

SYMMETRY STRUCTURES OF SUPERSTRING FIELD THEORIES

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A new extended formulation of free supersymmetric string field theories is presented. The hidden symmetries are exhibited and their quantization is given. The rigid space-time supersymmetry of the free theory is discussed.

1. Introduction

Long ago, a string theory involving both fermions and bosons was written down. This theory had two sectors: one for bosons [1] and one for fermions [2]. It was from this theory that supersymmetry was extracted [3] about the same time as it was independently discovered in Russia [4]. Some time later, it was shown that provided one projected the two sectors according to G parity, the corresponding string theory possessed an equal number of fermions and bosons at every level [5]. More recently, it was shown [6], as conjectured in ref. [5], that this superstring theory has 10-dimensional supersymmetry. The theory was also reformulated [7] so as to make the 10-dimensional supersymmetry manifest. The price for this success was the loss of manifest two-dimensional supersymmetry as well as Lorentz invariance.

Recently, there has been a considerable effort to find a gauge covariant formulation of string theories. At the level of the free theory, this has been solved for the bosonic [8–12] and supersymmetric strings [9,13]. It has been shown that strings possess an infinite number of local gauge symmetries. For superstrings these include the free analogues of general coordinate, Yang-Mills and supersymmetry transformations. More recently [14] a new gauge covariant formulation of the open bosonic string has been found. This formulation has a considerable simplicity and contains after gauge choices all previous formulations. It is the purpose of this paper to extend these latter results to the supersymmetric case, thus extending the gauge covariant formulations of refs. [9] and [13]. Although the exact connection is unclear, it is apparent that the infinite number of space-time string symmetries are

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closely connected with the two-dimensional symmetries of the world sheet. There are then considerable advantages in starting from the formulation [1,2] of superstrings which have the two-dimensional supersymmetry manifestly realized. We will find that the formulation of ref. [14] generalizes very naturally to the supersymmetric case. Larger gauge invariances will be found, the structure of which is such that the parameters of the gauge transformations can be subject to similar gauge transformations, in a repetitive manner. This structure leads, upon gauge-fixing and quantization, to the phenomenon of ghosts for ghosts and to the correct count for the physical degrees of freedom.

Space-time supersymmetry transformation laws are obtained through the examination of the Ward identities involving the fermion emission vertex [15,16], in conjunction with the well-known projection operators of Gliozzi, Olive and Scherk [5]. As the Ward identities involve an infinity of F_n and G_r gauges, the present formulation with an infinite number of auxiliary and supplementary fields will be seen to be well suited for the implementation of supersymmetry.

The organization of the rest of the paper is as follows. In sect. 2, we give a brief review of the previous work [9,13] and set the notations. Differential forms necessary for the discussion of the supergauge structure of the theory are developed in sect. 3. Although we shall describe the construction as a generalization of the bosonic tensor calculus introduced in ref. [10], a comment will be given, at the end of sect. 4, which clarifies its relation to the BRST approach. In sect. 4, using the compact language developed in the previous section, we describe the natural extension of the theory reviewed in sect. 2. By the introduction of further supplementary fields, the enlarged gauge symmetry structure becomes more transparent, including in particular, the existence of the tower of hidden local symmetries. These symmetries are, as in the bosonic string case [14], crucial in obtaining the correct quantization. Finally, in sect. 5, space-time supersymmetry transformation laws are described. An appendix is provided for the proofs of some of the formulae developed in sect. 3.

2. Previous results

In this section, we briefly review the covariant formulation of spinning strings in [9,13]. There are two sectors, the NS-sector containing only bosons and the R-sector containing only fermions. All physical degrees of freedom are contained in the expansions of the fundamental string fields

$$\Psi_{\text{NS}}(x^\mu(\sigma)) \equiv \left\{ \varphi(x) + A_\mu(x)b_{-1/2}^\mu + B_\mu(x)\alpha_{-1}^\mu + A_{\mu\nu}(x)b_{-1/2}^\mu b_{-1/2}^\nu + \cdots \right\} |0\rangle_{\text{NS}}, \quad (2.1)$$

$$\Psi_{\text{R}}(x^\mu(\sigma)) \equiv \left\{ \lambda(x) + \varphi_\mu(x)\alpha_{-1}^\mu + \psi_\mu(x)d_{-1}^\mu + \cdots \right\} |0\rangle_{\text{R}}. \quad (2.2)$$

As is by now well known, the covariant formulation of the corresponding free string theories requires the introduction of further supplementary and auxiliary string fields. These are

$$NS\text{-sector:} \quad \begin{pmatrix} \phi^r \\ \phi^m \end{pmatrix}, \quad \begin{pmatrix} \zeta^r_s & \zeta^r_n \\ \zeta^m_s & \zeta^m_n \end{pmatrix}, \quad (2.3)$$

$$R\text{-sector:} \quad \begin{pmatrix} \phi^p \\ \phi^m \end{pmatrix}, \quad \begin{pmatrix} \zeta^p_q & \zeta^p_n \\ \zeta^m_q & \zeta^m_n \end{pmatrix}. \quad (2.4)$$

Here we have introduced the following notation for later convenience: underlined indices are fermionic and indices without underlining are bosonic. Indices m, n, p, q are integer whereas indices r, s, t, \dots are half-integer. The field equations in the NS sector read

$$\begin{aligned} & (L_0 - \tfrac{1}{2})\Psi_{\text{NS}} + \sum_{r=1/2}^{\infty} G_{-r}\phi^r + \sum_{n=1}^{\infty} L_{-n}\phi^n = 0, \\ & G_r\Psi_{\text{NS}} + 2\phi^r - \sum_{s=1/2}^{\infty} G_{-s}\zeta^s_r + \sum_{n=1}^{\infty} L_{-n}\zeta^n_r + \sum_{s,n} (s + \tfrac{3}{2}n)\delta_{n,r-s}\zeta^n_s \\ & + 2\sum_{s,n} \delta_{n,r-s}\zeta^n_s = 0, \\ & L_n\Psi_{\text{NS}} + 2n\phi^n + \sum_{s=1/2}^{\infty} G_{-s}\zeta^s_n + \sum_{m=1}^{\infty} L_{-m}\zeta^m_n + \sum_{r,s} \tfrac{1}{2}(3r+s)\delta_{n,r+s}\zeta^r_s \\ & + \sum_{p,m} (2p+m)\delta_{n,p+m}\zeta^p_m = 0, \\ & G_r\phi^s = (L_0 + s + r - \tfrac{1}{2})\zeta^s_r + \tfrac{1}{2}(3r+s)\phi^{r+s}, \\ & L_n\phi^s = (L_0 + n + s - \tfrac{1}{2})\zeta^s_n - (s + \tfrac{3}{2}n)\phi^{n+s}, \\ & L_m\phi^n = (L_0 + m + n - \tfrac{1}{2})\zeta^n_m - (2m+n)\phi^{n+m}, \\ & G_r\phi^n = (L_0 + s + n - \tfrac{1}{2})\zeta^n_r - 2\phi^{n+r}, \end{aligned} \quad (2.5)$$

and are invariant under the infinite set of gauge transformations

$$\begin{aligned}
 \delta\Psi_{\text{NS}} &= \sum_{r=1/2}^{\infty} G_{-r}\Lambda^r + \sum_{n=1}^{\infty} L_{-n}\Lambda^n, \\
 \delta\phi^r &= -(L_0 + r - \frac{1}{2})\Lambda^r, \quad \delta\phi^n = -(L_0 + n - \frac{1}{2})\Lambda^n, \\
 \delta\xi^r_s &= -G_s\Lambda^r + \frac{1}{2}(3s + r)\Lambda^{r+s}, \\
 \delta\xi^r_n &= -L_n\Lambda^r - (r + \frac{3}{2}n)\Lambda^{r+n}, \\
 \delta\xi^r_r &= -G_r\Lambda^n - 2\Lambda^{r+n}, \\
 \delta\xi^r_m &= -L_m\Lambda^n - (2m + n)\Lambda^{n+m}.
 \end{aligned} \tag{2.6}$$

