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Self-consistent probabilities for gravitational lensing in inhomogeneous universes

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Summary. Gravitational bending of light influences observations of cosmologically distant sources, e.g. by producing an amplification bias in source counts. To assess the importance of any such effect one has to determine its probability, given a random distribution of "lenses". For this purpose we exhibit the assumptions underlying the Dyer-Roeder description of light propagation in a clumpy universe and point out its weaknesses. Assuming this description as a working hypothesis and combining it with the lens equation we derive a formula for the probability of lens effects which differs from the one hitherto accepted. We show that the previous formula is inconsistent with the assumed law of light propagation in a clumpy universe, and that, compared to the new formula, the former one significantly underestimates the importance of light bending.

Key words: gravitation – cosmology – source counts – gravitational lensing

1. Introduction

The theory of the gravitational lens effect is mainly concerned with (i) the investigation of properties of single, isolated mass distributions (e.g., Refsdal, 1964; Bourassa et al., 1973; Bourassa and Kantowski, 1975; Dyer and Roeder, 1980; Young et al., 1980, 1981; Schneider and Weiß, 1986) or (ii) the statistical properties of an ensemble of lenses, either at equal redshift – i.e., compact objects in the halo of a galaxy (Canizares, 1981; Vietri and Ostriker, 1983; Schneider, 1986a, b, c) – or randomly distributed throughout the universe (e.g., Press and Gunn, 1973; Bourassa and Kantowski, 1976; Turner, 1980; Canizares, 1982; Peacock, 1982; Turner et al., 1984; Dyer, 1984; Vietri, 1985; for a review, see Peacock, 1983). There is a fundamental difficulty intrinsic to the latter problem: gravitational lensing takes place only in a universe in which matter is clumped. However, no solution of the equations of General Relativity is known which is appropriate to describe such a clumpy universe.

A way out of this difficulty has now become very common and was first introduced by Press and Gunn (1973), following the work of Dyer and Roeder (1972, 1973). Starting from the focussing equation of geometrical optics in curved space-times (e.g., Misner et al., 1973; Chapter 22.5), they derived a differential equation for the diameter of a bundle of light rays, in terms of the affine parameter along the central ray, which propagates through a space-time in which the local density is diluted with respect to the standard Friedmann-Lemaitre (FL) model (e.g., Weinberg, 1972). Although this description of light propagation in a clumpy universe has not been derived by a perturbation approach or otherwise from general relativity, but is based on a number of *more or less plausible ad hoc assumptions*, these "Dyer-Roeder-distances" are usually applied to the gravitational lens equation, due to the lack of any more founded theory.

Dyer and Roeder assume that on a large scale even a clumpy universe can be described by a FL-metric, despite the fact that its density agrees almost nowhere with that of the FL-model. Although the validity of this assumption is, at least, questionable, it is the only way known out of the dilemma mentioned above. The absence of a reliable model of the metric fluctuations present in a clumpy universe, and thus of a realistic account of light propagation, is an obstacle not only for gravitational lens theory, but for cosmology in general (e.g., Ellis, 1984).

This paper deals with the effect on observable properties of background sources of randomly distributed lenses. In deriving the probability for a specific influence of this ensemble on background sources, one has to integrate a differential single lens probability over the ensemble. It is at this point where the Dyer-Roeder model enters. Whereas light bundles are described by the Dyer-Roeder differential equation (as a working hypothesis), the volume element dV of the universe has to be computed from the Friedmann-Lemaitre metric, since that is assumed to describe the large-scale structure. For the selfconsistency of statistical gravitational lens theory it is important to distinguish carefully between the following two measures of distance: the Dyer-Roeder angular-diameter distance (see Sect. 2 below), supposed to describe the propagation of ray bundles not going through clumps, and the Friedmann-Lemaitre angular-diameter distance used to calculate the total area of a sphere of constant redshift (e.g., Weinberg, 1972).

In Sect. 2 we outline and discuss the Dyer-Roeder model. Then, in Sect. 3 a general equation for the probabilities in statistical lens theory is derived, which deviates from similar equations used hitherto (e.g., Press and Gunn, 1973; Peacock, 1982; Canizares, 1982). It is shown that the difference arises from the different roles the Dyer-Roeder and Friedmann-Lemaitre distance play in the calculations, and that the "old" equations are

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inconsistent with the Dyer-Roeder description. Finally, we briefly summarize and discuss this result.

