$d = 11$ SUPERGRAVITY WITH LOCAL SO(16) INVARIANCE

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The transformation rules of $d = 11$ supergravity are given in a form which is manifestly covariant under local SO(16). The bosonic fields can be assigned to representations of E_8. This construction extends previous results where the SU(8) (and E_7) structure of $d = 11$ supergravity was exhibited and suggests further extensions involving infinite-dimensional symmetries.

Some time ago it was shown [1,2] that simple supergravity in eleven dimensions [3] admits a reformulation in which the tangent space group SO(1, 10) of the original version is replaced by SO(1, 3) × SU(8), furthermore it was shown that the bosonic fields could be assigned to representations of the noncompact group $E_{7(+7)}$, although this E_7 is not a symmetry of the new version. It is thus evident that the "hidden symmetries" that appear after reduction to lower dimensions [4] are not an artefact of the reduction but rather a property of the $d = 11$ theory itself, because all physical degrees of freedom are retained in the construction of refs. [1,2]. While the groups SU(8) and E_7 are linked with the reduction to four dimensions, other groups appear in the reduction to other dimensions [4], and it is therefore an obvious question whether the construction of refs. [1,2] can be extended to demonstrate the existence of yet more versions of $d = 11$ supergravity. In this paper it is shown that such an extension is indeed possible, and that the $d = 11$ theory has a hidden SO(16) (and E_8) structure as well [‡1]. In this way a further unification of symmetries beyond those apparent in ref. [3] is achieved, the $d = 11$ graviton and the three-index "photon", which are distinct fields in the formulation of ref. [3], are now fused into a single representation of the symmetry group, at least as far as their on-shell degrees of freedom are concerned. This means in particular that there is an entirely new (and certainly rather unusual) formulation of Einstein's theory of gravity which in this case also involves the (simply laced) exceptional Lie algebras and possibly their affine and hyperbolic extensions.

Our construction is based on a 3 + 8 split of the indices in the same way as the construction of refs. [1,2] was based on a 4 + 7 split. The necessary technology, conventions and notation have been explained at length in ref. [2], and therefore the description here will be brief (as in refs. [1,2], higher-order fermionic terms will be ignored throughout this paper). The fields of $d = 11$ supergravity are the elfbein $E_M{}^A$, a 32-component Majorana vector–spinor Ψ_M and a three-index gauge field A_{MNP} which appears only through its invariant field strength F_{MNPQ} in the equations of motion [3]. These fields depend on eleven coordinates z^M, which are subsequently split into $d = 3$ coordinates x^μ and $d = 8$ coordinates Y^m. Correspondingly, all $d = 11$ indices are decomposed into curved and flat $d = 3$ indices $\mu, \nu, ..$ and $\underline{\alpha}, \underline{\beta}$, ... [‡2], and curved and flat $d = 8$ indices m, n, ... and a, b, ..., respectively. To rewrite the theory into the new form one follows the "standard" prescription [4,6], which involves several redefinitions. One first uses the local SO(1, 10) invariance of the original theory to fix a gauge such that

‡1 For earlier speculations in this direction, see ref [5].

‡2 We underline flat $d = 3$ indices to distinguish them from the SO(8) spinor indices which will be introduced below

$$E_M{}^A = \begin{bmatrix} \Delta^{-1} e'_\mu{}^{\underline{\alpha}} & B_\mu{}^m e_m{}^a \\ 0 & e_m{}^a \end{bmatrix}, \tag{1}$$

where a Weyl rescaling factor has already been included; of course, $\Delta \equiv \det e_m{}^a$ [‡3]. The tangent space symmetry is thereby reduced to SO(1, 2) × SO(8) and compensating rotations are needed in the supersymmetry variations to maintain the gauge choice (1). As in refs. [1,2,6] our strategy will be to enlarge this symmetry to SO(1, 2) × SO(16) by the introduction of new gauge degrees of freedom.

