Graviton propagator in de Sitter space

B. Allen*
Department of Physics, Tufts University, Medford, Massachusetts 02155
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We consider the graviton propagator in a de Sitter background. The propagator depends upon the choice of a gauge-fixing term $L_{\text{gauge}}=\frac{1}{2}F^2$, and we consider the "$\epsilon$ gauges" with $F^v=\nabla_u(h^{uv}-\epsilon g^{uv}h^{\sigma}{}_{\sigma})$. We show that the propagator is completely finite and has no infrared divergences provided that ϵ is not given certain "exceptional" values. It is only for these "exceptional" values of ϵ that the propagator has an infrared divergence. We then show that in these exceptional cases the divergences are gauge artifacts and are not physical: they make no contribution to any physical tree-level scattering amplitudes. Furthermore, we show that at one-loop order the zero modes which arise (only) if ϵ is given one of the exceptional values are canceled by the Faddeev-Popov ghosts. There is thus no evidence that the de Sitter background is inconsistent when gravitational fluctuations are considered.

I. INTRODUCTION

The purpose of this paper is to examine the graviton propagator in a de Sitter background. In particular, we are concerned about whether the propagator is finite (for separated points) or whether it is infrared divergent and thus infinite (for separated points).[1] The situation is complicated slightly because the graviton propagator depends upon the choice of a gauge-fixing term in the gravitational action. We show in Sec. II that for "most" gauge-fixing terms there is no divergence in the graviton propagator and that only for certain "exceptional" gauge-fixing terms does an infrared divergence arise. In Sec. III we then demonstrate that in those special instances where divergences do arise, they are harmless gauge artifacts and make no contribution to physical scattering amplitudes. In Sec. IV we show how this cancellation takes place at one-loop order. For brevity we will give only the main arguments and results; details will appear in a later publication.[2] The infrared behavior of gravitons in de Sitter space has also been previously examined.[3–6]

To compute the two-point function we will work on the Euclidean version of de Sitter space, which is a four-sphere S^4 of radius a. The cosmological constant $\Lambda=3/a^2$. The two-point function is a maximally symmetric bitensor[7] and is a function of the distance $\mu(x,x')$ between the two points x and x'. *The same function of μ is also the two-point function for de Sitter space with a Lorentzian signature metric.* The only difference is the range of μ: in the Euclidean case $\mu^2\in[0,\infty)$ but in the Lorentzian case $\mu^2\in(-\infty,\infty)$. Thus we use the four-sphere as a tool for finding the propagator, but having obtained it, we have also found the propagator on the original Lorentzian physical spacetime.

II. THE FREE GRAVITATIONAL ACTION AND THE GRAVITON PROPAGATOR

We begin by introducing a complete set of pure spin-2, spin-1, and spin-0 functions on the four-sphere.[8,9] These functions are chosen to be eigenfunctions of the Laplacian $\Box$. The spin-0 scalar functions satisfy

$$\Box\phi_n=\lambda_n^{(0)}\phi_n,\quad \text{eigenvalues: } \lambda_n^{(0)}=-\frac{\Lambda}{3}n(n+3)\,,$$
$$\text{degeneracy: } (n+1)(n+2)(2n+3)/6\,. \tag{2.1}$$

The index n equals $0,1,2,\ldots$. The lowest eigenfunction $\phi_0=\text{const}$ and thus $\lambda_0^{(0)}=0$. When $n=1$ there are five degenerate functions ϕ_1^i $(i=1-5)$ with eigenvalue $\lambda_1^{(0)}=-4\Lambda/3$. These functions will become important later. They can be simply expressed: if we embed the sphere in R^5 as $a^2=X_1{}^2+\cdots+X_5{}^2$ then $\phi_1^i\propto X_i$. The derivatives of these functions $\nabla_\mu\phi_1^i$ are the five conformal Killing vectors on the four-sphere. They therefore obey the equation

$$\nabla_u\nabla_v\phi_1^i=-\frac{\Lambda}{3}g_{uv}\phi_1^i,\quad i=1,\ldots,5\,. \tag{2.2}$$