2. Clumpy universe and flux conservation

In this Section we briefly summarize the main results of standard and clumpy cosmology; for details, the reader is referred to Weinberg (1972) and Dyer and Roeder (1972, 1973).

Standard Friedmann-Lemaitre cosmology models the universe as being filled with a homogeneous, isotropic perfect fluid; the FL metric

$$ds^2 = c^2 dt^2 - R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right\} \tag{1}$$

is then an exact solution of Einstein's field equations, provided $R(t)$ satisfies the Friedmann equation $\dot{R}^2 = (8\pi G/3)R^2\rho - k$, where ρ is the density of matter and p the pressure at time t, and matter expands adiabatically, $(\rho c^2 R^3) + p(R^3) = 0$. (Here and in the following, we set the cosmological constant $\Lambda = 0$.) The metric (1) is parameterized by the present value of the Hubble-constant, H_0, and the present density of the universe, ρ_0, in units of the "critical" density $\rho_c = 3H_0^2/(8\pi G)$. The parameter k in (1) depends on the density parameter $\Omega \equiv \rho_0/\rho_c$; one has $k = +1$ $(0. -1)$ for $\Omega > 1$ $(=1. <1)$.
Distances in cosmology are defined "operationally" (see Weinberg, 1972): a source of diameter B at a redshift z is seen under the angle

$$\theta = B/D(z) \tag{2}$$

which *defines* the angular diameter distance $D(z)$. In an FL-dust universe ($p = 0$), one has

$$D(z) = D_1(z) = c/H_0 \cdot r_1(z)\,, \tag{3}$$

where

$$r_1(z) = 2\,\frac{\Omega z + (\Omega - 2)[(1+\Omega z)^{1/2} - 1]}{\Omega^2(1+z)^2}\,. \tag{4}$$

The radial physical distance interval dr_{prop} which corresponds to a redshift interval dz at z is

$$dr_{\text{prop}} = (c/H_0)\cdot(1+z)^{-2}(1+\Omega z)^{-1/2}\,dz\,; \tag{5}$$

e.g., a spherical shell of thickness dz at z has the volume

$$dV = 4\pi D_1^2(z)\cdot\frac{dr_{\text{prop}}}{dz}\,dz\,. \tag{6}$$

Now, the real universe certainly is not filled with a homogeneous perfect fluid. Model metrics which satisfy Einstein's equations and which could represent an inhomogeneous, but statistically homogeneous universe resembling reality, i.e. having a relatively large density in "small" regions and nearly vanishing density in "large" domains, are not known – the Swiss Cheese model (cf. Kantowski, 1969; Gunn, 1967) the globule model (e.g., Nottale, 1984) and FL models with infinitesimal perturbations are not appropriate for gravitational lens considerations. Zel'dovich (1964) noted that the propagation of ray bundles in an inhomogeneous universe must differ from that in the Fl model. Due to the lack of any realistic metric of an inhomogeneous (clumpy) universe, it is now usual to assume that a smoothed-out background metric (1) exists relative to which the inhomogeneities are randomly distributed and proper times of comoving observers and large spatial volumes coincide with those described by (1). In fact, Carfora and Marzuoli (1984) have devised a smoothing procedure which associates with any assumed, inhomogeneous model universe having a topologically spherical space-section and obeying Einstein's equation, an FL-metric which can reasonably be called a smoothed-out version of it. According to their procedure the density and pressure of the associated FL-model differ from the spatial averages of the corresponding quantities of the original, inhomogeneous model by terms due to spatial fluctuations of the expansion rate, shear motion and gravitational waves. Their results thus illustrate the problem under discussion, but their method does not enable one to construct metrics of inhomogeneous universes.

