Nuclear Physics B264 (1986) 573-587

GAUGE COVARIANT LOCAL FORMULATION OF FREE STRINGS AND SUPERSTRINGS

A. NEVEU, H. NICOLAI and P.C. WEST[1]

CERN, Geneva, Switzerland

Received 19 August 1985

A complete local gauge covariant formulation of all known free string theories is given. It only involves a finite number of supplementary string fields.

1. Introduction

A recurrent theme in modern physics is that apparent inconsistencies between different branches of physics have been overcome by introducing a unifying theory that possesses a local symmetry. What is equally remarkable is that these theories are specified entirely by a knowledge of the appropriate local gauge symmetry group and the fields upon which the symmetry is realized as well as the principles of causality and unitarity.

In the last few years it has become apparent that quantum mechanics and gravity are not reconcilable within the framework of point quantum field theory. Fortunately quantum field theories based on strings have been developed since around 1970 and it is now widely believed that these theories may provide a consistent theory of gravity quantum mechanics and Yang-Mills symmetries.

Despite the relatively long history of strings the local gauge symmetry underlying these theories has not been found, as a result there has not existed a covariant second-quantized description of string theories. Presumably this symmetry principle uniquely determines string theories and is responsible for their many known miraculous properties as well as many more properties to be discovered.

From a practical point of view it is desirable to have a symmetry manifest when quantizing a theory. In general, results are more easily proved in a second-quantized covariant formulation. Up to now, the most popular method of quantizing string theories has been the sum over histories in a first-quantized formalism [for reviews, see ref. [1]]. The analogue for point field theories is a sum over world trajectories of the point particle. This latter scheme has been superseded by a true second-quantized description of point field theories, where results are more easily available. There also exists a second-quantized form of strings in the light cone gauge [2] and more recently in more general gauges [3].

[1] Permanent address: King's College, London WC2, UK.

Recently, a gauge covariant formulation of bosonic string theories was given [4]. This formulation was explicitly found for the first six levels for the free theory and a systematic procedure to find the free action to all levels was given. The general technique to find the interacting theory was explained and the first interaction term was found*. Other attempts [6, 7] to find such a formulation started from the known projector [8] onto physical states. However, these formulations were non-local and we have argued that they cannot extend to the interacting theory.

The crucial observation to find the results of ref. [4] was that the string field itself did not provide enough degrees of freedom to describe its gauge covariant propagation. Supplementary string fields were needed and could be eliminated by an appropriate gauge choice to recover the well-known on-shell conditions. In this paper we find the local gauge covariant formulation of all known string theories *for all levels*. These formulations require only a finite number of supplementary string fields. All open strings require only six supplementary fields. Closed strings require twenty supplementary fields, except the Ramond-Ramond sector of closed superstrings, which requires twenty four.

It is remarkable that this also provides a complete and surprisingly simple solution to the problem of a causal, covariant and unitary description of massive fields of arbitrarily high "spin", at least at the non-interacting level.

2. Conformal (super)algebras in two dimensions

For the convenience of the reader, we briefly review some basic facts about the conformal and superconformal algebras which generate the conformal and superconformal transformations on the world-sheet [the interested reader may consult ref. [1] for further details]. It is well known that ordinary open strings are parametrized in terms of the normal mode operators α^{μ}_{m}, $m \in Z$ which satisfy the commutation relations ($\alpha^{\mu}_{-m} = \alpha^{\mu\dagger}_{m}$)

$$[\alpha^{\mu}_{m}, \alpha^{\nu}_{n}] = m\delta_{m+n,0}g^{\mu\nu} . \tag{2.1}$$

From these one constructs the Virasoro operators

$$L_m \equiv \tfrac{1}{2} \sum_{n=-\infty}^{+\infty} :\alpha_{m-n}\alpha_n: \tag{2.2}$$

which satisfy the algebra [1]

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}Dm(m^2-1)\delta_{m+n,0} . \tag{2.3}$$

A consistent bosonic string theory exists only for $D=26$, in which case one can prove the absence of ghosts. In this case, the physical states obey the generalized Klein-Gordon equation

$$(L_0 - 1)\psi = 0 , \tag{2.4}$$

* For an early attempt in this direction, see ref. [5].