The spin-1 vector functions ξ_n^u are transverse so $\nabla_u\xi_n^u=0$ and satisfy the eigenvalue equation

$$\Box\xi_n^u=\lambda_n^{(1)}\xi_n^u,\quad \text{eigenvalues: } \lambda_n^{(1)}=-\frac{\Lambda}{3}(n^2+5n+3)\,,$$
$$\text{degeneracy: } (n+1)(n+4)(2n+5)/2\,. \tag{2.3}$$

As before, the index n equals $0,1,2,\ldots$. When $n=0$ there are ten degenerate eigenvectors with eigenvalue $-\Lambda$. These $(\xi_0^u)^i$ $(i=1-10)$ are the ten Killing fields that generate rotations of the four-sphere. The spin-2 tensor harmonics h_n^{uv} are symmetric $h_n^{[uv]}=0$, traceless $g_{uv}h_n^{uv}=0$, and transverse $\nabla_u h_n^{uv}=0$. They satisfy the eigenvalue equation

$$\Box h_n^{uv}=\lambda_n^{(2)}h_n^{uv},\ \text{eigenvalues: } \lambda_n^{(2)}=-\frac{\Lambda}{3}(n^2+7n+8)\ ,$$
$$\text{degeneracy: } 5(n+1)(n+6)(2n+7)/6\ , \tag{2.4}$$

and again the index n equals $0,1,2,\ldots$. We will shortly expand an arbitrary metric perturbation in terms of these three complete sets.

Using the different spin eigenfunctions given above, we now construct a complete basis for all symmetric second-rank tensor fields on the four-sphere. First normalize the basis functions above so that

$$\delta_{ij}\delta_{nm}=\int \phi_{n,i}\phi_{m,j}d(\text{vol})=\int \xi^{\mu}_{n,i}\xi^{m,j}_{\mu}d(\text{vol})$$
$$=\int h^{vv}_{n,i}h^{m,j}_{vv}d(\text{vol})\ . \tag{2.5}$$

(In this equation we have added another index i that labels the degenerate set of eigenfunctions. In general we will continue to suppress this index whenever possible. Thus for example $\sum_n a_n\phi_n$ means $\sum_{n,i}a_{n,i}\phi_{n,i}$, i.e., all summations $\sum_n$ include an implicit suppressed degeneracy index.) Now define the symmetric tensor modes

$$V_n^{uv}=[-\tfrac{1}{2}(\lambda_n^{(1)}+\Lambda)]^{-1/2}\nabla^{(u}\xi_n^{v)}\ \text{ for } n=1,2,\ldots,$$
$$W_n^{uv}=[\lambda_n^{(0)}(\tfrac{3}{4}\lambda_n^{(0)}+\Lambda)]^{-1/2}(\nabla^u\nabla^v-\tfrac{1}{4}g^{uv}\Box)\phi_n$$
$$\text{for } n=2,3,\ldots, \tag{2.6}$$
$$X_n^{uv}=\tfrac{1}{2}g^{uv}\phi_n\ \text{ for } n=0,1,\ldots .$$

Together with the original transverse-traceless (TT) functions h_n^{uv}, these form the desired normalized basis. V_n^{uv} is the transverse (T) spin-1 part, W_n^{uv} is the longitudinal (L) spin-0 part, and X_n^{uv} is the pure trace (PT) spin-0 part. Notice that the index n does not start from zero for either V_n^{uv} or W_n^{uv}, since, for example, $\nabla^u\nabla^v\phi_0=\nabla^u\nabla^v(\text{const})=0$. Similarly, the right-hand side of expression (2.6) for W_n^{uv} vanishes for $n=1$ as a consequence of (2.2).

An *arbitrary* perturbation h^{uv} of the metric tensor can now be represented as

$$h^{uv}=\sum_0^\infty a_n h_n^{uv}+\sum_1^\infty b_n V_n^{uv}+\sum_2^\infty c_n W_n^{uv}+\sum_0^\infty e_n X_n^{uv}\ , \tag{2.7}$$

where the coefficients $\{a_n,b_n,c_n,e_n\}$ are a countably infinite set of constants that are uniquely determined by h^{uv}. [Note that these coefficients have a suppressed degeneracy index, as explained after Eq. (2.5).] Thus the measure on the space of all metric perturbations is

$$d[h^{uv}]=\left[\prod_0^\infty da_n\right]\left[\prod_1^\infty db_n\right]\left[\prod_2^\infty dc_n\right]\left[\prod_0^\infty de_n\right]\ .$$

The reader unhappy with this mode expansion may find it helpful to count degrees of freedom. The perturbation h^{uv} contains ten arbitrary functions. These are divided between the different components as h_n^{uv}:5, V_n^{uv}:3, W_n^{uv}:1, and X_n^{uv}:1.