Note that we have redefined the string field ξ^r_s by a factor (-1) with respect to ref. [13]; this is again for later convenience. In the R-sector, the field equations are

$$\begin{aligned}
 F_0\Psi_{\text{R}} + \sum_{n=1}^{\infty} F_{-n}\phi^n + \sum_{n=1}^{\infty} L_{-n}\phi^n &= 0, \\
 F_n\Psi_{\text{R}} + \sum_{m=1}^{\infty} (F_{-m}\xi^m_n + L_{-m}\xi^m_n) + \sum_{p+m=n} [2\xi^p_m + (\frac{3}{2}p + m)\xi^p_m] \\
 - 2F_0\phi^n + \frac{5}{2}n\phi^n &= 0, \\
 L_n\Psi_{\text{R}} + \sum_{m=1}^{\infty} [F_{-m}\xi^m_n + L_{-m}\xi^m_n] + \sum_{p+m=n} [(2p + m)\xi^p_m - \frac{1}{2}(m + 3p)\xi^p_m] \\
 + 2nF_0\phi^n + \frac{5}{2}n\phi^n &= 0, \quad \text{etc.},
 \end{aligned} \tag{2.7}$$

and the gauge transformations read

$$\begin{aligned}
 \delta\Psi_{\text{R}} &= \sum_{n=1}^{\infty} F_{-n}\Lambda^n + \sum_{n=1}^{\infty} L_{-n}\Lambda^n, \\
 \delta\phi^n &= F_0\Lambda^n - \frac{1}{2}n\Lambda^n, \quad \delta\phi^n = -F_0\Lambda^n - 2\Lambda^n, \quad \text{etc.}
 \end{aligned} \tag{2.8}$$

The proof of invariance of these sets of equations of motion with respect to the

transformations (2.6) and (2.8) requires several identities such as

$$\sum_{\substack{p+q=m \\ p, q \geq 1}} (m+p)(m+q) - \sum_{\substack{r+s=m \\ r, s \geq 1/2}} (\frac{1}{2}m+r)(\frac{1}{2}m+s) = \frac{1}{8}Dm^3 - 2m^2 - \frac{1}{4}m, \quad (2.9)$$

which are only valid for $D = 10$ space-time dimensions.

The so-called “finite set” [9] is the truncation of the above equations to the fields

$$\Psi_{\text{NS}}, \phi^{1/2}, \phi^{3/2}, \zeta^{1/2}_{1/2}, \zeta^{1/2}_{3/2}, \zeta^{3/2}_{1/2}, \zeta^{3/2}_{3/2}, \quad (2.10)$$

$$\Psi_{\text{R}}, \phi^1, \phi^1, \zeta^1_{11}, \zeta^1_{11}, \zeta^1_{11}, \zeta^1_{11}. \quad (2.11)$$

These sets are the minimal ones required for a covariant formulation. We remark that one may, in fact, truncate the systems (2.5)–(2.8) to any intermediate finite set by discarding all fields beyond a certain level.

3. Differential forms for superstrings*

The previous results, reviewed in the foregoing section, will now be expressed in terms of a compact differential geometrical language which will make transparent the symmetry structures of the superstring field theories. The formalism is an extension of the case of bosonic string [10], and makes use of the notion of superspace familiar in supergravity theories (the relation between the formulations of refs. [12] and [10] was studied in ref. [17]). Below, for definiteness, we shall exhibit the formalism for the open string. Adaptation to the case of the closed string is straightforward. Our notations and definitions will be such that they cover the NS- and the R-sectors at the same time.

First, we assemble the super Virasoro generators into \mathcal{L}_A , viz.,

$$\mathcal{L}_A \equiv \begin{cases} L_n & \text{if } A \text{ bosonic} \\ G_r \text{ or } F_n & \text{if } A \text{ fermionic} \end{cases} \quad (3.1)$$

When explicit indices are needed, we often use unbarred (barred) small latin letters for bosonic (fermionic) indices. The graded-antisymmetric (GAS) commutator is defined as

$$[\mathcal{L}_A, \mathcal{L}_B] \equiv \frac{1}{2}(\mathcal{L}_A \mathcal{L}_B - (-)^{AB} \mathcal{L}_B \mathcal{L}_A), \quad (3.2)$$

where in the phase it is understood that

$$A = \begin{cases} 0 & \text{if } A \text{ bosonic} \\ 1 & \text{if } A \text{ fermionic} \end{cases} \quad (3.3)$$

* In ref. [18], results very similar to the ones described in this section have been obtained independently.

We then write up the super-Virasoro algebra in the following manner:

$$[\mathcal{L}_A, \mathcal{L}_B] = [\mathcal{L}_A, \mathcal{L}_B]^C \mathcal{L}_C, \quad (3.4a)$$

$$[\mathcal{L}_{-A}, \mathcal{L}_{-B}] = [\mathcal{L}_{-A}, \mathcal{L}_{-B}]^{-C} \mathcal{L}_{-C}, \quad (3.4b)$$

$$\begin{aligned} [\mathcal{L}_A, \mathcal{L}_{-B}] &= [\mathcal{L}_A, \mathcal{L}_{-B}]^C \mathcal{L}_C + [\mathcal{L}_A, \mathcal{L}_{-B}]^{-C} \mathcal{L}_{-C} \\ &\quad + [\mathcal{L}_A, \mathcal{L}_{-B}]^0 \mathcal{L}_0 + [\mathcal{L}_A, \mathcal{L}_{-B}]^0 \mathcal{L}_0 + \eta_{AB} C(A). \end{aligned} \quad (3.4c)$$

In the above,

$$\mathcal{L}_0 \equiv L_0, \quad \mathcal{L}_0 \equiv F_0 \quad (\text{for R-sector only}), \quad (3.5)$$

$$\begin{aligned} \eta_{AB} &\equiv \frac{1}{2} [\mathcal{L}_A, \mathcal{L}_{-B}]^0 \\ &= \begin{cases} n\delta_{nm} & \text{for } A, B \text{ bosonic} \\ \delta_{\ell s} \text{ or } \delta_{nm} & \text{for } A, B \text{ fermionic} \end{cases}, \end{aligned} \quad (3.6)$$

$$C(n) = \begin{cases} \frac{1}{8} D(n^2 - 1) & \text{for NS} \\ \frac{1}{8} Dn^2 & \text{for R} \end{cases}, \quad (3.7a)$$

$$C(\underline{n}) = \frac{1}{2} D\underline{n}^2 \quad \text{for R},$$

$$C(\underline{r}) = \frac{1}{2} D\underline{r}^2 \quad \text{for NS}, \quad (3.7b)$$

(D = dimension of space-time) and the structure constants are denoted as $[\mathcal{L}_A, \mathcal{L}_B]^C$, etc., which are useful in dealing with various Jacobi identities. The generic Jacobi identity, to be frequently referred to, is of the form,

$$\begin{aligned} (-)^{AC} [[\mathcal{L}_A, \mathcal{L}_B] \mathcal{L}_C] + (-)^{BA} [[\mathcal{L}_B, \mathcal{L}_C] \mathcal{L}_A] \\ + (-)^{CB} [[\mathcal{L}_C, \mathcal{L}_A] \mathcal{L}_B] = 0. \end{aligned} \quad (3.8)$$

It is useful to remember

$$[\mathcal{L}_A, \mathcal{L}_B] = -(-)^{AB} [\mathcal{L}_B, \mathcal{L}_A], \quad (3.9a)$$

$$[\mathcal{L}_{-A}, \mathcal{L}_{-B}]^{-C} = -(-)^{AB} [\mathcal{L}_A, \mathcal{L}_B]^C = [\mathcal{L}_B, \mathcal{L}_A]^C, \quad (3.9b)$$

$$[[\mathcal{L}_A, \mathcal{L}_B] \mathcal{L}_C] = -(-)^{C(A+B)} [\mathcal{L}_C [\mathcal{L}_A, \mathcal{L}_B]], \quad (3.9c)$$

which follow from the definition (3.2).

Next, we introduce two kinds of bases, e^A, e_A , for constructing differential forms. Depending on the type of index A , each of them can be either bosonic or fermionic, viz.,

$$e^A = \begin{pmatrix} e^n \\ \theta^u \text{ or } \theta^r \end{pmatrix}, \quad e_A = \begin{pmatrix} e_n \\ \theta_u \text{ or } \theta_r \end{pmatrix}. \tag{3.10}$$

We assume the GAS properties,

$$e^A e^B = -(-)^{AB} e^B e^A, \tag{3.11a}$$

$$e_A e_B = -(-)^{AB} e_B e_A, \tag{3.11b}$$

and in addition we shall demand

$$e^A e_B = (-)^{AB} e_B e^A. \tag{3.12}$$

An $\binom{a}{b}$ form is then written as,

$$\omega = \omega_{B_1 \dots B_h}^{A_1 \dots A_a} e^{B_1} \dots e^{B_h} e_{A_1} \dots e_{A_a}, \tag{3.13}$$

where the coefficients $\omega_{B_1 \dots B_h}^{A_1 \dots A_a}$ are functionals of the string variable $x^\mu(\sigma)$, and are $\binom{0}{0}$ forms. The sets of indices $\{A_1, \dots, A_a\}$ and $\{B_1, \dots, B_h\}$ are always understood to be graded-antisymmetrized.