However, light rays are null geodesics not of the background metric (1) but of the unknown perturbed metric. Dyer and Roeder (1972, 1973) assumed that a mass fraction $\tilde{\alpha}$ of all matter is distributed homogeneously, whereas the rest is bound in clumps. Then, the matter density within a light bundle is only a fraction $\tilde{\alpha}$ of the average density, as long as the bundle does not pass through clumps. The focussing of rays is therefore reduced and, as a consequence, the angular diameter distance to a source whose line-of-sight is well away from all clumps at redshift z differs from (3). Starting from the focussing equation (cf. Misner et al., 1973, Chapter 22) which describes the change of the cross sectional area of a light bundle along its propagation, and assuming that (i) the shear on a ray bundle which is well away from all clumps is negligible and that (ii) the relation between the affine parameter λ along a beam and the coordinate time t agrees to a sufficient approximation with that in the FL model (i.e., $d\lambda \propto R(t)\,dt$), Dyer and Roeder derived a differential equation for the diameter of a ray bundle which propagates through a region of reduced density in a clumpy universe:

$$\frac{d^2D}{dz^2}(1+z)(1+\Omega z) + \frac{dD}{dz}\left(\frac{7}{2}\Omega z + \frac{\Omega}{2} + 3\right) + \frac{3}{2}\tilde{\alpha}\Omega D = 0\,. \tag{7}$$

The angular diameter distance is then

$$D_{\tilde{\alpha}}(z) = c/H_0 r_{\tilde{\alpha}}(z)\,, \tag{8}$$

where $r_{\tilde{\alpha}}(z)$ is the solution of (7) which satisfies $r_{\tilde{\alpha}}(z=0) = 0$ and $dr_{\tilde{\alpha}}/dz$ $(z=0) = 1$ (the Hubble law). Equation (7) implies that $D_\beta(z) > D_\gamma(z)$ if $\gamma > \beta$. Especially, $D_{\tilde{\alpha}}(z)$ is larger than $D_1(z)$ for all $\tilde{\alpha} < 1$. Since the luminosity distance $D_L(z)$, which is defined by (flux from a source at z) = (luminosity)/$(4\pi D_L^2(z))$, is generally $D_L = (1+z)^2 D$ (Etherington, 1933), a source with given luminosity and redshift appears fainter in a clumpy universe than it would appear in the smooth FL universe, if its line-of-sight is well away from all clumps. Sources whose light rays pass near a clump are amplified; since photons are neither created nor destroyed while they propagate through the universe this amplification must *exactly* compensate the dimming due to less matter in beams (Weinberg, 1976).

The assumptions on which the Dyer and Roeder calculations are based do not seem too plausible to us, for the metric of a clumpy universe agrees nowhere with (1); its density and curvature are smaller than those of its smoothed-out FL-background metric in most regions, whereas they are strongly enhanced in small regions (see also Alcock and Anderson, 1985). Therefore, particularly the assumption that the relation between affine parameter and coordinate time (or redshift) agrees with that in the FL universe appears to be questionable.

At any rate, the validity of the Dyer-Roeder equation for the propagation of light beams through a clumpy universe has not been established within General Relativity. The best one can achieve with this model is a *self-consistent* description of ray-bundles.

We define the amplification I of a source at redshift z with luminosity L as the ratio of its actual flux S and the flux S^* one obtains from the definition of the luminosity distance in the clumpy universe,

$$S^* = \frac{L}{4\pi D_L^2(z)} = \frac{L}{4\pi(1+z)^4 D_{\tilde{\alpha}}^2(z)}; \tag{9}$$

$$I = S/S^* . \tag{10}$$

According to the argument of Weinberg, the mean amplification $\langle I \rangle$ of an ensemble of sources at redshift z is

$$\langle I \rangle = \frac{D_{\tilde{\alpha}}^2(z)}{D_1^2(z)} . \tag{11}$$

Consider now a bundle of rays with vertex at the observer and solid angle $d\omega$. Following Dyer and Roeder we assume that its cross-sectional area at a redshift z is $dA = d\omega\, D_{\tilde{\alpha}}^2(z)$, if this bundle does not pass too close to intervening clumps. If all bundles from the observer to the surface with redshift z had this property, the area of that surface would be $4\pi D_{\tilde{\alpha}}^2(z)$. However since, according to the assumption concerning the role of the background metric (1), this area actually is $4\pi D_1^2(z)$, the light bundles have to be compressed, on the average, by a factor $\langle I \rangle^{-1}$ in area. Hence, as was to be expected, (7) is certainly not valid for beams encountering clumps, and (11) is necessary for self-consistency of the model.

