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How far can observable relations determine a Robertson-Walker metric?

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Summary. As has been pointed out by Weinberg, Ellis and others, even an exact determination of the apparent luminosity $S(z)$ (or – equivalently – of the angular size) of distant galaxies as a function of redshift z would not suffice to determine both the expansion factor and the curvature of the (purely kinematic) Robertson-Walker metric. Nor would a determination of the number $N(z)$ of galaxies per unit solid angle up to redshift z. But jointly these two functions do determine the metric *if* $L(t)$ and $n(t)$, the intrinsic luminosity of a standard galaxy and the proper number density of galaxies in the past, are provided by theory. In that case the model is in fact overdetermined, so that either of $L(t)$, $n(t)$ could be used to find the other. Here we examine these and similar relationships by geometric methods without recourse to series expansions. In particular, we find the curvature $K_0 = k/R_0^2$ from the formula

$$K = \frac{20\pi}{3} \lim_{A\to 0} \frac{36\pi V^2 - A^3}{A^4}$$

where the volume V and surface area A of a geodesic sphere around the observer are calculated respectively from the pairs $N(z)$, $n(t)$ and $S(z)$, $L(t)$.

Key words: cosmology – gravitation – evolution of galaxies – relativity

1. Introduction

Our aim in this paper is to examine if and how observable relations can in principle determine a suitable relativistic model of the universe we inhabit. We shall assume local isotropy everywhere, and thus a Robertson-Walker (RW) spacetime, but no field equations ("kinematic approach").

Usually one imposes Einstein's (or alternative) field equations ("dynamic approach") ab initio, parametrizes the resulting cosmological solutions, and tries to use the results of observations mainly to fit those few parameters. In the case of Friedmann-Lemaitre models with matter, radiation and cosmological constant, for example, four parameters are needed. In view of the uncertainty of the presently available data, that may be the most practical strategy. Nevertheless it is a fundamental question of cosmology which observable functions, if any, in principle determine a kinematical cosmological model and could, therefore, be used to test dynamical cosmological theories. In this less restricted kinematical approach one is led to focus on observable *functions*, not just on a few *parameters*, and in the long run this may even be of practical value.

Yet in this paper our interest is primarily theoretical; we wish to examine what information the various observations contain *in principle*, and what logical connections exist between them. Our main conclusion is that the kinematic information contained in the "one and only" past light cone reaching us here and now (cf. Fig. 1) not only suffices to determine, but even overdetermines the isotropic model provided it can be supplemented by suitably detailed theoretical predictions for the evolution of galaxies (their sizes and luminosities) and for the change in their number density for causes other than the cosmic expansion.[1] Whereas this holds in principle, in practice it will be very difficult indeed to carry through the programme observationally.

A relativistic model universe that is locally isotropic everywhere is also homogeneous and is characterized by the familiar Robertson-Walker (RW) metric which we write in the form

$$ds^2 = dt^2 - R^2(t)(d\psi^2 + \rho^2 d\omega^2) \tag{1}$$

in units such that $c = 1$, where ρ is given by

$$\rho = (\sin\psi, \psi, \sinh\psi) \quad \text{for } k = (1, 0, -1), \tag{2}$$

k being the curvature index. As usual $d\omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ denotes the metric of the unit sphere so that $4\pi R^2(t)\rho^2$ is the area of the sphere $\rho =$ const. at cosmic time t. The substratum representing idealized galaxy-worldlines corresponds to the congruence $t =$ var and "our galaxy" in particular shall correspond to $\psi = 0$ and the present time to $t = 0$. Instantaneous ruler distance from us at time t is given by $R(t)\psi$ and instantaneous "area-distance" by $R(t)\rho$. We assume the cosmic sections $t =$ const. to be simply connected and complete, so that for $k = 1$, ψ ranges over the closed interval $[0, \pi]$, otherwise over $[0, \infty)$.

2. Variables on the incoming ray

Figure 1 illustrates our past light cone "now", and in particular one of its rays. Once we have decided on isotropy, all our cos-

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[1] Strictly speaking galaxy evolution and cosmic expansion cannot be separated; this does not, however, affect the following analysis.