We now define the differential operator d , which turns an $\binom{a}{b}$ form into an $\binom{a}{b+1}$ form. As we shall demand it to possess a generalized derivation property, it suffices to define it on the basic elements. On a $\binom{0}{0}$ form ψ , we define

$$d\psi \equiv \mathcal{L}_A \psi e^A. \tag{3.14}$$

Requiring the cohomological property, $d^2\psi = 0$, we are led to define

$$de^A \equiv -\frac{1}{2}[\mathcal{L}_B, \mathcal{L}_C]^A e^B e^C. \tag{3.15}$$

$d^2e^A = 0$ then follows from the Jacobi identity. Eq. (3.15) further dictates

$$e^A de^B = (-)^{AB} de^B e^A. \tag{3.16}$$

To define a suitable derivation property for d , consider a $\binom{0}{2}$ form $\omega_{AB}e^Ae^B$ and act d on it. We get

$$d\omega = \mathcal{L}_C \omega_{AB} e^A e^B + \omega_{AB} de^A e^B + (-)^\phi \omega_{AB} e^A de^B,$$

where $(-1)^\phi$ is a phase produced when d goes through e^A to act on e^B . Now using (3.16) and GAS of ω_{AB} , it is easy to see that we should take $(-1)^\phi = -1$ so that the second and the third terms are equal. Thus, we define

$$d(e^A \dots) = (de^A) \dots - e^A d \dots \quad (3.17)$$

Finally, we need to define de_A , which should be a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ form. We choose

$$de_A = (-)^{BC} [\mathcal{L}_B, \mathcal{L}_{-A}]^{-C} e^B e_C. \quad (3.18)$$

The choice of the phase and the coefficient is dictated by the requirements (i) $d^2 e_A = 0$ and (ii) it is suitable in expressing the eqs. (2.5)–(2.8). In fact, on a $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ form

$$X \equiv X^A e_A = \phi^n e_n + \phi^r \theta_r. \quad (3.19)$$

(3.18), together with (3.14), gives, for the NS sector,

$$\begin{aligned} dX = & (L_m \phi^n + [L_m, L_{-k}]^{-n} \phi^k) e^m e_n + (L_m \phi^r + [L_m, G_{-s}]^{-r} \phi^s) e^m \theta_r \\ & + (G_s \phi^n + \{G_s, G_{-t}\}^{-n} \phi^t) \theta^t e_n + (G_t \phi^s - [G_t, L_{-n}]^{-s} \phi^n) \theta^t \theta_s, \end{aligned} \quad (3.20)$$

which corresponds exactly to eqs. (2.5) and (2.6). Using (3.12) and a reasoning similar to the one which led to (3.17), we establish

$$de_A e_B = -(-)^{AB} e_B de_A, \quad (3.21)$$

$$d(e_A \dots) = (de_A) \dots + e_A d \dots \quad (3.22)$$

It is now straightforward to work out the action of d on a general $\begin{pmatrix} a \\ b \end{pmatrix}$ form ω in (3.13). One gets

$$\begin{aligned} d\omega = & \left(\mathcal{L}_{B_0} \omega_{B_1 \dots B_b}^{A_1 \dots A_a} + a(-)^{B_0(A_1+B_1+\dots+B_b)} \right. \\ & \times [\mathcal{L}_{B_0}, \mathcal{L}_{-A_1}]^{-A_1} \omega_{B_1 \dots B_b}^{A_1 A_2 \dots A_a} \\ & \left. - \frac{1}{2} b [\mathcal{L}_{B_0}, \mathcal{L}_{B_1}]^{B_1} \omega_{B_1 B_2 \dots B_b}^{A_1 \dots A_a} \right) e^{B_0} e^{B_1} \dots e^{B_b} e_{A_1} \dots e_{A_a}. \end{aligned} \quad (3.23)$$

Before introducing the co-differential operator D , which turns an $\begin{pmatrix} a \\ b \end{pmatrix}$ form into an

$\binom{a-1}{b}$ form, we need to define the inner product of forms. A natural definition is

$$\begin{aligned} (e^{A_1} \dots e^{A_a}, e_{B_1 \dots B_b}) &= P_{B_1 \dots B_b}^{A_1 \dots A_a} \\ &\equiv \frac{1}{a!} \left(\delta_{B_1}^{A_1} \delta_{B_2}^{A_2} \dots \delta_{B_a}^{A_a} - (-)^{A_1 A_2} \delta_{B_1}^{A_2} \delta_{B_2}^{A_1} \dots \delta_{B_a}^{A_a} + \dots \right), \end{aligned} \quad (3.24)$$

where $P_{B_1 \dots B_b}^{A_1 \dots A_a}$ is the GAS projector. When mixed forms are involved, we demand that the order of the basis be properly matched before (3.24) is applied. For example, for $\omega = \omega^A_B e^B e_A$ and $\rho = \rho^{B'}_{A'} e^{A'} e_{B'}$

$$\begin{aligned} (\omega, \rho) &= (\omega^A_B, \rho^{B'}_{A'}) (e^B e_A, e^{A'} e_{B'}) \\ &= (\omega^A_B, \rho^{B'}_{A'}) (-)^{AB} (e_A e^B, e^{A'} e_{B'}) \\ &= (-)^{AB} (\omega^A_B, \rho^B_A), \end{aligned} \quad (3.25)$$

where the bracket for the coefficients denotes the Fock space inner product.

With this definition of inner product, D on an $\binom{a}{b}$ form ω is uniquely defined by

$$(D\omega, \rho) \equiv (\omega, d\rho), \quad (3.26)$$

where ρ is a $\binom{b}{a-1}$ form. Taking due care of the phase as in (3.25), one finds,

$$\begin{aligned} D\omega &= \left((-)^{A_1(B_1 + \dots + B_b)} \mathcal{L}_{-A_1} \omega_{B_1 \dots B_b}^{A_1 A_2 \dots A_a} \right. \\ &\quad \left. + b (-)^{A_1(B_2 + \dots + B_b)} [\mathcal{L}_{A_1}, \mathcal{L}_{-B_1}]^{-B_1} \omega_{B_1 B_2 \dots B_b}^{A_1 A_2 \dots A_a} \right. \\ &\quad \left. - \frac{1}{2} (a-1) [\mathcal{L}_{A_1}, \mathcal{L}_{A_2}]^{A_2} \omega_{B_1 \dots B_b}^{A_1 A_2 A_3 \dots A_a} \right) e^{B_1} \dots e^{B_b} e_{A_2} \dots e_{A_a}. \end{aligned} \quad (3.27)$$

An example is appropriate here. Let

$$Z \equiv \zeta^A_B e^B e_A = \zeta^n_m e^m e_n + \zeta^n_m e^m \theta_n + \zeta^n_m \theta^m e_n + \zeta^n_m \theta^m \theta_n. \quad (3.28)$$

Then, (3.27) gives, in the NS sector,

$$\begin{aligned} DZ &= \left(L_{-m} \zeta^n_m + [L_m, L_{-n}]^{-k} \zeta^m_k + G_{-r} \zeta^r_n + [G_r, L_{-n}]^{-s} \zeta^r_s \right) e^n \\ &\quad + \left(L_{-m} \zeta^m_r + [L_m, G_{-r}]^{-s} \zeta^m_s - G_{-s} \zeta^s_r + \{G_s, G_{-t}\}^{-n} \zeta^s_n \right) \theta^r. \end{aligned} \quad (3.29)$$

This is precisely the expression relevant in eqs. (2.5) and (2.6).