The fermionic fields must be redefined in a similar manner. The $d = 11$ Γ-matrices are represented by

$$\hat{\Gamma}^A = \gamma^{\underline{\alpha}} \otimes \hat{\Gamma}^9 \quad \text{or} \quad \mathbf{1} \otimes \hat{\Gamma}^a, \tag{2}$$

where the $\gamma^{\underline{\alpha}}$ are hermitean two-by-two matrices which generate the $d = 3$ Clifford algebra. The following relations are useful for the explicit reduction:

$$\gamma^{\underline{\alpha}\underline{\beta}\underline{\gamma}} = -\mathrm{i}\epsilon^{\underline{\alpha}\underline{\beta}\underline{\gamma}}, \quad \hat{\Gamma}^{a_1 \ldots a_8} = \epsilon^{a_1 \ldots a_8} \hat{\Gamma}^9. \tag{3}$$

For the 16-by-16 matrices $\hat{\Gamma}^a$ and $\hat{\Gamma}^9$ we choose the representation

$$\hat{\Gamma}^a = \begin{pmatrix} 0 & \Gamma^a_{\alpha\dot{\beta}} \\ \bar{\Gamma}^a_{\dot{\alpha}\beta} & 0 \end{pmatrix}, \qquad \hat{\Gamma}^9 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{4}$$

where $\Gamma^a_{\alpha\dot{\beta}}$ and $\bar{\Gamma}^a \equiv \Gamma^{a\mathrm{T}}$ are real. The usual Clifford algebra is implied by the relation

$$\Gamma^a_{\alpha\dot{\gamma}} \bar{\Gamma}^b_{\dot{\gamma}\beta} + \Gamma^b_{\alpha\dot{\gamma}} \bar{\Gamma}^a_{\dot{\gamma}\beta} = 2\delta^{ab}\delta_{\alpha\beta}.$$

Here and in the sequel, the indices $a, \alpha, \dot{\alpha}$ characterize the three fundamental SO(8) representations 8_v, 8_s and 8_c, respectively. To rewrite the theory we also need SO(16) vector and spinor indices $I, J, \ldots$ and $A, B, \ldots$ or $\dot{A}, \dot{B}, \ldots$, respectively. Under the SO(8) × SO(8) subgroup of SO(16) these representations reduce as follows:

$$16_\mathrm{v} \to (8_\mathrm{c}, 1) \oplus (1, 8_\mathrm{s}), \qquad 128_\mathrm{c} \to (8_\mathrm{s}, 8_\mathrm{c}) \oplus (8_\mathrm{v}, 8_\mathrm{v}),$$

$$128_\mathrm{s} \to (8_\mathrm{s}, 8_\mathrm{v}) \oplus (8_\mathrm{v}, 8_\mathrm{c})$$

(these decompositions differ from the usual ones by an SO(8) triality rotation). The tangent space group SO(8) which leaves the gauge (1) fixed should be identified with the diagonal subgroup of SO(8) × SO(8). We therefore decompose the SO(16) indices in the following manner:

$$I = (\alpha, \dot{\beta}), \qquad A = (\alpha\dot{\beta}, ab), \qquad \dot{A} = (\alpha a, b\dot{\beta}). \tag{5}$$

Using these decompositions one can find an explicit representation of the SO(16) Γ-matrices $\Gamma^I_{A\dot{A}}$ and $\bar{\Gamma}^I \equiv (\Gamma^I)^\mathrm{T}$:

$$\Gamma^\alpha_{\beta\dot{\gamma},\delta b} = \delta_{\beta\delta}\Gamma^b_{\alpha\dot{\gamma}}, \qquad \Gamma^\alpha_{ab,c\dot{\delta}} = \delta_{ac}\Gamma^b_{\alpha\dot{\delta}},$$

$$\Gamma^{\dot{\alpha}}_{ab,\beta c} = \delta_{bc}\Gamma^b_{\beta\dot{\alpha}}, \qquad \Gamma^{\dot{\alpha}}_{\beta\dot{\gamma},b\dot{\delta}} = -\delta_{\dot{\gamma}\dot{\delta}}\Gamma^b_{\beta\dot{\alpha}},$$

$$\text{all other components} = 0. \tag{6}$$

From these formulas it is readily checked that indeed

$$\Gamma^I_{A\dot{C}}\bar{\Gamma}^J_{\dot{C}B} + \Gamma^J_{A\dot{C}}\bar{\Gamma}^I_{\dot{C}B} = 2\delta^{IJ}\delta_{AB}. \tag{7}$$