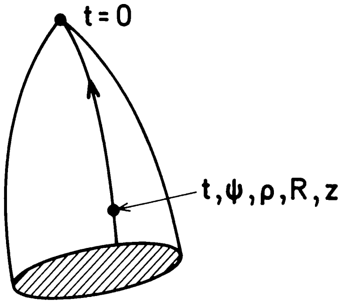

Fig. 1. Our past light cone with a light ray and the parameters associated with its events

mological information essentially rides on this one ray. Associated with any of its events are four intrinsic variables t, ψ, ρ, R, to which we add the redshift z of the galaxy at that event. By definition, $1 + z$ is the ratio of proper wave lengths at emission and reception which are proportional to the corresponding radii of the universe:

$$1 + z = \frac{R_0}{R}, \tag{3}$$

where variables without index here and below refer to the general event on the incoming ray and those with index 0 to their value at $t = 0$. Since the incoming ray is a radial null line ($d\theta = d\phi = 0$, $ds = 0$) with $d\psi/dt < 0$, Eq. (1) gives

$$\frac{d\psi}{dt} = -\frac{1}{R}. \tag{4}$$

We shall assume that the universe was expanding, $\dot{R}(t) > 0$, in at least some finite past $t_1 \le t \le 0$, which ensures that t, ψ, R, z are strictly monotonic functions of each other in their corresponding ranges; ρ can be expressed as a function of any one of these variables.

If the last period of expansion was preceded by one of contraction, we can also cover *that* by suitably patching it to the expanding one, and so on. Note that t, ψ, z and ρ vanish together, and that as a *ray*-variable ψ must not be limited above, even if $k = 1$, since a ray can come from beyond the antipode and can even have gone several times round the universe.

It is easy to convince oneself that if, instead of $R(t)$, *any* of the three variables ψ, t, R is known in terms of a second, Eq. (4) will yield the relation between the second and the third. More generally, if a relation of the form $F(\psi, t, R) = 0$ is known, (4) can be solved for each of its variables in terms of each. As a consequence then of Eqs. (2), (3) and (4) we have: given one additional (suitably solvable) relation between two or three variables belonging to different brackets in the following scheme

$$(t) \to (\psi, \rho; k) \to (R, z; R_0), \tag{5}$$

each of the five ray variables can be expressed in terms of each; however, such expressions will in general involve the parameters k and R_0, since these enter the transition from ψ to ρ and from R to z respectively.

Of the variables discussed so far, only z is directly observable here and now. The other three chief observables of cosmology are the (bolometric) flux S (total radiative energy crossing unit area in unit time) due to a distant galaxy, the solid angle ω subtended by a distant galaxy, and the number of galaxies per unit solid angle of sky with redshift no greater than z or flux no greater than S, denoted respectively by $N(z)$ and $N[S]$.

Each of S, ω, N has a progenitor: S is related to the (bolometric) luminosity L of a standard galaxy (total radiative energy emitted in unit time), ω to the average cross-sectional area F of a galaxy orthogonal to the line of sight, and N to the number n of galaxies per unit volume of space. In view of our knowledge of star formation and of the evolution of the brightness of stars, we expect L to change over cosmic times, as we do F, for dynamical reasons. Even $nR^3(t)$, which would remain constant if galaxies were conserved, is probably a variable, because of galaxy formation and galaxy cannibalism. We shall therefore consider all three of these galaxy parameters as a priori unknown functions of cosmic time: $L(t)$, $F(t)$, $n(t)$.

A theory of galaxy formation and evolution could throw light on these functions, and possibly determine them. We shall show that cosmology provides interconnections that might yield two of these function in terms of a third; thus in principle cosmological observations can test such theoretical relations.