To complete the transcription of the formulae in the previous section into the differential form language, we need a few more operators. Define the operator \Downarrow as follows:

$$\begin{aligned} \Downarrow \psi &= 0, & \Downarrow e^A &= 0, \\ \Downarrow e_A &\equiv \frac{1}{2} [\mathcal{L}_A, \mathcal{L}_B] e^B = \eta_{AB} e^B. \end{aligned} \tag{3.30}$$

When \Downarrow is acted upon a string of bases, we define

$$\begin{aligned} \Downarrow (\psi \dots) &= \psi \Downarrow \dots, \\ \Downarrow (e^A \dots) &= -e^A \Downarrow \dots, \\ \Downarrow (e_{A_1} \dots e_{A_u}) &= \frac{1}{a} \left[(\Downarrow e_{A_1}) e_{A_2} \dots e_{A_u} - e_{A_1} (\Downarrow e_{A_2}) \dots e_{A_u} + \dots \right]. \end{aligned} \tag{3.31}$$

In other words, \Downarrow is an antiderivation, except for the normalization convention in (3.31).

Next we introduce the generalized bosonic kinetic operator K defined by

$$K\psi \equiv \begin{cases} (L_0 - \frac{1}{2})\psi & \text{for NS-sector} \\ L_0\psi & \text{for R-sector} \end{cases}, \tag{3.32a}$$

$$Ke^A \equiv [\mathcal{L}_B, \mathcal{L}_0]^A e^B = Ae^A, \tag{3.32b}$$

$$Ke_A \equiv [\mathcal{L}_A, L_0]^B e_B = Ae_A. \tag{3.32c}$$

It is defined to be a derivation. In the R-sector, we naturally need a fermionic kinetic operator F as well. It is also a derivation and is defined similarly to K above:

$$F\psi \equiv F_0\psi\sigma, \tag{3.33a}$$

$$Fe^A \equiv [\mathcal{L}_B, F_0]^A e^B\sigma, \tag{3.33b}$$

$$Fe_A \equiv [\mathcal{L}_A, F_0]^B e_B\sigma. \tag{3.33c}$$

The only difference is that it creates a symbol σ , which keeps track of the statistics

of the form^{*}. σ has the properties,

$$\begin{aligned}\sigma e^A &= (-)^A e^A \sigma, & \sigma e_A &= (-)^A e_A \sigma, \\ \sigma^2 &= 1.\end{aligned}\tag{3.34}$$

When F acts an odd number of times, a σ remains in the expression. In such a case, we imagine an extra basis e , placed always to the right of all the other bases, which absorbs σ , i.e. $\sigma e = e$. This extra basis will not be explicitly displayed.

With the above definitions of K , F and \Downarrow , we can prove a number of useful properties. (Proofs of some of them are sketched in the appendix.)

$$F^2 = K,\tag{3.35}$$

$$\begin{aligned}(K\omega, \rho) &= (\omega, K\rho), & (F\omega, \rho) &= (\omega, F\rho), \\ (\Downarrow\omega, \rho) &= (\omega, \Downarrow\rho),\end{aligned}\tag{3.36}$$

$$[K, d] = [K, D] = 0,$$

$$[F, d] = [F, D] = 0,$$

$$[\Downarrow, K] = 0,\tag{3.37}$$

$$d\Downarrow + \Downarrow d = D\Downarrow + \Downarrow D = 0.\tag{3.38}$$

It is evident from (3.35) that F is a generalized Dirac operator. The properties above will be needed when we discuss the gauge symmetry structure in the next section.

A particular combination of \Downarrow and F , namely

$$\mathcal{F} \equiv F\Downarrow - \Downarrow F\tag{3.39}$$

is a useful operator. It is an antiderivation and anticommutes with F . Its action on the basis is easily worked out:

$$\mathcal{F}\psi = \mathcal{F}e^A = 0,$$

$$\mathcal{F}e_A = (-)^A [\mathcal{L}_B, \mathcal{L}_{-A}]^0 e^B \sigma.\tag{3.40}$$

Now we come to the most structured part of our machinery, the formula for $dD - Dd$. In the case of the bosonic string, it was shown [10] that

$$dD - Dd = 2K\Downarrow.\tag{3.41}$$

^{*} When one relates e^A and e_A to ghost creation and annihilation operators, σ is thus nothing but $(-)^N$ where N is the fermion number operator in Fock space.

This was due, among other things, to the identity valid only in 26 dimensions,

$$\eta_{nm}C(n) - [L_p, L_{-n}]^{-q}[L_m, L_{-q}]^p = 2n\eta_{nm}, \quad (3.42)$$

where $C(n) = \frac{26}{12}(n^2 - 1)$. In the case of the spinning strings, similar identities were encountered (see eq. (2.9)) in the construction of gauge covariant field theories. In our streamlined notation, such identities can be assembled into one equation, valid only in 10 dimensions,

$$\begin{aligned} \eta_{AB}C(A) - (-)^{CD}[\mathcal{L}_C, \mathcal{L}_{-A}]^{-D}[\mathcal{L}_B, \mathcal{L}_{-D}]^C \\ = [\mathcal{L}_A[\mathcal{L}_0, \mathcal{L}_{-B}]]^0 + [\mathcal{L}_A[\mathcal{L}_{-B}, F_0]]^{-0}, \end{aligned} \quad (3.43)$$

where the second term on the r.h.s. vanishes of course for the NS-sector. Using this identity, together with Jacobi identities, one can show (see appendix for a brief outline of the derivation)

$$dD - Dd = 2K \Downarrow + \mathcal{F}F. \quad (3.44)$$

The $\mathcal{F}F$ term is again understood to be absent in the NS-sector. (For the R-sector, the r.h.s. can be written more symmetrically as $F(F \Downarrow + \Downarrow F)$.) Let us give an explicit example. On a $\binom{1}{0}$ form $X \equiv \phi^n e_n + \phi^m \theta_n$, in the R-sector, we have

$$\mathcal{F}FX = \left(\frac{3}{2}nF_0\phi^n - \frac{3}{4}n^2\phi^n\right)e^n + \left(\frac{3}{2}nF_0\phi^n + 3n\phi^m\right)\theta^n, \quad (3.45a)$$

$$2K \Downarrow X = 2n(L_0 + n)\phi^n e^n + 2(L_0 + n)\phi^m \theta^n. \quad (3.45b)$$

On the other hand,

$$\begin{aligned} (dD - Dd)X = & \left[(2nL_0 + \frac{5}{4}n^3)\phi^n + \frac{3}{2}nF_0\phi^m \right. \\ & \left. + \delta_{mn} \sum_{p+q=m} \left((\frac{1}{2}m+p)(\frac{1}{2}n+q) - (m+p)(m+q) \right) \phi^m \right] e^n \\ & + \left[(2L_0 + \frac{5}{2}n^2)\phi^m + \frac{3}{2}nF_0\phi^n - 4 \sum_{p+q=m} (m + \frac{1}{2}p)\phi^m \delta_{mn} \right] \theta^n. \end{aligned} \quad (3.46)$$

After a trivial summation over p, q and m , this coincides with $(2K \Downarrow + \mathcal{F}F)X$.

This completes the development of the compact language and we are now ready to reveal the symmetry structure of the theory.

4. The master-set

The formalism developed in the preceding section now enables us to reexpress the formulas of sect. 2 in a very compact form. To do so we assign “level indices” to the string fields introduced there. For instance, in the NS-sector we have

$$\begin{aligned}\Psi_0^0 &\equiv \Psi_{\text{NS}}, & \phi_0^1 &\equiv \phi^A e_A, \\ \Psi_1^1 &\equiv \zeta^A{}_B e^B e_A, & \text{etc.}\end{aligned}\quad (4.1)$$

By means of the NS operators introduced in the previous section, the set of eq. (2.5) now becomes

$$\begin{aligned}K\Psi_0^0 + D\phi_0^1 &= 0, \\ d\Psi_0^0 + 2\Downarrow\phi_0^1 + D\Psi_1^1 &= 0, \\ K\Psi_1^1 - d\phi_0^1 &= 0,\end{aligned}\quad (4.2)$$

while the gauge transformations now read

$$\begin{aligned}\delta\Psi_0^0 &= D\Lambda_0^1, & \delta\Psi_1^1 &= -d\Lambda_0^1, \\ \delta\phi_0^1 &= -K\Lambda_0^1.\end{aligned}\quad (4.3)$$

This form of variations strongly suggests a further extension beyond (4.3). We therefore define new forms of arbitrarily high level in analogy with (4.1), viz.