Other SO(16) quantities such as $\Gamma^{IJ} \equiv \Gamma^{[I}\bar{\Gamma}^{J]}$ can be easily computed from (6); for instance,

$$\Gamma^{\alpha,\beta}_{\gamma\dot{\delta},\epsilon\dot{\xi}} = \delta_{\gamma\epsilon}\bar{\Gamma}^a_{\dot{\delta}[\alpha}\Gamma^a_{\beta]\dot{\xi}}, \qquad \Gamma^{\alpha,\beta}_{ab,cd} = \delta_{ac}\Gamma^{bd}_{\alpha\beta},$$

$$\Gamma^{\alpha,\dot{\beta}}_{\gamma\dot{\delta},ab} = -\Gamma^{\dot{\beta},\alpha}_{\gamma\dot{\delta},ab} = \Gamma^a_{\gamma\dot{\beta}}\Gamma^b_{\alpha\dot{\delta}}, \qquad \text{etc.} \tag{8}$$

The redefined fermionic fields must be assigned to representations of SO(16) such that the supersymmetry transformation parameter ϵ^I and the gravitino ψ^I_μ belong to the 16-dimensional vector representation and the remaining fermionic fields to the 128-dimensional spinor representation of SO(16). The correct form of the redefined fields can, of course, only be determined through a careful analysis of the supersymmetry transformation rules as in refs. [1,2]. Anticipating the final result, we have

$$\psi'_\mu = \Delta^{-1/2} e'_\mu{}^{\underline{\alpha}} (\Psi_{\underline{\alpha}} + \gamma_{\underline{\alpha}} \hat{\Gamma}^9 \hat{\Gamma}^a \Psi_a), \qquad \epsilon' = \Delta^{1/2}\epsilon. \tag{9}$$

As in (1) we temporarily use primes to distinguish the redefined fields from the unredefined ones; these primes will be dropped in the final expressions. The spinors in (9) still have 16 internal components (and, of course, two Dirac indices which we suppress). These we can split into two sets of eight components in accordance with (5). Explicitly,

$$\psi'_{\mu\alpha} = \Delta^{-1/2} e'_\mu{}^{\underline{\alpha}} (\Psi_{\underline{\alpha}\alpha} + \gamma_{\underline{\alpha}} \Gamma^a_{\alpha\dot{\beta}} \Psi_{a\dot{\beta}}),$$

$$\psi'_{\mu\dot{\alpha}} = \Delta^{-1/2} e'_\mu{}^{\underline{\alpha}} (\Psi_{\underline{\alpha}\dot{\alpha}} - \gamma_{\underline{\alpha}} \bar{\Gamma}^a_{\dot{\alpha}\beta} \Psi_{a\beta}). \tag{10}$$

[‡3] The Weyl rescaling factor in (1) leads to the standard Einstein action in three dimensions

The other fermionic fields are assembled into a 128-component spinor $\lambda_{\dot{A}}$, which according to (5) has the components $(\lambda_{\alpha a}, \lambda_{a\dot{\alpha}})$. The correct choice is

$$\lambda_{\alpha a} = \Delta^{-1/2}[2\Psi_{a\alpha} - (\Gamma_a \bar{\Gamma}^b \Psi_b)_\alpha],$$

$$\lambda_{a\dot{\alpha}} = \Delta^{-1/2}[2\Psi_{a\dot{\alpha}} - (\bar{\Gamma}_a \Gamma^b \Psi_b)_{\dot{\alpha}}]. \qquad (11)$$