The quadratic part of the gravitational action, including an adjustable gauge-fixing term, is[10]

$$S+S_{\text{gauge}}=(16\pi G)^{-1}$$
$$\times\int\{\tfrac{1}{4}\tilde{h}^{uv}(-\Box+\tfrac{2}{3}\Lambda)h_{uv}$$
$$+\tfrac{1}{6}\Lambda h^2-\tfrac{1}{2}(\nabla_u\tilde{h}^{uv})^2$$
$$+\tfrac{1}{2}[\nabla_u(h^{uv}-\epsilon g^{uv}h)]^2\}d(\text{vol})\ . \tag{2.8}$$

Here $h\equiv h^u{}_u$ and $\tilde{h}^{uv}\equiv h^{uv}-\tfrac{1}{2}g^{uv}h$. Thus for the general metric perturbation (2.7) on the four-sphere, the quadratic part of the action is

$$64\pi G(S+S_{\text{gauge}})=\sum_0^\infty a_n{}^2(-\lambda_n^{(2)}+\tfrac{2}{3}\Lambda)+\sum_1^\infty b_n{}^2(-\lambda_n^{(1)}-\Lambda)+\sum_2^\infty c_n{}^2(-\lambda_n^{(0)}-2\Lambda)$$
$$+\sum_0^\infty e_n{}^2[(1+4\epsilon-8\epsilon^2)\lambda_n^{(0)}+2\Lambda]+\sum_2^\infty 8(\epsilon-\tfrac{1}{2})[\lambda_n^{(0)}(\tfrac{3}{4}\lambda_n^{(0)}+\Lambda)]^{1/2}c_n e_n\ . \tag{2.9}$$

The way to think about this formula is as follows: Each possible field configuration $h^{uv}(x)$ is defined by an infinite set of constants $\{a_n,b_n,c_n,e_n\}$, which are like Fourier coefficients in (2.7). The value of the action for the configuration specified by these constants is then given by (2.9). Note that for the "standard" choice of gauge, which is $\epsilon=\tfrac{1}{2}$, the $c_n e_n$ cross term vanishes and the action is the sum of squares.

Now the two-point function can be easily obtained. The propagator is a sum of terms:

$$G^{abc'd'}(x,x')=\langle h^{ab}(x)h^{c'd'}(x')\rangle$$
$$=64\pi G(G_{\text{TT}}^{abc'd'}+G_T^{abc'd'}+G_L^{abc'd'}$$
$$+G_{\text{PT}}^{abc'd'}+G_{\text{CT}}^{abc'd'})\ . \tag{2.10}$$

The different parts are, respectively, the transverse-traceless (TT) spin-2 part, the transverse (T) spin-1 part, the longitudinal (L) spin-0 part, the pure-trace (PT) spin-0 part, and a spin-0 cross term (CT). The cross term arises because when $\epsilon\neq\tfrac{1}{2}$ there is a $c_n e_n$ cross term between the PT and L terms in the action (2.9). Note that on the four-sphere the points x and x' are always spacelike separated, and the field operators commute. Thus $G^{abc'd'}(x,x')=G^{c'd'ab}(x',x)$.

The propagator may be easily obtained from the action (2.9) by the following "inversion" process. One inserts the mode expansion (2.7) for $h^{ab}(x)$, and expression (2.9) for the action, into the definition

$$G^{abc'd'}=\frac{\int h^{ab}(x)h^{c'd'}(x')\exp[-(S+S_{\text{gauge}})]d[h^{uv}]}{\int \exp[-(S+S_{\text{gauge}})]d[h^{uv}]}\ ,$$

and evaluates the resulting Gaussian integrals, which are of the form

$$\int x^2 \exp(-\lambda_n x^2)dx \Big/ \int \exp(-\lambda_n x^2)dx = (2\lambda_n)^{-1} .$$