as well as the constraints

$$L_n\psi = 0 \qquad \text{for } n \geq 1 . \tag{2.5}$$

In a covariant formulation of string theories, (2.4) and (2.5) should emerge only after choosing a suitable gauge. Thus, to obtain such a formulation, one must relax these constraints and allow for appropriate gauge transformations. It is not difficult to see that these are generated by the operators L_{-n} with $n \geq 1$. It then follows from (2.3) that all operators in this gauge algebra can be generated out of the two operators L_{-1} and L_{-2} by taking successive commutators, as, e.g.,

$$L_{-3} = [L_{-1}, L_{-2}] , \qquad \text{etc.} \ldots . \tag{2.6}$$

If one considers closed bosonic strings, one simply has to double the number of operators since the string is now described by left-moving modes $\bar{\alpha}^{\mu}_m$ and right-moving modes α^{μ}_m. The corresponding operators L_m and $\bar{L}_m$ obey the same algebra as in (2.3); moreover $[L_m, \bar{L}_n] = 0$. The physical states satisfy constraints completely analogous to (2.4) and (2.5). In addition, the requirement that there should be no distinguished point on the string implies

$$(L_0 - \bar{L}_0)\psi = 0 . \tag{2.7}$$

For spinning strings, the algebra (2.3) is extended to the superconformal algebra in two dimensions. As is well known, one distinguishes two cases depending on the boundary conditions for the world-sheet fermions. The Ramond (R) sector [9] describes space-time fermions with normal mode operators α^{μ}_m and d^{μ}_m. The Neveu-Schwarz (NS) sector [10] describes space-time bosons with normal mode operators α^{μ}_m and b^{μ}_r where $r \in \mathbb{Z} + \frac{1}{2}$. In the R-sector, the superconformal algebra is generated by the operators L_m and F_m and is characterized by the (anti-)commutation relations

$$\begin{aligned} \{F_m, F_n\} &= 2L_{m+n} + \tfrac{1}{2}Dm^2\delta_{m+n,0} , \\ [L_m, F_n] &= (\tfrac{1}{2}m - n)F_{m+n} \qquad (m, n \in \mathbb{Z}) , \\ [L_m, L_n] &= (m-n)L_{m+n} + \tfrac{1}{8}Dm^3\delta_{m+n,0} . \end{aligned} \tag{2.8}$$

The physical states obey the generalized Dirac equation [9]

$$F_0\psi = 0 \tag{2.9}$$

and the constraints

$$F_n\psi = L_n\psi = 0 \qquad \text{for } n \geq 1 . \tag{2.10}$$

The algebra of gauge transformations in this sector is therefore generated by the operators F_{-n}, L_{-n} with $n \geq 1$. All operators in this gauge algebra can be generated out of F_{-1}, L_{-1} by taking successive commutators because

$$L_{-2} = F^2_{-1} , \qquad F_{-2} = 2[L_{-1}, F_{-1}] , \qquad \text{etc.} \ldots . \tag{2.11}$$

It is therefore sufficient to construct an action which is gauge invariant with respect to just F_{-1} and L_{-1} gauge transformations, where the gauge transformation parameters are completely unconstrained.

In complete analogy, the superconformal algebra in the NS sectors is generated by the operators L_m and G_r which satisfy the following (anti-) commutation relations

$$\{G_r, G_s\} = 2L_{r+s} + \tfrac{1}{2}D(r^2 - \tfrac{1}{4})\delta_{r+s,0}\,,$$
$$[L_m, G_r] = (\tfrac{1}{2}m - r)G_{m+r} \qquad (m, n \in \mathbb{Z}; \qquad r, s \in \mathbb{Z} + \tfrac{1}{2})\,,$$
$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{8}Dm(m^2-1)\delta_{m+n,0}\,. \tag{2.12}$$

The mass -shell condition reads

$$(L_0 - \tfrac{1}{2})\psi = 0\,, \tag{2.13}$$

while the constraints are now

$$L_n\psi = G_r\psi = 0\,, \qquad n \geqslant 1\,, \qquad r \geqslant \tfrac{1}{2}\,. \tag{2.14}$$

The algebra of gauge transformations in this sector is generated by L_{-n} and G_{-r}. As before, all of these operators can be generated out of just $G_{-1/2}$ and $G_{-3/2}$ (but *not* $G_{-1/2}$ and L_{-1}!), and the same comments apply as for the R-sector. We remind the reader that the absence of ghosts in the physical spectrum requires $D = 10$ in (2.8) and (2.12). Both the value $D = 26$ for the bosonic string and the value $D = 10$ for the spinning string will be crucial for the results to be presented in the following sections.