The next step is the evaluation of the supersymmetry variations of ψ'_μ and $\lambda_{\dot{A}}$. Before quoting the results, we introduce some further definitions to make the formulas less cumbersome. As in refs. [1,2], we make use of the modified derivative operator

$$\mathcal{D}_\mu \equiv \partial_\mu - B_\mu{}^m \partial_m, \qquad (12)$$

and the coefficients of anholonomity

$$\Omega'_{\alpha ab} \equiv e'_{\underline{\alpha}}{}^\mu (e_a{}^m \mathcal{D}_\mu e_{mb} - e_a{}^m \partial_m B_\mu{}^n e_{nb}), \qquad (13)$$

$$\Omega'_{\underline{\alpha}ab} \equiv 2e'_{\underline{\alpha}}{}^\mu e'_{\underline{\beta}}{}^\nu \mathcal{D}_{[\mu} B_{\nu]}{}^m e_{ma}. \qquad (14)$$

Furthermore, it is convenient to define

$$F_a \equiv \mathrm{i}\epsilon^{\underline{\alpha}\underline{\beta}\underline{\gamma}} F_{\underline{\alpha}\underline{\beta}\underline{\gamma}a}, \qquad (15)$$

$$\Omega'_{\underline{\alpha}a} \equiv \mathrm{i}\epsilon_{\underline{\alpha}}{}^{\underline{\beta}\underline{\gamma}} \Omega'_{\underline{\beta}\underline{\gamma}a}, \qquad (16)$$

$$F_{\underline{\alpha}ab} \equiv \mathrm{i}\epsilon_{\underline{\alpha}}{}^{\underline{\beta}\underline{\gamma}} F_{\underline{\beta}\underline{\gamma}ab} \qquad (17)$$

A straightforward although lengthy calculation yields the following results

$$\begin{aligned}\delta\psi'_\mu = {} & [\mathcal{D}_\mu - \tfrac{1}{4}\omega'_{\mu\underline{\alpha}\underline{\beta}}\gamma^{\underline{\alpha}\underline{\beta}} - \tfrac{1}{2}\gamma'_\mu \gamma'^\nu \partial_m B_\nu{}^m]\epsilon' \\ & + e'_\mu{}^{\underline{\alpha}}[(\tfrac{1}{4}\Omega'_{\underline{\alpha}ab} - \tfrac{1}{16}\sqrt{2}\,\Delta^{-1}F_{\underline{\alpha}ab}\hat{\Gamma}^9)\hat{\Gamma}^{ab} \\ & + \tfrac{1}{8}\Delta\Omega'_{\underline{\alpha}a}\hat{\Gamma}^9\hat{\Gamma}^a - \tfrac{1}{24}\sqrt{2}\,\Delta^{-1}F_{\underline{\alpha}abc}\hat{\Gamma}^{abc}]\epsilon' \\ & + \gamma'_\mu \Delta^{-1}[\hat{\Gamma}^9\hat{\Gamma}^m(\partial_m - \Delta^{-1}\partial_m\Delta - \tfrac{1}{4}\omega_{mbc}\hat{\Gamma}^{bc}) \\ & - \tfrac{1}{24}\sqrt{2}\,F_a\hat{\Gamma}^a - \tfrac{1}{96}\sqrt{2}\,F_{abcd}\hat{\Gamma}^9\hat{\Gamma}^{abcd}]\epsilon' \\ & + \tfrac{1}{4}e'_\mu{}^{\underline{\alpha}}\Delta^{-1}(-\mathrm{i}\epsilon_{\underline{\alpha}}{}^{\underline{\beta}\underline{\gamma}} + 2\delta_{\underline{\alpha}}^{\underline{\beta}}\gamma^{\underline{\gamma}})e'_{\underline{\beta}}{}^\nu \partial_m e'_{\nu\underline{\gamma}}\hat{\Gamma}^9\hat{\Gamma}^m\epsilon',\end{aligned} \qquad (18)$$