One thus obtains[2]

$$G_{\rm TT}^{abc'd'} = \frac{1}{2}\sum_0^\infty \frac{h_n^{ab}(x)h_n^{c'd'}(x')}{-\lambda_n^{(2)}+\frac{2}{3}\Lambda} ,$$

$$G_T^{abc'd'} = \sum_1^\infty \frac{\nabla^{(a}\xi_n^{b)}(x)\nabla^{(c'}\xi_n^{d')}(x')}{(\lambda_n^{(1)}+\Lambda)^2} ,$$

$$G_L^{abc'd'} = (\nabla^a\nabla^b - \tfrac{1}{4}g^{ab}\Box)(\nabla^{c'}\nabla^{d'} - \tfrac{1}{4}g^{c'd'}\Box')g_\epsilon(x,x') , \tag{2.11}$$

$$G_{\rm PT}^{abc'd'} = g^{ab}g^{c'd'}f_\epsilon(x,x') ,$$

$$G_{\rm CT}^{abc'd'} = [g^{ab}(\nabla^{c'}\nabla^{d'} - \tfrac{1}{4}g^{c'd'}\Box') + g^{c'd'}(\nabla^a\nabla^b - \tfrac{1}{4}g^{ab}\Box)]h_\epsilon(x,x') .$$

The TT and T sums are evaluated in Ref. 11 using the methods of Ref. 7; they are ϵ independent and finite. The scalar functions f_ϵ, g_ϵ, and h_ϵ depend upon the gauge-fixing parameter ϵ, and are

$$f_\epsilon = \frac{\Lambda}{384\pi^2} + \frac{3}{256\Lambda}(\epsilon-\tfrac{1}{4})^{-2}\sum_{i=1}^{5}\phi_1^i(x)\phi_1^i(x') + \frac{1}{32}\sum_2^\infty \frac{\lambda_n^{(0)}+2\Lambda}{[(1-\epsilon)\lambda_n^{(0)}+\Lambda]^2}\phi_n(x)\phi_n(x') ,$$

$$g_\epsilon = -\frac{1}{8}\sum_2^\infty \frac{(1+4\epsilon-8\epsilon^2)\lambda_n^{(0)}+2\Lambda}{[(1-\epsilon)\lambda_n^{(0)}+\Lambda]^2(\frac{3}{4}\lambda_n^{(0)}+\Lambda)\lambda_n^{(0)}} \times \phi_n(x)\phi_n(x') , \tag{2.12}$$

$$h_\epsilon = \tfrac{1}{4}(\epsilon-\tfrac{1}{2})\sum_2^\infty \frac{\phi_n(x)\phi_n(x')}{[(1-\epsilon)\lambda_n^{(0)}+\Lambda]^2} .$$

From these expressions we can easily understand the infrared behavior of the propagator.

We will now show that the two-point function has an infrared divergence if and only if one of the denominators in (2.12) vanishes. If all of the denominators are nonzero then the two-point function is finite for $x \neq x'$. Thus, provided that the gauge-fixing parameter ϵ does not take on one of the "exceptional" values $\epsilon = \frac{1}{4}, \frac{7}{10}, \frac{5}{6}, \ldots$,

$$\epsilon_{\rm exceptional} = \frac{n^2+3n-3}{n(n+3)}, \quad n=1,2,\ldots , \tag{2.13}$$

the propagator is finite and free of infrared divergences.

The two-point function for a scalar field of mass m is[7]

$$G(m^2,Z) = \sum_0 \frac{\phi_n(x)\phi_n(x')}{-\lambda_n^{(0)}+m^2} = \frac{\Gamma(\frac{3}{2}+v)\Gamma(\frac{3}{2}-v)}{16\pi^2 a^2}F(\tfrac{3}{2}+v,\tfrac{3}{2}-v;2;Z(x,x')) , \tag{2.14}$$

where $v = (\frac{9}{4} - a^2m^2)^{1/2}$ and $Z(x,x') = \cos^2[\mu(x,x')/2a]$. Here $\mu(x,x')$ is the geodesic distance between the two spacetime points x and x', and a is the radius of the four-sphere, so $\Lambda = 3/a^2$. One can see that $G(m^2,Z)$ is finite provided that all of the denominators in the mode sum (2.14) are nonzero. This is because the nonpositive values of m^2 for which the denominators vanish are the same as the values of m^2 for which $\frac{3}{2}-v$ is a negative integer and the gamma function $\Gamma(\frac{3}{2}-v)$ diverges.