3. Evaluation of probabilities

In this Section, the basic equations for gravitational lens statistics applied to a cosmologically distributed ensemble of lenses are reconsidered. It is found that the equation for the probability distribution derived below differs from that used in some previous papers (e.g., Canizares, 1982; Peacock, 1982; Turner et al., 1984). This difference derives from the different parts the distances $D_{\tilde{\alpha}}(z)$ and $D_1(z)$ play in the problem: While the angular diameter distances of a clumpy universe $D_\alpha(z)$ are those distances which enter the lens equation (e.g., Press and Gunn, 1973; Schneider, 1985), the distances $D_1(z)$ are used to describe "large" (i.e. on scales $\gg$ galaxy clusters) volume elements and areas of spheres of constant redshift [see Eq. (6)], since it is assumed that the large scale geometry of the universe is still appropriately described by (1). In view of the discussion of the foregoing section, a comparison of the two equations for the probability function cannot decide which one (if any) is *correct*, but only which one is *consistent* with the assumptions of the Dyer-Roeder model. (Our criticism does not apply to (Dyer, 1984) where, however, no general probability distribution is derived but other aspects of lens statistics are emphasized.)

Consider an ensemble of cosmologically distributed lenses; for simplicity of the discussion, only a single population of lenses is considered. Let the lens mass be M and assume that lenses are conserved, i.e. their number density is $n = n_0(1+z)^3$, where n_0 is the present value of n, related to the cosmological parameters by

$$n_0 = \Omega(1-\tilde{\alpha})\frac{3H_0^2}{8\pi GM} . \tag{12}$$

Next, the lens equation is written in a general form. Let $\boldsymbol{\chi}$ and $\boldsymbol{\xi}$ be position vectors in the source and lens plane, respectively, and let $\boldsymbol{\xi}_*$ describe the position of the lens. The lens property is given by the deflection $\hat{\boldsymbol{\alpha}}(\Delta\boldsymbol{\xi})$ a light ray experiences if it traverses the lens plane at $\boldsymbol{\xi} \equiv \boldsymbol{\xi}_* + \Delta\boldsymbol{\xi}$. The lens equation relates the source position to the impact vector of a corresponding light ray in the lens plane.

$$\boldsymbol{\chi} = \frac{D_s}{D_d}\boldsymbol{\xi} - D_{ds}\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi} - \boldsymbol{\xi}_*) , \tag{13}$$

where

$$D_d \equiv D_{\tilde{\alpha}}(z_d) \equiv c/H_0 r_d , \tag{14a}$$

$$D_s \equiv D_{\tilde{\alpha}}(z_s) \equiv c/H_0 r_s , \tag{14b}$$

$$D_{ds} \equiv c/H_0 r_{ds} ; \tag{14c}$$

here, z_d and z_s are the redshifts of lens and source, respectively, and r_{ds} is the value at z_s of the solution to (7) which satisfies $r(z_d) = 0$ and $(dr/dz)(z_d) = (1+z_d)^{-2}(1+\Omega z_d)^{-1/2}$.

It is convenient to introduce dimensionless quantities in the source and lens plane by defining the length

$$\xi_0 = \left[\frac{4GM}{c^2}\frac{D_d D_{ds}}{D_s}\right]^{1/2} \tag{15}$$

and

$$\boldsymbol{x} = (\boldsymbol{\chi}/\xi_0) D_d/D_s , \tag{16a}$$

$$\boldsymbol{r} = \boldsymbol{\xi}/\xi_0 , \tag{16b}$$

$$\boldsymbol{r}_* = \boldsymbol{\xi}_*/\xi_0 , \tag{16c}$$

$$\Delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}_* , \tag{16d}$$

$$\boldsymbol{\alpha}(\Delta\boldsymbol{r}) = \frac{D_d D_{ds}}{D_s \xi_0}\hat{\boldsymbol{\alpha}}(\xi_0 \Delta\boldsymbol{r}) ; \tag{16e}$$

then, the lens equation takes the form

$$(\boldsymbol{x} - \boldsymbol{r}_*) = \Delta\boldsymbol{r} - \boldsymbol{\alpha}(\Delta\boldsymbol{r}) . \tag{17}$$