To proceed, we next introduce string functionals $\psi(x^\mu(\sigma))$ which are in one-to-one correspondence with the states created by applying the normal mode operators. For the open bosonic string, we have

$$\psi_V(x^\mu(\sigma)) = A(x)|0\rangle + \alpha^\mu_{-1}A_\mu(x)|0\rangle$$
$$+ \{\alpha^\mu_{-1}\alpha^\nu_{-1}A_{\mu\nu}(x) + \alpha^\mu_{-2}A'_\mu(x)]|0\rangle + \cdots\,, \tag{2.15}$$

with bosonic fields $A(x)$, $A_\mu(x), \ldots$. In the NS-sector, the corresponding expansion reads

$$\psi_{NS}(x^\mu(\sigma)) = \varphi(x)|0\rangle + b^\mu_{-1/2}\varphi_\mu(x)|0\rangle$$
$$+ [b^\mu_{-1/2}b^\nu_{-1/2}\varphi_{\mu\nu}(x) + \alpha^\mu_{-1}\varphi'_\mu(x)]|0\rangle + \cdots\,, \tag{2.16}$$

with bosonic fields $\varphi(x)$, $\varphi_\mu(x), \ldots$.. In the R-sector, the expansion is

$$\psi_R(x^\mu(\sigma)) = \psi(x)|0\rangle + \alpha^\mu_{-1}\psi_\mu(x)|0\rangle + \alpha^\mu_{-1}\psi'_\mu(x)|0\rangle + \cdots\,, \tag{2.17}$$

where the fields $\phi(x)$, $\phi_\mu(x), \ldots$ are now *spinors* in ten space-time dimensions. Obviously, the functional fields ϕ_V, ϕ_{NS} and ϕ_R contain fields of arbitrarily high spin. The supplementary fields and gauge transformation parameters to be introduced below will be understood to have similar expansions. For closed strings, one can write down analogous expansions which, however, must satisfy the constraint (2.7) or its analogues for the spinning string. It is important that these constraints, which relate left and right movers, are purely *algebraic* because the kinetic operator cancels in (2.7). Thus, taking the closed bosonic string as an example, we have

$$\psi_{\mathrm{V},\bar{\mathrm{V}}}(x^\mu(\sigma)) = h(x)|0\rangle + \alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}h_{\mu\nu}(x)|0\rangle + \cdots . \tag{2.18}$$

Similarly, for closed spinning strings, we introduce the notations $\phi_{\mathrm{NS},\overline{\mathrm{NS}}}$, $\phi_{\mathrm{R},\overline{\mathrm{NS}}}$, $\phi_{\mathrm{R},\bar{\mathrm{R}}}$, and for the heterotic string $\phi_{\mathrm{NS},\bar{\mathrm{V}}}$ and $\phi_{\mathrm{R},\bar{\mathrm{V}}}$, to distinguish the various sectors in a self-explanatory notation.

Finally, we recall that in order to recover the superstring in the "old" formalism, one must introduce appropriate projection operators in the NS- and R-sectors before coupling them together [11]. In the NS-sector, the projector is given by

$$P_{\mathrm{NS}} \equiv \tfrac{1}{2}[1-(-1)^{\Sigma_{r\geq 1/2} b^\mu_{-r} b^\mu_r}] , \tag{2.19}$$

whereas in the R-sector, it is

$$P_{\mathrm{R}} \equiv \tfrac{1}{2}[1-\gamma^*(-1)^{\Sigma_{n\geq 1} d^\mu_{-n} d^\mu_n}] , \tag{2.20}$$

where γ^* is the γ^5 matrix in ten dimensions.

3. Open superstrings

In this section we give a local gauge covariant formulation of the free open superstring. We do this within the context of the original Lorentz covariant formalism of refs. [9] and [10]. One could also consider the more recent formulation of ref. [12]. However, even at the non-interacting level, the action of ref. [12] is not bilinear in the fields. In the original formulation the supersymmetry is realized by a vertex operator [11]. We find this to be an encouraging feature since the new group theory of strings is based on vertex operators [13]. This has emerged for strings based on group lattices [14].