If we define a mode sum which is identical to (2.14) except that it starts at $n=2$, we obtain

$$\tilde{G}(m^2,Z) = G(m^2,Z) - \tfrac{3}{8}\pi^{-2}m^{-2}a^{-4} - \tfrac{15}{8}\pi^{-2}a^{-2}(4+m^2a^2)^{-1}(2Z-1) . \tag{2.15}$$

Now expanding the denominator of (2.12) using partial fractions, and observing that $(\partial/\partial m^2)G(m^2,Z)$ is the mode sum with "squared" denominators, we obtain the following closed forms for f, g, and h:

$$f_\epsilon = -\tfrac{1}{32}(1-\epsilon)^{-3}\left[\Lambda(1-2\epsilon)\frac{\partial}{\partial k} + (1-\epsilon)\right]G(k,Z) ,$$

$$g_\epsilon = \tfrac{1}{4}\Lambda^{-2}\tilde{G}(0,Z) + \tfrac{1}{2}\Lambda^{-2}\tilde{G}(k,Z) - \tfrac{3}{4}\Lambda^{-2}\tilde{G}(-\tfrac{4}{3}\Lambda,Z) - \tfrac{1}{2}\Lambda^{-1}(1-2\epsilon)(1-\epsilon)^{-1}\frac{\partial}{\partial k}\tilde{G}(k,Z) , \tag{2.16}$$

$$h_\epsilon = -\tfrac{1}{4}(\epsilon-\tfrac{1}{2})(1-\epsilon)^{-2}\frac{\partial}{\partial k}\tilde{G}(k,Z) ,$$

where $k = \Lambda(\epsilon-1)^{-1}$. The reader can easily confirm that these quantities are finite and free of infrared divergences provided that ϵ is not given one of the exceptional values defined in (2.13). For example if we take the "standard" gauge $\epsilon = \frac{1}{2}$ then f, g, and h are all finite.

In a certain sense we have finished, because we have shown that the graviton propagator is finite, unless one makes a bad choice for the gauge-fixing parameter ϵ. In the next section we show what happens in these exceptional cases, for example, if $\epsilon = \frac{1}{4}$. We will see that although the propagator diverges in that case, the scattering amplitude remains finite and ϵ independent. This proves that the infrared divergences that occur for the exceptional choices of ϵ are gauge artifacts.

III. GAUGE DEPENDENCE OF THE GRAVITON PROPAGATOR AND TREE-LEVEL SCATTERING

Suppose that we add to our action for free gravity an interaction term

$$S_I = \tfrac{1}{2}\int [(\nabla_u\phi)^2 + m^2\phi^2]d(\text{vol}) , \tag{3.1}$$

which is the action of a massive scalar field ϕ. We can now calculate, for example, the tree-level scattering of two scalar particles, as shown in Fig. 1. The amplitude for this process is

$$A = \left\langle \text{in} \left| \int\int T_{uv}(x)G^{uv\alpha'\beta'}(x,x')T_{\alpha'\beta'}(x')\sqrt{g}\,d^4x\sqrt{g'}\,d^4x' \right| \text{out} \right\rangle , \tag{3.2}$$

where $|\,\mathrm{in}\rangle$ and $|\,\mathrm{out}\rangle$ represent the arbitrary in and out states of the scalar field. The stress tensor T_{uv} is the operator

$$T_{uv}(x)=\frac{1}{\sqrt{g}}\frac{2\delta S_I}{\delta g^{uv}(x)}=(\nabla_u\phi)(\nabla_v\phi)-\tfrac{1}{2}g_{uv}[(\nabla_\sigma\phi)^2+m^2\phi^2] \tag{3.3}$$

which is the variation of the interaction action with respect to the variation of the background metric.