One can now ask for a certain property Q of the lens mapping; e.g. "to cause an amplification greater than I", or "to cause multiple images such that the flux ratio of the brightest images is less than q". Since the properties of the lens mapping, for fixed z_d, z_s, lens and source model, only depend on the relative positions of observer, source and lens, the property Q is satisfied if the relative position $\boldsymbol{x} - \boldsymbol{r}_*$ lies in some region $A(Q) \in R^2$. The area of $A(Q)$, a_Q, is the dimensionless Q-cross-section of the lens under consideration. In general, a_Q depends on the redshifts of source and lens, as well as on the lens and source model. The corresponding dimensional cross section in the lens plane is

$$A_d = \xi_0^2 a_Q , \tag{18}$$

and in the source plane

$$A_s = \xi_0^2 (D_s/D_d)^2 a_Q . \tag{19}$$

We now outline the derivation of the probability $P(Q)$ that the lens population causes the property Q for a certain kind of sources at redshift z_s.

3.1. The "standard" derivation

Here, one considers a random line-of-sight to a source at z_s. The expected number of lenses in the redshift interval dz_d around z_d which cause the property Q is taken to be

$$dP = n_0(1+z_d)^3 A_d \frac{dr_{prop}}{dz_d} dz_d\,,$$

and the total number expected is obtained by integrating dP from 0 to z_s. Using (5), (12), (14), (15), and (18), this is

$$P_1(Q) = \int_0^{z_s} dp = \frac{3}{2\pi}\Omega(1-\tilde{\alpha})\int_0^{z_s} dz_d \frac{1+z_d}{(1+\Omega z_d)^{1/2}} \frac{r_d r_{ds}}{r_s} a_Q\,, \tag{20}$$

(cf. Press and Gunn, 1973; Peacock, 1982; Canizares, 1982; Turner et al., 1984). If the expected number of lenses $P_1(Q)$ is $\ll 1$, then $P_1(Q)$ can be considered as the probability defined below Eq. (19).

3.2. The "new" derivation

Instead of integrating along the line-of-sight, we now calculate the total Q-cross-section of all lenses on the source sphere $z = z_s$. This is obtained by integration of A_s over all lenses. In a shell of thickness dz_d at z_d, there are $n_0(1+z_d)^3(dr_{prop}/dz_d)4\pi D_1^2(z_d)\,dz_d$ lenses; hence the total Q-cross-section is, by (3) and (5),

$$A_{tot} = 4\pi n_0\left(\frac{c}{H_0}\right)^3 \int_0^{z_s} dz_d \frac{1+z_d}{(1+\Omega z_d)^{1/2}} r_1^2(z_d) A_s\,, \tag{21}$$

as long as the single cross sections A_s do not overlap. Since the lenses are assumed to be distributed randomly, the condition of no significant overlap simply is $A_{tot} \ll$ area of the sphere $(z = z_s) = 4\pi D_1^2(z_s)$. The probability is then simply the ratio of the total cross section A_{tot} and the area of the sphere $z = z_s$,

$$\begin{aligned} P_2(Q) &= \frac{3}{2\pi}\Omega(1-\tilde{\alpha})\left[\frac{r_s}{r_1(z_s)}\right]^2 \int_0^{z_s} dz_d \frac{1+z_d}{(1+\Omega z_d)^{1/2}} \\ &\quad \left[\frac{r_1(z_d)}{r_d}\right]^2 \frac{r_d r_{ds}}{r_s} a_Q \\ &= \frac{3}{2\pi}\Omega(1-\tilde{\alpha})\int_0^{z_s} dz_d \frac{1+z}{(1+\Omega z_d)^{1/2}} \frac{r_d r_{ds}}{r_s} a_Q \frac{\langle I\rangle_s}{\langle I\rangle_d}\,. \end{aligned} \tag{22}$$

The two expressions (20) and (22) differ by the factors $[r_s/r_1(z_s)]^2$ and $[r_1(z_d)/r_d]^2$ or $\langle I\rangle_s/\langle I\rangle_d$, respectively. Hence, the difference must be closely connected to the cosmological model involved. But where does the difference arise?