$$\begin{aligned}\Psi_k^k &\equiv \zeta^{A_1\dots A_k}{}_{B_1\dots B_k} e^{B_1}\dots e^{B_k} e_{A_1}\dots e_{A_k}, \\ \phi_k^{k+1} &\equiv \phi^{A_1\dots A_{k+1}}{}_{B_1\dots B_k} e^{B_1}\dots e^{B_k} e_{A_1}\dots e_{A_{k+1}},\end{aligned}\quad (4.4)$$

and replace (4.2) by the more general system

$$\begin{aligned}K\Psi_k^k - d\phi_{k-1}^k + D\phi_k^{k+1} &= 0, \\ d\Psi_k^k + 2\Downarrow\phi_k^{k+1} + D\Psi_{k+1}^{k+1} &= 0.\end{aligned}\quad (4.5)$$

This is not only invariant under the obvious generalization of (4.3) but under a still bigger set of gauge transformations

$$\begin{aligned}\delta\Psi_k^k &= -d\Lambda_{k-1}^k + D\Lambda_k^{k+1} + 2\Downarrow\Lambda_{k-1}^{k+1}, \\ \delta\phi_k^{k+1} &= -K\Lambda_k^{k+1} + d\Lambda_{k-1}^{k+1} + D\Lambda_k^{k+2}.\end{aligned}\quad (4.6)$$

For the proof, we need all the identities worked out in the foregoing section. The reader will undoubtedly recognize the formal similarity of (4.5) and (4.6) with the corresponding set of equations for the bosonic string. In fact, everything is the same, except that for the algebra to work, we now need $D = 10$ rather than $D = 26$. Just as in the bosonic case [14], we can now identify the gauge invariances of the gauge invariances

$$\begin{aligned} \delta\Lambda_{k-1}^k &= D\tilde{\Lambda}_{k-1}^{k+1} + 2\Downarrow\tilde{\Lambda}_{k-2}^{k+1} + d\tilde{\Lambda}_{k-2}^k, \\ \delta\Lambda_{k-1}^{k+1} &= K\tilde{\Lambda}_{k-1}^{k+1} - d\tilde{\Lambda}_{k-2}^{k+1} + D\tilde{\Lambda}_{k-1}^{k+2}. \end{aligned} \tag{4.7}$$

The r.h.s. of (4.7) is again invariant under yet another set of gauge invariances, and so on.

Similar results can be derived for the R-sector. The only difference is that the equations now contain the first-order operator F defined in (3.33) rather than the second-order operator K . Furthermore, the relation (3.41), which is valid also in the NS sector, has to be replaced by (3.44) which we now write in the symmetric form

$$dD - Dd = F(F\Downarrow + \Downarrow F). \tag{4.8}$$

In complete analogy with (4.1), we then introduce the following forms

$$\begin{aligned} \Psi_0^0 &\equiv \Psi_R, & \phi_0^1 &\equiv \phi^A e_A, \\ \Psi_1^1 &\equiv \zeta^A{}_B e^B e_A, & \text{etc.}, \end{aligned} \tag{4.9}$$

to rewrite eqs. (2.7) in the form

$$\begin{aligned} F\Psi_0^0 + D\phi_0^1 &= 0, \\ d\Psi_0^0 + D\Psi_1^1 + (F\Downarrow + \Downarrow F)\phi_0^1 &= 0, \\ F\Psi_1^1 - d\phi_0^1 &= 0. \end{aligned} \tag{4.10}$$

The gauge invariances (2.8) can now be re-expressed as

$$\begin{aligned} \delta\Psi_0^0 &= D\Lambda_0^1, & \delta\Psi_1^1 &= -d\Lambda_0^1, \\ \delta\phi_0^1 &= -F\Lambda_0^1. \end{aligned} \tag{4.11}$$

As before a further extension of these results is possible. Introducing forms of arbitrary rank as in (4.4), we can replace (4.10) by the more general system

$$\begin{aligned} F\Psi_k^k - d\phi_{k-1}^k + D\phi_k^{k+1} &= 0, \\ d\Psi_k^k + D\Psi_{k+1}^{k+1} + (F\Downarrow + \Downarrow F)\phi_k^{k+1} &= 0, \end{aligned} \tag{4.12}$$

which is invariant under

$$\begin{aligned}\delta\mathcal{V}_k^k &= -d\Lambda_{k-1}^k + D\Lambda_k^{k+1} + (F\Downarrow + \Downarrow F)\Lambda_{k-1}^{k+1}, \\ \delta\phi_k^{k+1} &= -F\Lambda_k^{k+1} + d\Lambda_{k-1}^{k+1} + D\Lambda_k^{k+2}.\end{aligned}\quad (4.13)$$

The gauge invariances of the gauge invariances are given by

$$\begin{aligned}\delta\Lambda_{k-1}^k &= D\tilde{\Lambda}_{k-1}^{k+1} + d\tilde{\Lambda}_{k-2}^k + (F\Downarrow + \Downarrow F)\tilde{\Lambda}_{k-2}^{k+1}, \\ \delta\Lambda_{k-1}^{k+1} &= F\tilde{\Lambda}_{k-1}^{k+1} - d\tilde{\Lambda}_{k-2}^{k+1} + D\tilde{\Lambda}_{k-1}^{k+2}.\end{aligned}\quad (4.14)$$

As in (4.7), the r.h.s. of (4.14) is invariant under yet another set of gauge invariances, and so on.

Before concluding this section, we wish to comment on the appearance of BRST related structures in the string field theory of ref. [14] *before* it is gauge fixed and corresponding ghost fields are added. In the papers of ref. [19] it is explained how one can covariantly impose the first class constraints of a relativistic system. This achieved by the introduction of a pair of anticommuting coordinates for each of the constraints. For the (super)string, the constraints are the \mathcal{L}_A 's which satisfy the (super)conformal algebra. Correspondingly we must introduce further anticommuting (commuting) coordinates $(C^r, \bar{C}_{rn}, r = 1, 2)$.

The starting point for the classical string field theory is the first-quantized string theory. The same holds for the classical point particle field theory. Carrying out this first quantization we have a configuration space of $x^\mu(\sigma)$ and $c(\sigma), \bar{c}(\sigma)$ upon which the string fields are defined. The well-known action [20] of the operator Q ($Q^2 = 0$) is none other than the action of conformal symmetry in the configuration space.

The action of the conformal group on an arbitrary string functional is $\delta\chi = Q\chi$. However, Q carries a ghost number and so rotates different ghost number sectors of χ into each other. In particular, we find $\delta\chi^{(0)} = Q\chi^{(-1)}$. In the classical string field theory we consider only string fields of shifted ghost number zero and so we rewrite the variation as $\delta\chi^{(0)} = Q\Lambda$ where Λ is now a parameter which has an appropriate ghost number. The action which is invariant under this transformation is

$$\int \mathcal{D}x \mathcal{D}c \mathcal{D}\bar{c} \chi^{(0)} Q\chi^{(0)}. \quad (4.15)$$

We wish to stress that there has been no mention of gauge fixing and ghost fields and we are dealing with a gauge and *not* a BRST symmetry in the string field theory. The BRST symmetry in the first-quantized theory becomes the gauge symmetry in the classical string field theory. We also note that this is entirely in analogy with the point particle case [21]. The point particle field theory is defined

on (x^μ, c) and has $\delta\phi = Q\Lambda$ where $Q \sim c(\partial^2 + m^2)$ and the action is given by $\int dx dc \phi Q\phi$.

The same logic is easily applied to the contents of this paper. The final actions and transformation laws we write down are of the form $(\chi, Q\chi)$ and $\delta\chi = Q\Lambda$ respectively, but now we not only restrict the ghost number of χ , but also consistently restrict the power in χ of the commuting zero mode ghost associated with F_0 . In this sense consistent means that if you take the equation of motion $Q\chi = 0$ without restricting the power of the commuting ghost, one finds that one has extra equations. These extra equations are algebraically deducible from the same equation $Q\chi = 0$ when the occurrence of the commuting zero mode ghost in χ is restricted*.