$$\begin{aligned}\delta[\Delta^{-1/2}(2\Psi_a - \hat{\Gamma}_a\hat{\Gamma}^b\Psi_b)] = {} & \Delta^{-1}\hat{\Gamma}^m\hat{\Gamma}_a[\partial_m - \tfrac{1}{2}\Delta^{-1}\partial_m\Delta - \tfrac{1}{4}\omega_{mbc}\hat{\Gamma}^{bc}]\epsilon' \\ & + [\tfrac{1}{48}\sqrt{2}\,\Delta^{-1}F_{bcde}\hat{\Gamma}_a{}^{bcde} \\ & + \tfrac{1}{24}\sqrt{2}\,\Delta^{-1}F_b(\hat{\Gamma}^b\hat{\Gamma}_a - 4\delta_a^b)]\epsilon' \\ & + \gamma^{\underline{\alpha}}[-\tfrac{1}{2}\Omega'_{\underline{\alpha}(bc)}\hat{\Gamma}^9\hat{\Gamma}^b\hat{\Gamma}_a\hat{\Gamma}^c + \tfrac{1}{4}\sqrt{2}\,\Delta^{-1}F_{\underline{\alpha}ab}\hat{\Gamma}^b \\ & + \tfrac{1}{8}\Delta\Omega'_{\underline{\alpha}b}\hat{\Gamma}^b\hat{\Gamma}_a + \tfrac{1}{24}\sqrt{2}\,\Delta^{-1}F_{\underline{\alpha}bcd}\hat{\Gamma}^9\hat{\Gamma}^{bcd}\hat{\Gamma}_a]\epsilon' \\ & - \tfrac{1}{4}\mathrm{i}\gamma^{\underline{\alpha}}\epsilon_{\underline{\alpha}}{}^{\underline{\beta}\underline{\gamma}}\Delta^{-1}e'_{\underline{\beta}}{}^\nu\partial_m e'_{\nu\underline{\gamma}}\hat{\Gamma}^m\hat{\Gamma}_a\epsilon'.\end{aligned} \qquad (19)$$

As before, we can decompose these 16-component equations into two sets of SO(8) spinor equations. However, the expressions are still rather unwieldy in the above form, and their further simplification now requires the identification of the proper SO(16) covariant bosonic quantities. For this step, an educated guess is necessary, but taking the hints from refs. [1, 2], one suspects that the 56-bein (e^m_{AB}, e^{mAB}) of refs. [1,2] must now be replaced by a "zweihundertachtundvierzigbein" (e^m_{IJ}, e^m_A) with flat indices in the 248-representation of E_8. In the special SO(16) gauge corresponding to (1), this 248-bein has the following components.

$$e^m_{IJ}:\quad e^m_{\alpha,\beta} = e^m_{\dot{\alpha},\dot{\beta}} = 0,$$

$$e^m_{\alpha,\dot{\beta}} = -e^m_{\dot{\beta},\alpha} = \Delta^{-1}e_a{}^m\Gamma^a_{\alpha\dot{\beta}},$$

$$e^m_A:\quad e^m_{ab} = 0,\qquad e^m_{\alpha\dot{\beta}} = \Delta^{-1}e_a{}^m\Gamma^a_{\alpha\dot{\beta}}. \qquad (20)$$

The Weyl rescaling factor Δ^{-1} in (20) is just as essential here as the corresponding factor of $\Delta^{-1/2}$ in refs. [1,2]. The 248-bein in itself is not yet sufficient to render (18) and (19) more transparent. In addition, one also needs an "E_8-gauge connection" $(Q_M^{IJ}, P_M{}^A)$ with $M = (\mu, m)$ assuming all values 1, ..., 11. The explicit expressions are found following the procedure described in ref. [2]. For $M = \mu$, one finds [‡4]

$$Q_\mu{}^{\alpha,\beta} = e'_\mu{}^{\underline{\alpha}}(\tfrac{1}{4}\Omega'_{\underline{\alpha}ab} - \tfrac{1}{16}\sqrt{2}\,\Delta^{-1}F_{\underline{\alpha}ab})\Gamma^{ab}_{\alpha\beta},$$

$$Q_\mu{}^{\dot{\alpha},\dot{\beta}} = e'_\mu{}^{\underline{\alpha}}(\tfrac{1}{4}\Omega'_{\underline{\alpha}ab} + \tfrac{1}{16}\sqrt{2}\,\Delta^{-1}F_{\underline{\alpha}ab})\Gamma^{ab}_{\dot{\alpha}\dot{\beta}}, \qquad (21)$$