We begin with the Ramond sector. The ϕ field equation for the Ramond sector is of the form $F_0\psi + \cdots = 0$. Because of the gauge invariance of ψ it must be annihilated by the action of F_1 and L_1 when the other equations of motion are used. Consequently we require equations of the form $L_1\psi = \cdots$ and $F_1\psi = \cdots$, and so supplementary fields $\phi^{(1)}$ and $\chi^{(1)}$. The ψ equation is now of the form

$$F_0\psi + F_{-1}\phi^{(1)} + L_{-1}\chi^{(1)} + \cdots = 0 .$$

We then require L_1 and F_1 on $\phi^{(1)}$ and $\chi^{(1)}$, i.e., four more supplementary fields $\phi^{(2)}$, $\bar{\phi}^{(2)}$, $\chi^{(2)}$ and $\bar{\chi}^{(2)}$ which yield the equations $F_1\phi^{(1)} = \cdots$, $F_1\chi^{(1)} = \cdots$, $L_1\phi^{(1)} = \cdots$, $L_1\chi^{(1)} = \cdots$, respectively. One might expect that one requires F_1 and L_1 on these level-two supplementary fields, but since they do not occur in the ψ equation of motion this is not clear. In fact explicit calculation shows that these six supplementary fields are sufficient to obtain gauge invariance at all levels. The result of this argument is the set of equations

$$F_0\psi + F_{-1}\phi^{(1)} + L_{-1}\chi^{(1)} = 0\,, \tag{3.1}$$

$$F_1\psi = 2F_0\phi^{(1)} - \tfrac{5}{2}\chi^{(1)} - L_{-1}\bar{\phi}^{(2)} - F_{-1}\phi^{(2)}\,, \tag{3.2}$$

$$\begin{aligned}
L_1\psi &= -2F_0\chi^{(1)} - \tfrac{5}{2}\phi^{(1)} - L_{-1}\bar{\chi}^{(2)} - F_{-1}\chi^{(2)}\,,\\
F_1\phi^{(1)} &= -F_0\phi^{(2)} + \tfrac{1}{2}\bar{\phi}^{(2)} - 2\chi^{(2)}\,,\\
F_1\chi^{(1)} &= -F_0\bar{\phi}^{(2)} - 2\phi^{(2)} + 2\bar{\chi}^{(2)}\,,\\
L_1\phi^{(1)} &= -F_0\chi^{(2)} + \tfrac{1}{2}\bar{\chi}^{(2)} + \tfrac{1}{2}\phi^{(2)}\,,\\
L_1\chi^{(1)} &= F_0\bar{\chi}^{(2)} + 2\chi^{(2)} + \tfrac{1}{2}\bar{\phi}^{(2)}\,,
\end{aligned} \tag{3.3}$$

which follow from the action [(,) indicates the scalar product in Fock space]

$$\begin{aligned}
S = {}& \tfrac{1}{2}(\psi, F_0\psi) + (\psi, F_{-1}\phi^{(1)}) + (\psi, L_{-1}\chi^{(1)})\\
& + \tfrac{5}{2}(\phi^{(1)}, \chi^{(1)}) - (\phi^{(1)}, F_0\phi^{(1)}) + (L_1\phi^{(1)}, \bar{\phi}^{(2)}) + (F_1\phi^{(1)}, \phi^{(2)})\\
& + (\chi^{(1)}, L_1\psi) + (L_1\chi^{(1)}, \bar{\chi}^{(2)}) + (\chi^{(1)}, F_0\chi^{(1)}) + \tfrac{1}{2}(\phi^{(2)}, F_0\phi^{(2)})\\
& - \tfrac{1}{2}(\phi^{(2)}, \bar{\phi}^{(2)}) - 2(\phi^{(2)}, \chi^{(2)})\,.
\end{aligned} \tag{3.4}$$

We emphasize that all higher-level equations of motion are a consequence of (3.1)-(3.3). For instance,

$$\begin{aligned}
F_2\psi &= -2[L_1, F_1]\psi\\
&= 2\phi^{(2)} - 3\bar{\chi}^{(2)}\,, \qquad \text{etc.}\ldots.
\end{aligned} \tag{3.5}$$

It is rather straightforward now to prove the invariance *for all levels* of the equations of motion (3.1)-(3.3) or, equivalently, the action (3.4) under the gauge transformations

$$\begin{aligned}
&\delta\psi = F_{-1}\Lambda\,, \qquad \delta\phi^{(1)} = F_0\Lambda\,, \qquad \delta\chi^{(1)} = -2\Lambda\,,\\
&\delta\phi^{(2)} = F_1\Lambda\,, \qquad \delta\chi^{(2)} = -L_1\Lambda\,, \qquad \delta\bar{\phi}^{(2)} = \delta\bar{\chi}^{(2)} = 0
\end{aligned} \tag{3.6}$$

("F_1 gauges") and

$$\begin{aligned}
&\delta'\psi = L_1\Lambda'\,, \qquad \delta'\phi^{(1)} = -\tfrac{1}{2}\Lambda'\,, \qquad \delta'\chi^{(1)} = -F_0\Lambda'\,,\\
&\delta'\bar{\phi}^{(2)} = -F_1\Lambda'\,, \qquad \delta'\bar{\chi}^{(2)} = -L_1\Lambda'\,, \qquad \delta'\phi^{(2)} = \delta'\chi^{(2)} = 0
\end{aligned} \tag{3.7}$$