Now inserting the graviton propagator (2.10) into the amplitude (3.2) one sees that $G_{\mathrm{TT}}^{abc'd'}$ and $G_T^{abc'd'}$ give ϵ-independent and finite contributions to A. The ϵ-dependent terms can be written as

$$A_L+A_{\mathrm{PT}}+A_{\mathrm{CT}}=64\pi G\left\langle \mathrm{in}\left|\int\int T^a{}_a(x)\rho_\epsilon(x,x')T^{b'}{}_{b'}(x')\sqrt{g}\,d^4x\sqrt{g'}\,d^4x'\right|\mathrm{out}\right\rangle , \tag{3.4}$$

where $\rho_\epsilon=f_\epsilon+\Box\Box' g_\epsilon/16-(\Box+\Box')h_\epsilon/4$. In obtaining (3.4) we have assumed that the stress tensor is conserved, i.e., that $\nabla_u T^{uv}=0$. This is true for the matrix element of T^{uv} between (on-shell) physical states such as $|\,\mathrm{in}\rangle$ and $|\,\mathrm{out}\rangle$. We have also integrated by parts. The boundary terms vanish since T^{uv} is conserved and symmetric.[12]

The function $\rho_\epsilon(x,x')$ that appears in the integrand of (3.4), which represents the ϵ-dependent contribution to the amplitude A, can be obtained from (2.12). It is

$$\rho_\epsilon=\frac{\Lambda}{384\pi^2}+\frac{3}{256\Lambda}(\epsilon-\tfrac{1}{4})^{-2}\sum_{i=1}^{5}\phi_1^i(x)\phi_1^i(x')+\frac{1}{12}\sum_{2}^{\infty}\frac{\phi_n(x)\phi_n(x')}{\lambda_n^{(0)}+\frac{4}{3}\Lambda}$$
$$=\frac{\Lambda}{384\pi^2}+\frac{5\Lambda}{2048\pi^2}(\epsilon-\tfrac{1}{4})^{-2}(2Z-1)-\tfrac{1}{12}\tilde{G}(-\tfrac{4}{3}\Lambda,Z) \;. \tag{3.5}$$

From this expression one can see that the scattering amplitude A contains a single term that appears to depend upon ϵ, and which appears to diverge when $\epsilon=\frac{1}{4}$. We will now show that this term contributes *nothing* to A, for *any* value of ϵ.

In the amplitude (3.4) the ϵ-dependent term is

$$\frac{3}{256\Lambda}(\epsilon-\tfrac{1}{4})^{-2}\sum_{i=1}^{5}\left\langle \mathrm{in}\left|\int\int T^a{}_a(x)\phi_1^i(x)T^{a'}{}_{a'}(x')\phi_1^i(x')\sqrt{g}\sqrt{g'}\,d^4x\,d^4x'\right|\mathrm{out}\right\rangle . \tag{3.6}$$

Inserting a complete set of (physical) states of the ϕ field, $1=\sum_C|\,C\rangle\langle C\,|$ we obtain for the (apparently) ϵ-dependent term of A

$$\frac{3}{256\Lambda}(\epsilon-\tfrac{1}{4})^{-2}\sum_C\sum_i\int\langle \mathrm{in}\,|\,T^a{}_a(x)\,|\,C\rangle\phi_1^i(x)\sqrt{g}\,d^4x\int\langle C\,|\,T^{a'}{}_{a'}(x')\,|\,\mathrm{out}\rangle\phi_1^i(x')\sqrt{g'}\,d^4x' \;. \tag{3.7}$$

We will now show that this term vanishes, because the ϕ_1^i obey Eq. (2.2) which is

$$\nabla_u\nabla_v\phi_1^i=-\frac{\Lambda}{3}g_{uv}\phi_1^i \;.$$

Because the stress tensor between physical states is conserved, we have

$$0=\int\langle \mathrm{in}\,|\,\nabla_v T^{uv}(x)\,|\,C\rangle\nabla_u\phi_1^i\sqrt{g}\,d^4x \tag{3.8}$$

but integrating by parts this means that from (2.2)

$$0=\int\langle \mathrm{in}\,|\,T^{uv}\,|\,C\rangle\nabla_u\nabla_v\phi_1^i d(\mathrm{vol})$$
$$=-\frac{\Lambda}{3}\int\langle \mathrm{in}\,|\,T^u{}_u(x)\,|\,C\rangle\phi_1^i d(\mathrm{vol}) \;. \tag{3.9}$$

Thus we see that as long as the interaction stress tensor is conserved, the tree-level scattering amplitude is gauge invariant because the term proportional to $(\epsilon-\frac{1}{4})^{-2}$ contributes nothing. This means that the scattering amplitude is finite for any value of ϵ. It does not matter if ϵ is given one of the exceptional values, or even if $\epsilon=\frac{1}{4}$, because the only ϵ-dependent term in the amplitude vanishes. Thus we conclude that the apparent infrared divergences that occur for the exceptional values of ϵ are gauge artifacts, and are not physical.