5. Space-time supersymmetry

It has been known for a long time that the physical spectrum of the combined NS- and R-model is supersymmetric after suitable “chiral” projections in each sector [5]. However, apart from the proof that there is an equal number of bosons and fermions at each level in the light cone gauge, space-time supersymmetry has remained rather obscure in the “old” formalism, and this has prompted Green and Schwarz to develop the “new” formalism [6]. On the other hand, although supersymmetry is manifest in the “new” formalism, the original explicit Lorentz invariance has been lost. It is clearly desirable to have a formalism in which *both* space-time supersymmetry *and* Lorentz invariance are explicit. In this section, we will demonstrate that this can be accomplished by use of the covariant formalism developed in the foregoing sections, at least up to a certain level. Since we encounter several stringent consistency checks already at the level which we have considered, we hope that these results will suffice to convince the reader of the power of the formalism. We will also present further (and independent) evidence that the master sets described in sect. 4 are best suited for the task of making space-time supersymmetry explicit.

Our basic strategy can be described as follows. We will attempt to define supersymmetry transformations in such a way that the first of eqs. (2.5) is transformed into a linear combination of eqs. (2.7) and vice versa. Starting from the super-Yang-Mills transformation laws [5]

$$\delta\lambda = \sigma^{\mu\nu} F_{\mu\nu} \epsilon, \quad \delta A_\mu = \bar{\epsilon} \gamma_\mu \lambda, \quad (5.1)$$

where λ and A_μ are the massless fields in (2.1) and (2.2) after the GOS-projection, we are led to a unique answer for $\delta\Psi_R$ and $\delta\Psi_{NS}$. Since there is no need to impose

* If one were to not restrict the occurrence of the commuting zero mode ghost associated with F_0 , one would find another tower of auxiliary string fields [18]. However, the superconformal invariance can be implemented without these fields.

the equations of motion this result is “off-shell”. The auxiliary and supplementary fields play a crucial role in this construction. In the supersymmetry transformation rules, level number will not be conserved since the fundamental string fields Ψ_R and Ψ_{NS} in (2.1) and (2.2) vary into the auxiliary and supplementary fields (2.3) and (2.4). Therefore, the mismatch in the Fermi-Bose count that arises at any finite level after switching on the off-shell degrees of freedom is irrelevant: the supersymmetry variation of any field contains infinitely many fields of arbitrarily high level! Needless to say that the well-known no-go theorems for the non-existence of an off-shell formulation of $N = 1$ $d = 10$ super-Yang-Mills theory [5] no longer apply in this case.

The two basic ingredients of our construction are GOS projectors [5] and the fermion emission vertex [15]. The projectors are defined by

$$P_{NS} \equiv \frac{1}{2} \left(1 + (-)^{\sum_{r=1/2}^{\infty} b_r^* b_r} \right), \quad (5.2)$$

$$P_R \equiv \frac{1}{2} \left(1 + (-)^{\sum_{n=1}^{\infty} d_n^* \gamma^*} \right) \quad (5.3)$$

in the NS- and R-sectors, respectively; γ^* is the analogue of the γ^5 matrix in 10 dimensions. These projectors can be applied to all equations in sect. 2. While the L_n operators commute with P_R and P_{NS} , the fermionic operators G_r and F_m flip the “chirality”; thus, the “chirality” of the fields ϕ_{NS}^r and ϕ_R^m in (2.5) and (2.7) is opposite to that of Ψ_{NS} , ϕ_{NS}^n and Ψ_R and ϕ_R^m (we now use subscripts R and NS to distinguish between the two sectors). The other important object is the fermion emission vertex which converts NS-states into R-states. It is given by [15]

$$W_F(0) = \exp(+L_{-1}^R) \tilde{W}_F(0), \quad (5.4)$$

where*

$$\begin{aligned} \tilde{W}_F(0) =_{NS} \langle 0 | \exp \left(\sum_{n=0}^{\infty} \sum_{r=1/2}^{\infty} \Gamma_{-n}^{\mu} B_{nr}(1) \gamma^* b_{r\mu} \right) | 0 \rangle_R \\ \times \exp \left(\frac{1}{2} \sum_{r,s=1/2}^{\infty} b_r^{\mu} A_{rs}(1) b_{s\mu} \right), \end{aligned} \quad (5.5)$$

and the numerical coefficients $A_{rs}(1)$ and $B_{nr}(1)$ are given in ref. [22]. It is evident from (5.5), that $\tilde{W}_F(0)$ converts an NS-state into an R-state. Consequently, the L_{-1} operator in (5.4) has to be the one in the R-sector and we have indicated this by the superscript R. From now on, it will be understood that all operators are to be taken

* We use the conventions and notation of ref. [22].

in the appropriate sectors, and superscripts will not always be indicated. Note also that the operator which converts an R-state into an NS-state is the adjoint of (5.4), i.e. $W_F(0)^+$.

In the following, we will not need the explicit form (5.5) but rather certain Ward identities that tell us how to pull L , F and G operators through W_F . To exhibit them, we introduce an angular variable $0 \leq \theta \leq 2\pi$ and define the θ -dependent vertex by

$$W(\theta) \equiv e^{i\theta L_0^R} W_F(0) e^{-i\theta L_0^{NS}}. \tag{5.6}$$

Here L_0^{NS} is the usual L_0 operator in the NS-sector whereas L_0^R is defined such that

$$L_0^R \equiv F_0^2 + \frac{5}{8}. \tag{5.7}$$

The ‘‘anomalous’’ term in (5.7) is precisely the shift that makes the superconformal algebras in the NS- and R-sectors coincide in $D = 10$ and is necessary for our further considerations. The operator (5.6) now obeys the Ward-identity [15]

$$[L_n, W(\theta)] = e^{in\theta} \left(\frac{1}{8}(5n + 1) - i \frac{d}{d\theta} \right) W(\theta). \tag{5.8}$$

Another important property of $W(\theta)$ is [15]

$$P_R W(\theta) = W(\theta) P_{NS}. \tag{5.9}$$

The necessity of imposing the GOS projection can be seen as follows. As a function of the complex variable $z = e^{i\theta}$, W is *not* single-valued but has a square root branch cut. Multiplication by the operator P_{NS} from the right projects out just the right terms to render (5.9) single valued*. One can then define Fourier components of $W(\theta)P_{NS}$ by contour integrals in $z = e^{i\theta}$. Thus, we define

$$W_n \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} e^{in\theta} W(\theta) P_{NS}. \tag{5.10}$$

Since the operator is now single-valued, we can now drop ‘‘surface terms’’, and it is straightforward to prove the relations

$$[L_m, W_n] = \left(\frac{1}{8} - \frac{3}{8}m - n \right) W_{m+n} \tag{5.11}$$

from (5.8). In particular, (5.11) implies

$$F_0^2 W_0 = W_0 \left(L_0 - \frac{1}{2} \right), \tag{5.12}$$

* For the light-cone version of (5.6), this was already noted in ref. [6].

which explicitly confirms the necessity of the redefinition (5.7) because, with this definition, the kinetic operators in the NS- and R-sectors are connected through W_0 .

The Ward identities involving F 's and G 's are more complicated [16]. For arbitrary $N \in \mathbb{Z}$ and $n \geq 0$, they read

$$\sum_{m=0}^{\infty} \beta_{nm} F_{-m} W_{m+N} = -i W_{N+n+1/2} G_{-n-1/2} - i \sum_{s=1/2}^{\infty} W_{N-s} \alpha_{n+1/2, s} G_s, \quad (5.13)$$

where the coefficients α_{rs} and β_{nm} are given by

$$\begin{aligned} \alpha_{rs} &\equiv (-)^{r+s} \sum_{\substack{p+q=r+s \\ p \leq r-1/2}} \binom{-\frac{1}{2}}{p} \binom{\frac{1}{2}}{q}, \\ \beta_{nm} &\equiv (-)^{m+n} \sum_{\substack{p+q=m \\ p \leq n}} \binom{-\frac{1}{2}}{n-p} \binom{\frac{1}{2}}{q}. \end{aligned} \quad (5.14)$$

These coefficients are directly related to the ones which appear in the fermion emission vertex (5.4) [16, 22]. Observe that, in (5.13), we have omitted a factor γ^* on the r.h.s. because W is defined with the GOS projector. From (5.13), it is also obvious that the infinite set is more convenient than the finite one of ref. [9] because of the occurrence of an infinite number of F 's and G 's.