3. The information contained in a single data pair

The following well-known equations (cf. Weinberg, 1972; Rindler, 1977) connect the observables with their respective progenitors at area-distance $R_0\rho$ and redshift z:

$$S = \frac{L}{(1+z)^2 4\pi R_0^2 \rho^2}, \tag{6}$$

$$\omega = \frac{F(1+z)^2}{R_0^2\rho^2}, \tag{7}$$

$$dN = \frac{nR_0^3\rho^2\, d\psi}{(1+z)^3}. \tag{8}$$

Given an RW-model $[R(t); k]$ we can calculate via (5) $t(z)$ and $\rho(z)$, so if $L(t)$ is known it can be converted to $L(t(z))$, and (6) then reads $S = S(z)$: the RW-model together with $L(t)$ predicts an S-z relation. Analogously it predicts ω-z and N-z relations.

The questions of primary interest to us, however, are converse ones, e.g. do *observable* relations $S(z)$, $\omega(z)$, $N(z)$ alone determine the shape, size and motion of an RW-model? The answer to this question is: no. *Any* three such functions are compatible with *any* RW-model $[R(t); k]$; such a model uniquely determines $z(t)$ and $\rho(t)$ and (6) then determines $L = L(t)$; similarly (7) and (8) respectively determine $F = F(t)$ and $n = n(t)$. The observations have thus completed the model to $[R(t); k; L(t), F(t), n(t)]$ – but they have *not* restricted its kinematics.

We can go much farther with a data pair $[S(z), L(t)]$, its first component given by observation, its second by theory. Equation (6) then constitutes a relation between z, t and ρ, and according to (5) this yields a unique $R(t)$ for any choice of the parameters R_0 and k, at least for sufficiently small $|t|$ and ρ. We shall demonstrate this in detail. First we invert the explicitly z-dependent part of (6), $(1+z)^2 S(z)$, [piecewise, if necessary, in the intervals in which $(1+z)^2 S(z)$ is monotonic – a limitation relevant in the case $k = 1$, when $S(z)$ becomes infinite at $\psi = \pi$, 2π, etc.] Such inversion yields z as a function of t and ψ and the model-parameters R_0, k:

$$z = z[t, \psi; R_0, k]. \tag{9}$$

With (3) and (4) we then get a differential equation of the form

$$\frac{d\psi}{dt} = -\frac{1}{R_0}(1+z) = f[t,\psi;R_0,k], \tag{10}$$

which can be uniquely solved, since $\psi_0 = 0$:

$$\psi = \psi(t;R_0,k). \tag{11}$$

Lastly we get $R(t;R_0,k) = -(d\psi/dt)^{-1}$, using (4).

Thus, knowledge of $[S(z), L(t)]$ determines, for each choice of R_0 and k, a corresponding expansion function $R(t)$ as far back as the data allow. But R_0 and k are left free *unless*, for example, for some finite z_A, $S(z_A)$ is infinite (practically: is exceptionally large in all directions, a contingency not observed in *our* universe!). For that can occur only in a closed universe, and then it occurs for the first time at the antipode $\psi = \pi$. With $t_A = t(z_A)$ we have in that case (cf. (3), (4)) the following two equations

$$\psi(t_A;R_0,1) = \pi, \tag{12}$$

$$\frac{d}{dt}\psi(t_A;R_0,1) = -\frac{1}{R_0}(1+z_A), \tag{13}$$

from which t_A can be eliminated and R_0 found, giving us a unique model. (This presupposes that our particle horizon has already swept over the antipode, thus making it "visible".) As a matter of fact, the functions $S(z)$, $L(t)$ belonging to a closed universe cannot in general be reproduced by an open model beyond a certain point closer than the antipode. In a closed model $(1+z)^2S(z)$ goes from ∞ to ∞ in the range $0 \leq \psi \leq \pi$, so it has a minimum in that range, and with it L/ρ^2 (cf. (6)). For simplicity, let us assume that $L(t)$ is constant. At the minimum of L/ρ^2 we have $d\rho/dz = (d\rho/dt)(dt/dz) = 0$ and since dt/dz is not zero, $d\rho/dt = 0$. This implies $k = 1$ since in an open universe $d\rho/dt = -(d\rho/d\psi)\cdot(1/R) < 0$. With this information, the "correct" value of R_0 could be found analogously to (12), (13).