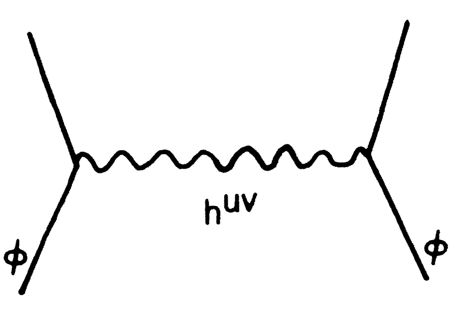

FIG. 1. Tree-level scattering of a scalar field ϕ by a graviton h^{uv}.

IV. THE GRAVITON ZERO MODES ARE CANCELED BY GHOSTS IF $\epsilon=\frac{1}{4}$

In this section we compute the one-loop partition function Z. This partition function can be expressed as a functional determinant which can then be evaluated by various regularization schemes, for example, by ζ-function regularization.[8–10,13] The details of this calculation may be found in Ref. 2 but the results alone are very simple.

The partition function Z may be expressed as a path integral

$$Z=\Delta\int d[h^{uv}]\exp(-S[h^{uv}]-S_{\text{gauge}}[h^{uv}]) , \qquad (4.1)$$

where the measure $d[h^{uv}]$ on the space of metric perturbations is defined just after Eq. (2.7), the gauge-fixed action is given in (2.8), and Δ is the Faddeev-Popov ghost determinant which compensates for the effects of the arbitrarily chosen gauge-fixing term. The path integral in (4.1) (without the Δ factor) is a product of Gaussians, and equals[14]

$$\text{Det}_0^\infty(-\lambda_n^{(2)}+\tfrac{2}{3}\Lambda)^{-1/2}\text{Det}_1^\infty(-\lambda_n^{(1)}-\Lambda)^{-1/2} \times\text{Det}_2^\infty[(1-\epsilon)\lambda_n^{(0)}+\Lambda]^{-1}[\Lambda(\epsilon-\tfrac{1}{4})]^{-1} . \qquad (4.2)$$

The first two factors arise from the TT and T parts of the action. The remaining factors arise from the combined effects of the L and PT parts of the action. Notice that as $\epsilon\rightarrow\frac{1}{4}$ the PT determinant has a zero mode which makes the path integral diverge.

The Faddeev-Popov ghost determinant Δ on the other hand is

$$\Delta=\text{Det}_1^\infty(-\lambda_n^{(1)}-\Lambda)\text{Det}_2^\infty[(1-\epsilon)\lambda_n^{(0)}+\Lambda] \times[\Lambda(\epsilon-\tfrac{1}{4})] \qquad (4.3)$$

and it has a zero mode that makes it vanish as $\epsilon\rightarrow\frac{1}{4}$. The product of (4.2) and (4.3), which is the partition function Z, is entirely independent of ϵ:

$$Z=\frac{\text{Det}_1^\infty(-\lambda_n^{(1)}-\Lambda)^{1/2}}{\text{Det}_0^\infty(-\lambda_n^{(2)}+\frac{2}{3}\Lambda)^{1/2}} . \qquad (4.4)$$

If there had been a true infrared divergence then Z would have had a zero mode in the denominator, and would have diverged. This result can also be found in the work of Yasuda.[15]

V. CONCLUSION

We have shown that the graviton propagator in de Sitter space is free of infrared divergences if the gauge-fixing term is correctly chosen. This same conclusion has also been reached by Higuchi.[6] In addition, we have shown that the infrared divergence which occurs if the gauge-fixing parameter ϵ is given one of the "exceptional" values is not a physical divergence: it is a gauge artifact. Thus a bad choice of ϵ, for example $\epsilon=\frac{1}{4}$, can lead to formidable problems,[1] especially if one is not aware that the resulting divergences are not physical.