To implement (5.1), we start with

$$\delta \Psi_R = F_0 W_0 \Psi_{NS} + \dots \quad (5.15)$$

and use (5.13) and (5.11) such that the first of eqs. (2.7) is varied into a linear combination of eqs. (2.5). After some calculation, this leads to the result

$$\begin{aligned} \delta \Psi_R &= F_0 W_0 \Psi_{NS} + i \sum_{n=0}^{\infty} \beta_{n0} W_{-n-1/2} \phi_{NS}^{n+1/2} + \sum_{r,s=1/2}^{\infty} \alpha_{rs} F_0 W_{-r-s} \zeta_{NS, s}^r, \\ \delta \chi_R^n &= W_0 \phi_{NS}^n, \\ \delta \phi_R^m &= i \sum_{n=0}^{\infty} \beta_{nm} W_{m-n-1/2} \phi_{NS}^{n+1/2}. \end{aligned} \quad (5.16)$$

The reader may verify from (5.5) that the first two terms in $\delta \Psi_R$, i.e.

$$\delta \Psi_R = F_0 W_0 \Psi_{NS} + i W_{-1/2} \phi_{NS}^{1/2} + \dots \quad (5.17)$$

indeed reproduce (5.1) after the elimination of the auxiliary field $\phi_{NS}^{1/2}$. In a similar

fashion, one can determine the supersymmetry variations of the bosons; they are ($\chi_{NS}^n \equiv \phi_{NS}^n$, $\chi_R^n \equiv \phi_R^n$)

$$\begin{aligned} \delta \Psi_{NS} &= W_0^+ \Psi_R + \sum_{n=1}^{\infty} \gamma_{n0} W^+ \phi_R^n + \sum_{m,n=1}^{\infty} \gamma_{nm} W_{m+n}^+ \zeta_{Rm}^n, \\ \delta \phi_{NS}^r &= i \sum_{n=1}^{\infty} \delta_{ns} W_{n-s}^+ \left(-\frac{1}{2} n \chi_R^n + F_0 \phi_R^n \right), \\ \delta \chi_{NS}^n &= W_0^+ (F_0 \chi_R^n + 2 \phi_R^n), \end{aligned} \tag{5.18}$$

where the coefficients are given by expressions similar to the ones given in (5.14). To verify (5.16) and (5.18), one must make use of the special properties of the coefficients (5.14); for instance, one needs the relation [16]

$$\sum_{r+s=n} \left(s + \frac{1}{2} n \right) \alpha_{rs} = -\frac{1}{8} (3n + 1). \tag{5.19}$$

Although, in (5.16) and (5.18), the infinite set is obviously the appropriate one, one can in principle reformulate these results in terms of the finite set. This is, however, rather awkward since it requires one to re-express all G and F operators occurring in (5.13) through $G_{\pm 1/2}$, $G_{\pm 3/2}$ and $F_{\pm 1}$, $L_{\pm 1}$. One may also try to calculate the variations of the supplementary fields by repeating the above procedure. It then turns out that the master set is still “better” for the following reason. When the first term on the right-hand side of (5.13) acts on a supplementary field ζ_s^r (this corresponds to some component of $D\Psi_1^1$), the Ward identity (5.13) produces terms of the type $G_r \zeta_s^r$ (corresponding to some component of $d\Psi_1^1$). But the infinite set does *not* provide an equation for $d\Psi_1^1$, see (4.2), whereas the master set does provide such an equation, see (4.5). Thus, although one can pass from one set to another by successive gauge fixing, the supersymmetry transformations will become more and more complicated owing the compensating transformations that are necessary to maintain the chosen gauge. We will not pursue this matter here, but would like to point out that the full supersymmetry transformation can presumably be written in a very compact form by absorbing all string fields into a functional field which depends on some (commuting and anticommuting) ghost coordinates $c^\alpha(\sigma)$ [11]. The supersymmetry transformations are then represented by

$$\delta \Psi_R(x^\mu(\sigma), \dots) = W \Psi_{NS}(x^\mu(\sigma), \dots), \tag{5.20}$$

with some operator W . From (5.16) and (5.17) one infers that this operator must depend on the ghosts because (5.20), when expanded, must give rise to mixing between the fundamental string field and the supplementary fields as in (5.16) and (5.17); this observation is in accord with the results of ref. [23]. A possible form of

W is suggested by the form of the recently constructed off-shell vertex for the bosonic theory [24].

Note added

We give here some details on the truncation of the Ramond string field ψ_R , mentioned at the end of sect. 4. When expanded out on the commuting (θ^0, θ_0) and anticommuting (e^0, e_0) ghost zero mode, we restrict Ψ_R to be of the form

$$\Psi_R = \psi_R|0_R\rangle + \theta^0\phi_R|0_R\rangle + e^0F\phi_R|0_R\rangle.$$

In the notation of eq. (3.10)

$$\{e^0, e_0\} = 1 = [\theta^0, \theta_0]$$

and one takes

$$e_0|0_R\rangle = \theta_0|0_R\rangle = 0.$$

The interesting feature of such a restriction on Ψ_R is that $Q_R\Psi_R$ has the same form:

$$Q_R\Psi_R = \{(D + d)\psi_R + (F\Downarrow + \Downarrow F)\phi_R + \theta^0[F\psi_R + (D - d)\phi_R] + e^0F[F\psi_R + (D - d)\phi_R]\}|0_R\rangle.$$

Of course, $Q_R\psi_R = 0$ reproduces the equations of motion (4.12). Thus, if we also restrict the gauge transformation string field to be of the same form, the structures of the supplementary string fields of the Ramond and Neveu-Schwarz sectors become similar, as indeed the explicit equations (2.5) and (2.7) suggest.

One then defines a left-vacuum $\langle 0_R|$ satisfying

$$\langle 0_R|e^0 = 0, \quad \langle 0_R|\theta^0 = 0, \quad \langle 0_R|0_R\rangle = 1.$$

For the inner product of two Ramond string fields Ψ_R^1 and Ψ_R^2 both of the above restricted form, we then take

$$\langle 0_R|(-\theta_0)e_0\Psi_R^{1\dagger}\Psi_R^2|0_R\rangle.$$

It is straightforward to check that our rules give, upon doing the zero mode algebra:

$$\langle 0_R|(-\theta_0)e_0\Psi_R^{1\dagger}\Psi_R^2|0_R\rangle = \langle \psi_R^1, \phi_R^2 \rangle + \langle \phi_R^1, \psi_R^2 \rangle.$$

Hence the inner product is hermitian. The Ramond sector action is then

$$S_R = \frac{1}{2} \langle 0_R | (-\theta_0) e_0 \Psi_R^\dagger Q_R \Psi_R | 0_R \rangle.$$

Expanding ψ_R and ϕ_R on the ghost occupation number basis, one then retrieves for the first supplementary fields the hermitian action of ref. [13], which gives rise to the equations of motion (2.7).

Appendix

In the following, we shall sketch the proofs of the useful properties of the operators introduced in sect. 3. With the exception of (3.44), the properties such as those listed in (3.35)–(3.38) are relatively easy to prove and we shall pick (3.35) and (3.38) as examples.

Proof of $F^2 = K$. Since F and K are both derivations, we shall first prove it on the basic elements. On a $\binom{0}{0}$ from ψ , the proof is trivial. On e_A , using the definition of F , we get

$$\begin{aligned} F^2 e_A &= [F_0, \mathcal{L}_{-A}]^B F e_B \sigma \\ &= [F_0, \mathcal{L}_{-A}]^{-B} [F_0, \mathcal{L}_{-B}]^{-C} e_C \sigma^2 \\ &= [F_0 [F_0, \mathcal{L}_{-A}]]^{-C} e_C. \end{aligned} \tag{A.1}$$

In the last line, we used the fact that the “intermediate states” of the Jacobi triple commutator is, in this case, saturated by the Virasoro operators with negative indices. Now we can use the Jacobi identity to get

$$[F_0 [F_0, \mathcal{L}_{-A}]]^{-C} = [L_0, \mathcal{L}_{-A}]^{-C} = A \delta_A^C. \tag{A.2}$$

Thus, $F^2 e_A = A e_A = K e_A$. In a similar fashion, we get $F^2 e^A = K e^A$. Next it is easy to show $\sigma F e^A = -(-1)^A F e^A \sigma$ and $\sigma F e_A = -(-1)^A F e_A \sigma$. Using this, one proves $\sigma F = -F \sigma$ on any form. This in turn means that when F , as a derivation, goes through another F , we get a minus sign, since F itself contains a σ . Now let e be any basis (including a zero form) and let FF act on a string of bases starting with e . Then,

$$\begin{aligned} F^2(e \dots) &= F((Fe) \dots + eF \dots + \dots) \\ &= (F^2 e) \dots - (Fe)F + (Fe)F + \dots \\ &= K e \dots + e F^2 \dots \end{aligned} \tag{A.3}$$

The process continues and, due to the pairwise cancellation exhibited above, we have K acting on each basic element. This proves $F^2 = K$ on any form.

