned}0=f'_{ij}\nabla_i\nabla_j H_1&+f''_{ijkl}\nabla_m h_{ij}\nabla_m h_{kl}\\&+f'_{ij}(R_{kikm}h_{mj}+R_{kijm}h_{mk}).\end{aligned}$$

Now write

$$\sigma_{ij}=R_{ijij} \quad \text{(no sum)}$$

for the sectional curvatures on M^n and rotate at a given point the coordinate system such that $\{h_{ij}\}$ is diagonal. Then the last equation reads

$$0=\sum_i \frac{\partial f}{\partial \lambda_i} \nabla_i \nabla_i H_1 + \sum_{i,k} \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_k} \nabla_m h_{ii} \nabla_m h_{kk} + \sum_{i,k} \frac{\partial f}{\partial \lambda_i} (\sigma_{ik}\lambda_i - \sigma_{ik}\lambda_k).$$

The second term on the RHS is non-positive due to the concavity of f. The last term can be written as

$$\frac{1}{2}\sum_{i,k}\left(\frac{\partial f}{\partial \lambda_k} - \frac{\partial f}{\partial \lambda_i}\right)(\lambda_k - \lambda_i)\sigma_{ik}.$$

When restricted to the variables (λ_i, λ_k), f satisfies the conditions of Lemma 2 for each pair (i, k). Since $\sigma_{ik} \geqq 0$ by assumption, the last expression is less than or equal to zero. Hence we finally conclude

$$\sum_i \frac{\partial f}{\partial \lambda_i} \nabla_i \nabla_i H_1 \geqq 0.$$

Our ellipticity assumption on f and the strong maximum principle then yield $H_1 \equiv \text{const}$, completing the proof of Theorem 1.

We may now proceed as in [10, Sect. 4] to classify all isoparametric hypersurfaces with at most two distinct principal curvatures which satisfy our curvature assumptions. The only difference appears in the case $c>0$, where we get a much larger class of examples. The additional examples are generalized Clifford tori which arise since we are not restricted to hypersurfaces with non-negative principal curvatures. This partially generalizes the result of Cheng and Yau in [3], where the special case $f=(H_2)^{1/2}$ was considered.

Using similar notation as in [10], we define the family of hypersurfaces $\mathscr{H}_c$ in $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}(c)$, and $\mathbb{S}^{n+1}(c)$ as follows

For $c>0$: $\mathscr{H}_c$ is the family of all small umbilic hyperspheres and generalized Clifford tori in $\mathbb{S}^{n+1}(c)$.

For $c=0$: $\mathscr{H}_c$ is the family of all hyperspheres and orthogonal spherical hypercylinders in $\mathbb{R}^{n+1}$.

For $c<0$: $\mathscr{H}_c$ is the family of all geodesic distance spheres, horospheres, and geodesic hypercylinders in $\mathbb{H}^{n+1}(c)$.

Furthermore, given a complete space $N^{n+1}(c)$ of constant curvature c, let $\varrho(\mathscr{H}_c)$ be the image of the family $\mathscr{H}_c$ under the associated universal covering ϱ. Then proceeding as in [10] we obtain the following consequence of Theorem 1.

3. Corollary. *Let $F: M^n \to N^{n+1}(c)$ be an isometric immersion into a complete Riemannian manifold of constant sectional curvature c such that the sectional curvature of M^n is non-negative. Let the function $f=f(\lambda)$ be as in Theorem 1 with $\lambda(p_0) \in \Gamma(f)$ for some $p_0 \in M^n$. If $f(\lambda) \equiv \text{const} > 0$ on $F(M^n)$, then $F(M^n) \in \varrho(\mathscr{H}_c)$.*

In the special case where $c>0$ and $N^{n+1}(c)=\mathbb{P}^{n+1}(c)$ is the $(n+1)$-dimensional real projective space, we conclude from [10, Lemma 4.7]:

4. Corollary. *Let $F: M^n \to \mathbb{P}^{n+1}(c)$ be an isometric immersion into $\mathbb{P}^{n+1}(c)$ and let $f = f(\lambda)$ be as in Theorem 1. If $F(M^n)$ has non-negative sectional curvature, $\lambda(p_0) \in \Gamma(f)$ for some point $p_0 \in M^n$ and if $f(\lambda) \equiv \mathrm{const} > 0$ on $F(M^n)$, then F is an embedding onto a distance sphere of radius $< \pi/2\sqrt{c}$ or a covering map onto a Clifford quadric in $\mathbb{P}^{n+1}(c)$.*

Finally we can extend Corollary 4.C in [10] to general functions $f(\lambda)$ without assuming a lower bound on the sectional curvature.

5. Corollary. *Let $F: M^n \to N^{n+1}(c)$ be an isometric hypersurface immersion into a Riemannian manifold $N^{n+1}(c)$ of constant sectional curvature c and let $f = f(\lambda)$ be as in Theorem 1. If $F(M^n)$ has strictly positive sectional curvature, $\lambda(p_0) \in \Gamma(f)$ for some point $p_0 \in M^n$ and if $f(\lambda) \equiv \mathrm{const} > 0$ on $F(M^n)$, then $F(M^n)$ is umbilic and has constant curvature.*

Proof. This is an immediate consequence of Corollary 3 since positive sectional curvature rules out all examples with a product structure.

References

1. Caffarelli, L., Nirenberg, L., Spruck, J.: Nonlinear second order elliptic equations IV, Starshaped compact Weingarten hypersurfaces. Current topics in partial differential equations. Y. Ohya, K. Kasahara, N. Shimakura (eds.), pp. 1–26. Tokyo: Kinokunze 1986
2. Cartan, E.: Familles de surfaces isoparamétrique dans les espaces à courbure constante. Ann. Mat. Pura Appl. **17**, 177–191 (1938)
3. Cheng, S.Y., Yau, S.T.: Hypersurfaces with constant scalar curvature. Math. Ann. **247**, 81–93 (1980)
4. Garding, L.: An inequality for hyperbolic polynomials. J. Math. Mech. **8**, 957–965 (1959)
5. Korevaar, N.J.: Sphere theorems via Alexandrov for constant Weingarten curvature hypersurfaces – Appendix to a note of A. Ros. J. Differ. Geom. **27**, 221–223 (1988)
6. Lawson, H.B., Jr.: Local rigidity theorems for minimal hypersurfaces. Ann. Math. **89**, 187–197 (1969)
7. Mitrinovic, D.S.: Analytic inequalities. Grundlehren der Math. Wissenschaften. Berlin Heidelberg New York: Springer 1970
8. Nomizu, K., Smyth, B.: A formula of Simons' type and hypersurfaces with constant mean curvature. J. Differ. Geom. **3**, 367–377 (1969)
9. Ros, A.: Compact hypersurfaces with constant scalar curvature and a congruence theorem. J. Differ. Geom. **27**, 215–220 (1988)
10. Walter, R.: Compact hypersurfaces with a constant higher mean curvature function. Math. Ann. **270**, 125–145 (1985)
11. Wente, H.C.: Counterexample to a conjecture of H. Hopf. Pac. J. Math. **121**, 193–243 (1986)
12. Yau, S.T.: Submanifolds with constant mean curvature, I, II. Am. J. Math. **96**, 346–366 (1974); **97**, 76–100 (1975)

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