The reason why the exceptional values of the gauge-fixing parameter $\epsilon=(n^2+3n-3)/(n^2+3n)$ $(n=1,2,\ldots)$ cause infrared divergences in the graviton propagator is easy to understand. For the exceptional values of ϵ the classical gauge-fixing condition

$$F^v=\nabla_u(h^{uv}-\epsilon g^{uv}h^\sigma{}_\sigma)=0 \qquad (5.1)$$

fails to determine h^{uv} uniquely: it fails to fix the gauge. This is because under a gauge transformation $h^{uv}\rightarrow h^{uv}+k^{(u;v)}$, the gauge-fixing term transforms as $F^v\rightarrow F^v+\nabla_u(k^{(u;v)}-\epsilon g^{uv}k^\sigma{}_{;\sigma})$. Thus the gauge condition determines h^{uv} uniquely if and only if all solutions k^u to the equation $\nabla_u(k^{(u;v)}-\epsilon g^{uv}k^\sigma{}_{;\sigma})=0$ satisfy $k^{(u;v)}=0$. This is in fact the case *if* ϵ is not given one of the "exceptional" values. If, however, ϵ *is* given one of the exceptional values $(n^2+3n-3)/(n^2+3n)$ then $k^u=\nabla^u\phi_n$ is a solution to the above equation, but has $k^{(u;v)}\neq 0$. Thus for the exceptional values of ϵ the gauge-fixing condition fails to fix the gauge.

The reader who is not sure if these Euclidean results obtained on S^4 apply equally well to Lorentzian de Sitter is urged to reread the Introduction. As we stressed there, the four-sphere is used to obtain the two-point function for spacelike separations $(\mu^2>0)$. The same function is *also* the two-point function on the Lorentzian spacetime, only there the range of μ^2 is $-\infty<\mu^2<\infty$. Formula (2.14) illustrates this nicely; it is explained in detail in Ref. 7.

In the case of a massive scalar field, the requirement of de Sitter invariance does *not* select a unique vacuum state. There *is* however a unique de Sitter-invariant vacuum state which is specified by either of the two *additional* requirements:[4] (1) The propagator has Hadamard form, or (2) the propagator is finite for x and x' antipodal points. This state is commonly called the "Euclidean" vacuum state.

The situation for the graviton is identical. In our calculation we have used the path-integral method, and integrated over all field configurations which are regular on S^4. Thus we have obtained the graviton two-point function in the "Euclidean" vacuum state. It satisfies both of the above requirements.

The alert reader will have noted that we have given nice closed forms (2.16) for the spin-0 scalar part of the graviton propagator but that we have not given the corresponding formulas for the spin-2 (TT) and spin-1 (T) parts (2.11). These quantities are maximally symmetric bitensors and can be found using the same techniques as in the vector case.[7] The solutions will be published shortly.[11]

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*Present address: DAF, Observatoire de Paris, Meudon, 5, Place Janssen, 92195 Meudon Principal Cedex, France.

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[12]A typical boundary term is of the form $\int T_{ab}(\nabla^b\psi)d\Sigma^a$ where $\Sigma=\Sigma_{in}-\Sigma_{out}$ is composed of two three-surfaces bounding the interaction region, and ψ is a scalar function. Upon further integration by parts, the conservation equation $\nabla^b T_{ab}=0$ implies that this boundary term is of the form $\int T_{ab}d\sigma^{ab}$ where $\sigma=\partial\Sigma=0$ since $\partial^2=0$. Thus the boundary terms all vanish.

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[14]Our symbolic notation is defined as in Ref. 13. Thus

$$\text{Det}_{n=k}^{\infty}(\lambda_n)\equiv[-\zeta'(0)+\zeta(0)\ln m^2] ,$$

where

$$\zeta(z)=\sum_{n=k}^{\infty}\lambda_n^{-z}$$

and m^2 is a regularization mass. It can be thought of formally as the product $\prod_{n=k}^{\infty}\lambda_n$, and it vanishes when any $\lambda_n\to 0$.

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