Proof of $d\Downarrow + \Downarrow d = 0$. The strategy of the proof is exactly the same as in the previous one. The only non-trivial part is the demonstration of $(d\Downarrow + \Downarrow d)e_A = 0$, which we shall now sketch. From the definition of d and \Downarrow , one gets

$$(d\Downarrow + \Downarrow d)e_A = -\frac{1}{2}([\mathcal{L}_B, \mathcal{L}_C]^{A'} \eta_{A'A} - [\mathcal{L}_C, \mathcal{L}_{-A}]^{-B'} \eta_{B'B} + (-)^{BC} [\mathcal{L}_B, \mathcal{L}_{-A}]^{-C'} \eta_{C'C}) e^B e^C. \quad (A.4)$$

This vanishes due to a Jacobi identity. To see this, start from the generic Jacobi identity

$$0 = [[\mathcal{L}_B, \mathcal{L}_C] \mathcal{L}_{-A}] + (-)^{A(B+C)} [[\mathcal{L}_{-A}, \mathcal{L}_B] \mathcal{L}_C] + (-)^{B(A+C)} [[\mathcal{L}_C, \mathcal{L}_{-A}] \mathcal{L}_B] \quad (A.5)$$

and look at the part involving L_0 . For instance, in the last triple commutator, such a part is

$$(-)^{B(A+C)} [\mathcal{L}_C, \mathcal{L}_{-A}]^{-B'} [\mathcal{L}_{-B'}, \mathcal{L}_B] = (-)^{B(A+C)} [\mathcal{L}_C, \mathcal{L}_{-A}]^{-B'} (-)^{1+BB'} [\mathcal{L}_B, \mathcal{L}_{-B'}], \quad (A.6)$$

where we have used (3.9a). This expression contains $-[\mathcal{L}_C, \mathcal{L}_{-A}]^{-B'} 2\eta_{B'B} L_0$. Collecting the coefficients of L_0 from the other two terms, one easily sees that the r.h.s of (A.4) vanishes. Now as before, on a string of bases, one shows

$$(d\Downarrow + \Downarrow d)(e \dots) = (d\Downarrow + \Downarrow d)e \dots + e(d\Downarrow + \Downarrow d) \dots, \quad (A.7)$$

where the signs are $+$ for ψ or e^A , $-$ for e_A . This shows that $d\Downarrow + \Downarrow d$ vanishes on any form.

Proof of $dD - Dd = 2K\mathcal{F}$. When one tries to compute the l.h.s., on a general $\binom{a}{b}$ form ω , using the formulas given in sect. 3, one encounters numerous terms of varying structures. Eventually most of them cancel and the remainder gives precisely the r.h.s. However, a part of the cancellation mechanism is not transparent and it should be helpful to give some guides.

First, the terms involving L_0 and F_0 operators are easily seen to match those on the r.h.s. It is also not difficult to check that the terms proportional to \mathcal{L}_A and \mathcal{L}_{-A} all cancel among each other if one recalls the GAS of appropriate sets of indices. It is the numerical terms handling of which is not trivial. Below we give a typical example of how one should deal with them.

Consider a term, call it $(-Dd\omega)^{22}$, which is produced by operating the second term in (3.27) onto a second term in (3.23). Explicitly, it is of the form (suppressing the basis),

$$\begin{aligned} (-Dd\omega)^{22} &= -(b+1)a(-)^{A(B_1+\dots+B_b)}(-)^{B(A+B_1+\dots+B_b)} \\ &\quad \times [\mathcal{L}_{A_1}, \mathcal{L}_{-B_0}]^{-B} [\mathcal{L}_B, \mathcal{L}_{-C}]^{-A} \omega_{B_1\dots B_b}^{CA_2\dots A_a}. \end{aligned} \quad (\text{A.8})$$

Because of the graded-antisymmetry assumed *before* and *after* D acts on $d\omega$, the above expression actually represents the following:

$$(-Dd\omega)^{22} = C_0 + C_1 + C_2, \quad (\text{A.9})$$

$$\begin{aligned} C_0 &= -b(a-1)^{A(B_1+\dots+B_b)+BB_1+AA_2+B_1(A_2+B+B_2+\dots+B_b)} \\ &\quad \times [\mathcal{L}_A, \mathcal{L}_{-B_0}]^{-B} [\mathcal{L}_{B_1}, \mathcal{L}_{-C}]^{-A_2} \omega_{BB_2\dots B_b}^{CA_3\dots A_a}, \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} C_1 &= b(-)^{A(B_1+\dots+B_b)+BB_1+B_1(A+B+B_2+\dots+B_b)} \\ &\quad \times [\mathcal{L}_A, \mathcal{L}_{-B_0}]^{-B} [\mathcal{L}_{B_1}, \mathcal{L}_{-C}]^{-A} \omega_{BB_2\dots B_b}^{CA_2\dots A_a} \\ &\quad + (a-1)(-)^{A(B_1+\dots+B_b)+AA_2+B(A_2+B_1+\dots+B_b)} \\ &\quad \times [\mathcal{L}_A, \mathcal{L}_{-B_0}]^{-B} [\mathcal{L}_B, \mathcal{L}_{-C}]^{-A_2} \omega_{B_1\dots B_b}^{CA_3\dots A_a}, \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned} C_2 &= -(-)^{A(B_1+\dots+B_b)+B(A+B_1+\dots+B_b)} \\ &\quad \times [\mathcal{L}_A, \mathcal{L}_{-B_0}]^{-B} [\mathcal{L}_B, \mathcal{L}_{-C}]^{-A} \omega_{B_1\dots B_b}^{CA_2\dots A_a}. \end{aligned} \quad (\text{A.12})$$

They are classified according to the number of contractions among the indices of the structure constants. This classification is useful because there are definite patterns associated with these types. C_0 , which has no such contraction, gets completely cancelled. (The reader should find the relevant term in $dD\omega$.) C_1 , on the other hand, has one contraction. For example, consider the term proportional to $[\mathcal{L}_C, \mathcal{L}_{-B_1}]^A [\mathcal{L}_A, \mathcal{L}_{-B_0}]^{-B}$. If one looks through the other parts of $(dD - Dd)\omega$, one finds a term which is of exactly the same structure except that A is replaced by $-A$. Together they form an incomplete triple commutator, i.e.

$$\begin{aligned} &[\mathcal{L}_C, \mathcal{L}_{-B_1}]^A [\mathcal{L}_A, \mathcal{L}_{-B_0}]^{-B} + [\mathcal{L}_C, \mathcal{L}_{-B_1}]^{-A} [\mathcal{L}_{-A}, \mathcal{L}_{-B_0}]^{-B} \\ &= \left[[\mathcal{L}_C, \mathcal{L}_{-B_1}] \mathcal{L}_{-B_0} \right]^{-B} - [\mathcal{L}_C \mathcal{L}_{-B_1}]^0 [L_0, \mathcal{L}_{-B_0}]^{-B} \\ &\quad - [\mathcal{L}_C, \mathcal{L}_{-B_1}]^0 [F_0, \mathcal{L}_{-B_0}]^{-B}. \end{aligned} \quad (\text{A.13})$$

Now it is easy to recognize that the two terms on the r.h.s., for which the intermediate operators are L_0 and F_0 , are precisely those which occur in $(2K + \mathcal{F}F)\omega$. The triple commutator term, on the other hand, finds its partners from the other parts of $(dD - Dd)\omega$ and cancel due to the Jacobi identity. Finally, consider C_2 , in which two indices are contracted among the structure constants. This is, in fact, the only term in $(dD - Dd)\omega$ with such a feature. Together with the central charge term produced from the appropriate commutator in $dD - Dd$, it takes part in the identity (3.43), valid only in 10 dimensions. Again the terms on the r.h.s. of (3.43) are recognized to occur in $2K \Downarrow + \mathcal{F}F$. With the knowledge of the general pattern sketched above, the reader should complete the proof without difficulty.

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