‡4 We define $X_{[ab]} \equiv \tfrac{1}{2}(X_{ab} - X_{ba})$, $X_{(ab)} \equiv \tfrac{1}{2}(X_{ab} + X_{ba})$

$$Q_\mu{}^{\alpha,\dot\beta} = -Q_\mu{}^{\dot\beta,\alpha} \qquad (21\text{ cont'd})$$
$$= e'_\mu{}^{\underline\alpha}(\tfrac{1}{8}\Delta\Omega'_{\underline\alpha a}\Gamma^a_{\alpha\dot\beta} - \tfrac{1}{24}\sqrt{2}\,\Delta^{-1}F_{\underline\alpha abc}\Gamma^{abc}_{\alpha\dot\beta}),$$
$$P_\mu{}^{\alpha\dot\beta} = e'_\mu{}^{\underline\alpha}(\tfrac{1}{16}\Delta\Omega'_{\underline\alpha a}\Gamma^a_{\alpha\dot\beta} + \tfrac{1}{48}\sqrt{2}\,\Delta^{-1}F_{\underline\alpha abc}\Gamma^{abc}_{\alpha\dot\beta}),$$
$$P_\mu{}^{ab} = e'_\mu{}^{\underline\alpha}(-\tfrac{1}{2}\Omega'_{\underline\alpha(ab)} + \tfrac{1}{4}\delta_{ab}\Omega'_{\underline\alpha c}{}^{c} - \tfrac{1}{8}\sqrt{2}\,\Delta^{-1}F_{\underline\alpha ab}). \qquad (22)$$

For $M = m$, one obtains

$$Q_m{}^{\alpha,\beta} = (\tfrac{1}{4}e_a{}^n\partial_m e_{nb} + [\sqrt{2}/(7\cdot 24)]\,e_{ma}F_b)\Gamma^{ab}_{\alpha\beta},$$
$$Q_m{}^{\dot\alpha,\dot\beta} = (\tfrac{1}{4}e_a{}^n\partial_m e_{nb} - [\sqrt{2}/(7\cdot 24)]\,e_{ma}F_b)\Gamma^{ab}_{\dot\alpha\dot\beta},$$
$$Q_m{}^{\alpha,\dot\beta} = -Q_m{}^{\dot\beta,\alpha} = -\tfrac{1}{96}\sqrt{2}\,F_{mabc}\Gamma^{abc}_{\alpha\dot\beta}, \qquad (23)$$
$$P_m{}^{\alpha\dot\beta} = +\tfrac{1}{192}\sqrt{2}\,F_{mabc}\Gamma^{abc}_{\alpha\dot\beta},$$
$$P_m{}^{ab} = -\tfrac{1}{2}e_{(a}{}^n\partial_m e_{nb)} + \tfrac{1}{4}\delta_{ab}e_c{}^n\partial_m e_{nc} + [\sqrt{2}/(7\cdot 12)]\,e_{m[a}F_{b]}\,. \qquad (24)$$

Observe that $P_\mu{}^{ab}$ and $P_m{}^{ab}$ have both a symmetric and an antisymmetric part in ab and that the terms in $Q_m{}^{\alpha,\beta}$ and $Q_m{}^{\dot\alpha,\dot\beta}$ corresponding to the original tangent space group SO(8) agree, but do not coincide with the SO(8) spin connection ω_{mab}.