For the data pair $[\omega(z), F(t)]$ the arguments and results are analogous. For $[N(z), n(t)]$ the results are again analogous but the argument is slightly more complicated. We re-write (8), using (10)(i):

$$(1+z)^2N'(z)\frac{dz}{dt} = R_0^3 n(t)\rho^2(1+z)^{-1}\frac{d\psi}{dt} = -R_0^2\rho^2 n(t). \tag{14}$$

Now from (10)(i) it is clear firstly that any function of z, for example the coefficient of dz/dt in (14), can be converted to a function of $d\psi/dt$; and secondly that $dz/dt = -R_0 d^2\psi/dt^2$. Thus (14) yields a differential equation of the form

$$\frac{d^2\psi}{dt^2} = f\left[t,\psi,\frac{d\psi}{dt};R_0,k\right],$$

which, together with the initial conditions $\psi_0 = 0$, $(d\psi/dt)_0 = -1/R_0$, can be solved uniquely for $\psi(t;R_0,k)$; and that, as before, yields $R(t;R_0,k) = -(d\psi/dt)^{-1}$.

Closed models are characterized not only by infinites of S but also of ω at $\psi = 0$, π, 2π, etc. and corresponding zeros of dN. In between, these functions have turning points, which, as in the case of $S(z)$, can lead to the unique determination of the model.

An independent method of determining the present curvature $K_0 = k/R_0^2$ and thus of k and R_0 has been suggested by Weinberg (1970). (Together with *one* of three pairs $[S,L]$, $[\omega,F]$, $[N,n]$, this would then give the model uniquely.) The method hinges on the possibility of measuring the parallax ϵ of a single distant galaxy, and it has been proposed that a line from earth to an artificial solar satellite might serve as base line b. This parallax can be related either with the brightness or angular size of that galaxy as follows (Weinberg, 1970, 1972; see also the Appendix below):

$$\frac{b}{\epsilon} = \frac{\sqrt{L}}{\sqrt{4\pi S(1+z)^2 - K_0 L}} = \frac{(1+z)\sqrt{F}}{\sqrt{\omega - K_0 F(1+z)^2}}, \tag{15}$$

and since these relations involve K_0, that would be determined if, for example, the L, S and z of that one galaxy were also known.

4. The information contained in two data pairs

One does not need to go outside the data pairs $[S,L]$, $[\omega,F]$, $[N,n]$ to obtain the full model. But we need $[N,n]$ *and* one of the other pairs to determine K_0. For this purpose it is convenient to make use of the following result from differential geometry (cf. Rindler 1977, Eqs. (7.4)): the Gaussian curvature K of a 3-space of constant curvature is given by

$$K = \frac{20\pi}{3}\lim_{A\to 0}\frac{36\pi V^2 - A^3}{A^4}, \tag{16}$$

where V is the volume and A the surface area of a geodesic sphere, in our case a sphere $\rho = \text{const}$ at $t = 0$. Its volume $4\pi R_0^3 \int \rho^2\, d\psi$ (cf. Eq. (1)) as a function of z could be obtained by integrating (8), $V = 4\pi\int n^{-1}(t)(1+z)^3 N'(z)\,dz$ if we knew $t(z)$ so as to be able to convert $n(t)$ to $n(t(z))$. Its area $4\pi R_0^2\rho^2$ is given directly by (6) or (7), but here again translations from the given functions $L(t)$ or $F(t)$ to functions of z are needed.

For this and other purposes, we derive three important model-independent identities by eliminating $R_0^2\rho^2$ pairwise between Eqs. (6), (7), (8) – using the form (14) for (8):

$$\frac{4\pi(1+z)^4 S}{\omega} = \frac{L}{F}, \tag{17}$$

$$\omega(z)N'(z)\,dz = -n(t)F(t)\,dt, \tag{18}$$

$$4\pi(1+z)^4 S(z)N'(z)\,dz = -n(t)L(t)\,dt. \tag{19}$$

For definiteness, we assume we know $[S(z), L(t)]$ and $[N(z), n(t)]$; the argument for $[\omega, F]$ in place of $[S,L]$ is analogous. Integrating (19) (with the initial condition $z_0 = 0$) then yields an equation of the form

$$f(z) = g(t),$$

which can be solved uniquely for either variable:

$$t = t(z), \tag{20}$$

$$z = z(t). \tag{21}$$

The function (20) provides the translation $L(t) \to L(t(z))$ and $n(t) \to n(t(z))$, which was all we still needed to determine K_0 using (16). The easiest way then to determine the expansion function is via (3) and (21): $R(t) = R_0(1+z(t))^{-1}$.