In the derivation of P_1 an assumption is implicitely made, which is not used in the derivation of P_2: in a), a *random* line-of-sight to a source is considered, and it is implicitely assumed that such a line-of-sight behaves like an *average* line-of-sight. Especially, this means that all randomly chosen lines-of-sight are taken to have equal statistical weight. This, however, is clearly *inconsistent* with the adopted Dyer-Roeder model, as explained in the last section. There, a line-of-sight either is well away from all clumps, or the differential Eq. (7) is not valid. Hence, the direction to a source is not a random variable in the clumpy universe. What is random, however, is the position of a source on the sphere $z = z_s$, which is the random variable used to derive $P_2(Q)$. Therefore, we conclude that $P_2(Q)$ is the probability which is *consistent* with the adopted cosmological model and which should be used for statistical lens considerations.

Table 1. Mean amplification $\langle I\rangle$ and relative difference P_1 and P_2, $\Delta P/P$

Source redshift z_s	$\langle I\rangle_s$	$\Delta P/P$ [%]
0.1	1.0046	0.32
0.2	1.0167	1.17
0.3	1.0349	2.43
0.4	1.0579	4.02
0.5	1.0849	5.87
0.6	1.1153	7.95
0.7	1.1488	10.02
0.8	1.1849	12.63
0.9	1.2233	15.19
1.0	1.2640	17.86
1.2	1.3512	23.53
1.4	1.4455	29.55
1.6	1.5462	35.86
1.8	1.6528	42.43
2.0	1.7650	49.22
2.2	1.8826	56.22
2.4	2.0053	63.40
2.6	2.1329	70.75
2.8	2.2653	78.26
3.0	2.4025	85.93

It is now shown that the difference between $P_1(Q)$ and $P_2(Q)$ is by no means negligible. For this, we consider an ensemble of point masses of mass M and restrict our consideration to point sources. One can then calculate the probability that a source at z_s is amplified by more than I. The corresponding cross section

$$a_I = 2\pi\left[\frac{I}{(I^2-1)^{1/2}} - 1\right] \tag{23}$$

(cf. Canizares, 1981) is independent of redshifts and can thus be taken out of the integrals in (20) and (22). Then, the relative difference $\Delta P/P_1 = (P_2 - P_1)/P_1$ depends only on the cosmological model $(\Omega, \tilde{\alpha})$ and the redshift of the source. In Table 1 we have listed $\Delta P/P_1$, as well as the mean amplification $\langle I\rangle$ [cf. Eq. (11)] for the critically closed, completely clumpy universe ($\Omega = 1$, $\tilde{\alpha} = 0$). It is seen that the difference between P_1 and P_2 is really significant, being $\sim 18\%$ at $z_s = 1$, 49% at $z_s = 2$ and $\sim 86\%$ at $z_s = 3$. Since one has always $P_2(Q) \geq P_1(Q)$, this means that if the Dyer-Roeder model of light propagation in a clumpy universe is valid, then *statistical lens calculations made hitherto underestimate the relevance of lensing.* We have compared both expressions (20) and (22) with the Monte-Carlo simulations of Refsdal (1970) and found good agreement with his results from (22), whereas (20) leads to values which are systematically too low.

4. Summary

After some critical remarks about the Dyer-Roeder description of the propagation of light bundles through an inhomogeneous (clumpy) universe, which is commonly used in the theory of gravitational lenses, we derived a general equation for the probabilities of any specific effect of an ensemble of cosmologically distributed lenses on background sources. For this, the Dyer-Roeder description was taken as a working hypothesis. The equation obtained differs from similar ones used hitherto. It was shown that the new equation is consistent with the Dyer-Roeder model, whereas the old ones are not, due to the different measures of distance which are inevitably inherent in the Dyer-Roeder description of light propagation through a clumpy universe.

Besides being of principal interest, the self-consistent equation shows that hitherto the effects of gravitational lenses have been underestimated. The amount of the difference depends on the effect under consideration (e.g., influence on source counts, expected number of observable lens cases), as well as on the cosmological parameters and the redshift of the sources; it can be significant (see Table 1). Therefore, future work on statistical lens theory should take the self-consistent evaluation of probabilities into account, if it employs the Dyer-Roeder distance assignment.

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