The expressions (21)–(24) are not completely independent but subject to the SO(16) covariant constraint ("generalized vielbein postulate")

$$\mathcal{D}_\mu e^m_{IJ} + \partial_n B_\mu{}^m e^n_{IJ} + \partial_n B_\mu{}^n e^m_{IJ} + 2Q_{\mu K[I}e^m_{J]K} + \Gamma^{IJ}_{AB}P_\mu{}^A e^{mB} = 0, \qquad (25)$$
$$\partial_m e^n_{IJ} + 2Q_{mK[I}e^n_{J]K} + \Gamma^{IJ}_{AB}P_m{}^A e^{nB} = 0, \qquad (26)$$

and a similar one for $e^m{}_A$ (in contrast to refs. [1,2], the position of the indices does not matter here).

The consistency of the construction now requires that all supersymmetry variations of the original $d = 11$ theory can be cast into a manifestly SO(16) covariant form. The SO(16) covariance of the field equations then follows by the usual arguments [1,2]. To simplify the notation, we introduce the SO(1, 2) × SO(16) covariant derivatives

$$D_\mu\epsilon^I \equi\ \ {}_\mu\epsilon^I - \tfrac{1}{4}\hat\omega_{\mu\underline{\alpha\beta}}\gamma^{\underline{\alpha\beta}}\epsilon^I + Q^{IJ}_\mu\epsilon^J, \qquad (27)$$
$$D_m\epsilon^I \equiv \partial_m\epsilon^I + Q^{IJ}_m\epsilon^J + \tfrac{1}{4}e'_{\underline\alpha}{}^{\nu}\partial_m e'_{\underline\beta\nu}\gamma^{\underline{\alpha\beta}}\epsilon^I, \qquad (27)$$

with

$$\hat\omega_{\mu\underline{\alpha\beta}} \equiv \omega'_{\mu\underline{\alpha\beta}} + 2e'_{\mu[\underline\alpha}e'_{\underline\beta]}{}^{\nu}\partial_m B_\nu{}^m. \qquad (28)$$

Omitting all primes, we are now able to rewrite the supersymmetry variations of $d = 11$ supergravity in the following form (to arrive at (33) we have to discard a local SO(16) rotation):

$$\delta e_\mu{}^{\underline\alpha} = \tfrac{1}{2}\bar\epsilon^I\gamma^{\underline\alpha}\psi^I_\mu, \qquad (29)$$
$$\delta\psi^I_\mu = (D_\mu - \tfrac{1}{2}\partial_m B_\mu{}^m)\epsilon^I + \gamma_\mu(e^m_{IJ}D_m\epsilon^J + \tfrac{1}{2}e^m_A\Gamma^{IJ}_{AB}P^B_m\epsilon^J), \qquad (30)$$
$$\delta B_\mu{}^m = \tfrac{1}{2}e^m_{IJ}\bar\epsilon^I\psi^J_\mu + \tfrac{1}{8}e^m_A\Gamma^I_{A\dot A}\bar\epsilon^I\gamma_\mu\lambda_{\dot A}, \qquad (31)$$
$$\delta\lambda_{\dot A} = 2\bar\Gamma^I_{\dot AA}\gamma^\mu\epsilon^I P^A_\mu + \bar\Gamma^I_{\dot AA}e^m_A D_m\epsilon^I + \tfrac{1}{4}e^m_{IJ}\Gamma^{IJ}_{\dot A\dot B}\bar\Gamma^K_{\dot BC}P^C_m\epsilon^K + e^m_{IJ}\bar\Gamma^I_{\dot AA}P^A_m\epsilon^J, \qquad (32)$$
$$\delta e^m_{IJ} = -\tfrac{1}{8}\Gamma^{IJ}_{AB}e^m_A\Gamma^K_{B\dot B}\bar\epsilon^K\lambda_{\dot B},$$
$$\delta e^m_A = -\tfrac{1}{8}\Gamma^{IJ}_{AB}e^m_{IJ}\Gamma^K_{B\dot B}\bar\epsilon^K\lambda_{\dot B}. \qquad (33)$$