But, of course, knowing K_0, we could alternatively determine $R(t)$ from either $[S(z), L(t)]$ or $[N(z), n(t)]$ as shown in the last section. In a concrete case, this overdetermination of the model could serve as a check on our data and assumptions, in particular of homogeneity. Of greater interest perhaps is the now apparent possibility to *derive* two of the "theoreticals" $L(t)$, $F(t)$, $n(t)$ from the third and from the observables $S(z)$, $\omega(z)$, $N(z)$. If K_0 can be found independently, e.g. by the parallax method or in a closed universe, then, as we have seen, a single data pair such as $[S, L]$ will determine the model uniquely, hence also $z(t)$; and with that, the relevant two equations of (17)–(19) yield the other two theoreticals. If K_0 is *not* known independently, a second theoretical must be known at least for "small" values of t, so as to allow the determination of K_0 from (16), and then again all three theoreticals are fully determined.

5. Conclusion

The following schemes, in an obvious notation, summarize our findings:

$[R(t), k, L(t)] \Rightarrow S(z)$

$[R(t), k, F(t)] \Rightarrow \omega(z)$

$[R(t), k, n(t)] \Rightarrow N(z)$

$[S(z), L(t), N(z), n(t), \text{for small } t, z] \Rightarrow K_0$

$[\omega(z), F(t), N(z), n(t), \text{for small } t, z] \Rightarrow K_0$

$[S(z), L(t), K_0] \Rightarrow R(t)$

$[\omega(z), F(t), K_0] \Rightarrow R(t)$

$[N(z), n(t), K_0] \Rightarrow R(t)$

$[S(z), N(z), L(t) \text{ resp. } n(t), K_0] \Rightarrow n(t) \text{ resp. } L(t)$

$[\omega(z), N(z), F(t) \text{ resp. } n(t), K_0] \Rightarrow n(t) \text{ resp. } F(t)$

$[S(z), \omega(z), L(t) \text{ resp. } F(t), K_0] \Rightarrow F(t) \text{ resp. } L(t)$

Our results complement those of Weinberg (1970, 1972), who considered single data pairs and constant galactic functions, and those of Ellis et al. (1985) who considered a more general kinematic model (isotropy about "us" only), with less detailed results.

Appendix: distance by parallax

We here give a simple derivation of formula (15) above, under the assumption that the baseline is normal to the line of sight; the general case is only trivially different. Figure 2 shows two rays issuing from a galaxy G at angle $d\omega$ and meeting our baseline with a separation b and with total parallax ϵ. From Eq. (1), $b = R_0 \rho \, d\omega$. Local distance normal to b is measured by $R_0 \psi$, so

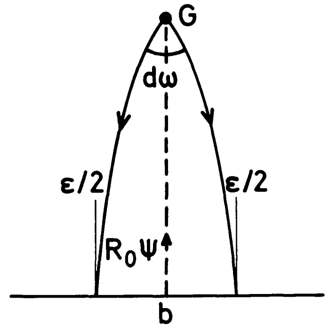

Fig. 2. Distance by parallax in an RW model

$$\epsilon = db/d(R_0\psi) = (d\rho/d\psi)\, d\omega = \sqrt{1 - k\rho^2}\, d\omega\,.$$

It follows that

$$\frac{b}{\epsilon} = \frac{R_0 \rho}{\sqrt{1 - k\rho^2}}\,,$$

and the two parts of Eq. (15) then result from substituting for ρ from (6) and (7) respectively.

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