From these results one can, of course, recover the transformation rules of $N = 16$ supergravity in three dimensions [7] by dropping all terms with ∂_m, Q_m and P_m, which vanish in the torus reduction. In the usual formulation of the $N = 16$ theory, the field $B_\mu{}^m$ does not occur since it is converted into a set of scalar fields by duality transformations. In the present context, this fact is expressed by the SO(16)-invariant constraint

$$e^m{}_A P_\mu{}^A = \mathrm{i}\epsilon_\mu{}^{\nu\rho}\mathcal{D}_\nu B_\rho{}^m, \qquad (34)$$

which can be easily verified from (16) and (22). However, the field $B_\mu{}^m$ cannot be eliminated in general because of its explicit occurrence in $\mathcal{D}_\mu$, see (12); it is only in the torus reduction that $B_\mu{}^m$ appears only through its associated field strength $\partial_{[\mu}B_{\nu]}{}^m$ and can be dualized.

The results described here provide further evidence that "hidden symmetries" appear not just in the reduction of $d = 11$ supergravity to lower dimensions but are present in the $d = 11$ theory itself; this was already one of the main conclusions of refs. [1,2]. An important question concerns the role of E_8 in our construction (and the role of E_7 in refs. [1,2]). Although many relations look "E_8 covariant", the theory clearly lacks E_8 invariance. The main reason for this is, of course, that the fermions belong to representations of SO(16) but not E_8, moreover, the constraint (34) also violates the putative E_8 invariance. One possibility already suggested in ref. [2] is that further gauge degrees of freedom may have to be added to unveil this invariance (if it is there). Further progress in this direction will also require a better understanding of what has happened to the usual formulation of Einstein gravity.

It is rather tempting at this point to speculate about the possibility of further extensions. Our results suggest the existence of a vastly larger symmetry in $d = 11$ supergravity than hitherto expected. There is little doubt that yet more versions of the theory exist involving E_6, $E_5 = SO(5, 5)$, etc. However, these are less interesting as they are, in a sense, already contained in the results obtained so far. It would be far more gratifying if one could carry the procedure still further. The next step would very likely involve the infinite-dimensional algebras E_9 and E_{10} (and perhaps E_{11} if we carry the counting to the extreme?). The emergence of E_9 in the dimensional reduction to two dimensions and the possible relevance of E_{10} were already pointed out in ref. [8]. However, it is rather doubtful that the direct dimensional reduction will yield much insight below $d = 2$ because more and more information is lost as one drops the dependence on more and more coordinates, a defect from which a construction along the lines of refs. [1,2] would not suffer. Another way to see that not much is gained by a direct reduction is to note that the tangent space group SO(9) expected in this reduction is already contained in SO(16) via the non-regular embedding of SO(9) into SO(16). More specifically, the relevant decompositions are $16_v \to 16$, $128_s \to 128$ (the SO(9) vector–spinor) and $128_c \to 44 \oplus 84$ [4,8] [‡5]. This SO(9) coincides with the transverse subgroup of SO(1, 10) that classifies the on-shell states of $d = 11$ supergravity. In the present formulation there are extra fields $e_\mu{}^{\alpha}$, $\psi_\mu{}^I$ and $B_\mu{}^m$, which, in the reduction to three dimensions, carry no dynamical degrees of freedom or are dependent. We are thus led to conjecture that extensions beyond SO(16) will not only lead to infinite-dimensional symmetries but also involve *off-shell* classifications. Thus, clarifying the role of the exceptional groups in the present construction may also shed new light on the still unsolved problem of extending $d = 11$ supergravity off-shell. Finally, all of this hints at a theory "beyond $d = 11$ supergravity" (not necessarily a string theory!) whose spectrum forms a single irreducible representation of the relevant infinite-dimensional symmetry group. Clearly, much work remains ahead.

‡5 To be completely explicit, the 44 and 84 representations of SO(9) are given by

$$\{X_{(ab)}, \Gamma^{a}_{\alpha\dot\beta} X_{\alpha\dot\beta}\} \quad \text{and} \quad \{X_{[ab]}, \Gamma^{abc}_{\alpha\dot\beta} X_{\alpha\dot\beta}\},$$

respectively, if the SO(8) components of the 128_c representation of SO(16) are denoted by $\{X_{\alpha\dot\beta}, X_{ab}\}$.

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