("L_1 gauges"). To show that this set of equations correctly describes the R-sector, we now have to show that, after imposing suitable gauge conditions, we are led back to (2.9) and (2.10); in addition, we must show that the supplementary fields can be gauged away so there are no new physical degrees of freedom besides those contained in ψ. From (3.6) and (3.7), we see that the supplemenatry fields $\phi^{(1)}$ and $\chi^{(1)}$ can be gauged away by use of Λ and Λ', i.e., we can put

$$\phi^{(1)} = \chi^{(1)} = 0\,. \tag{3.8}$$

Substituting (3.8) into (3.1)-(3.3), one shows that (2.9) is recovered indeed, and the remaining supplementary fields are forced on shell, i.e.,

$$(L_0+2)\phi^{(2)}=(L_0+2)\bar{\phi}^{(2)}=(L_0+2)\chi^{(2)}$$
$$=(L_0+2)\bar{\chi}^{(2)}=0\,. \tag{3.9}$$

The remaining system of equations possesses a residual on-shell gauge invariance

$$\begin{aligned}
\hat{\delta}\psi&=(L_{-2}+L_{-1}^2)\Omega^++(L_{-1}F_{-1}+\tfrac{3}{4}F_{-2})\Omega^-\,,\\
\hat{\delta}\phi^{(2)}&=(-\tfrac{5}{4}-F_{-1}F_1)\Omega^+-\tfrac{3}{2}L_{-1}F_1\Omega^-\,,\\
\hat{\delta}\bar{\phi}^{(2)}&=-L_{-1}F_1\Omega^++(\tfrac{1}{2}+\tfrac{5}{2}F_{-1}F_1)\Omega^-\,,\\
\hat{\delta}\chi^{(2)}&=-F_{-1}L_1\Omega^+-(\tfrac{9}{8}-\tfrac{3}{2}L_{-1}L_1)\Omega^-\,,\\
\hat{\delta}\bar{\chi}^{(2)}&=-(1+L_{-1}L_1)\Omega^+-\tfrac{5}{2}F_{-1}L_1\Omega^-\,,
\end{aligned} \tag{3.10}$$

where the parameters $\Omega^{\pm}$ are subject to the on-shell condition

$$F_0\Omega^++2\Omega^-=\Omega^+-F_0\Omega^-=0\,. \tag{3.11}$$

(This condition is necessary for the proof of gauge invariance.) The above on-shell gauge invariance is quite analogous to the one that occurs in ordinary electrodynamics: the Landau gauge $\partial^\mu A_\mu=0$ is preserved by $\delta A_\mu=\partial_\mu\Omega$ with $\Box\Omega=0$. To see that the remaining on-shell supplementary fields can be gauged away by use of $\Omega^{\pm}$, we define the linear combinations

$$\phi'^{(2)}\equiv-\tfrac{1}{4}\bar{\phi}^{(2)}+\chi^{(2)}\,,\qquad \chi'^{(2)}\equiv\tfrac{1}{2}(\bar{\chi}^{(2)}+\phi^{(2)})\,, \tag{3.12}$$

such that these fields satisfy the same equations as $\Omega^{\pm}$, i.e.,

$$\begin{aligned}
F_0\phi^{(2)}+2\phi'^{(2)}&=F_0\phi'^{(2)}-\phi^{(2)}=0\,,\\
F_0\chi'^{(2)}+2\chi^{(2)}&=F_0\chi^{(2)}-\chi'^{(2)}=0\,.
\end{aligned} \tag{3.13}$$

$\phi^{(2)}$ and $\chi^{(2)}$ can now be gauged away, and (3.13) then implies that also $\phi'^{(2)}=\chi'^{(2)}=0$. Altogether, we have thus shown the existence of a gauge where all supplementary fields vanish. In this gauge, (3.1) and (3.2) are then equivalent to (2.9) and (2.10). An alternative, quicker, method of gauge fixing which arrives at the same constraints is given for the Veneziano model in the next section.

The treatment of the NS-sector is entirely analogous. One introduces six supplementary fields $\phi^{(1/2)}$, $\phi^{(3/2)}$, $\zeta^{(1)}$, $\zeta^{(2)}$, $\zeta'^{(2)}$ and $\zeta^{(3)}$. The equations of motion read

$$(L_0-\tfrac{1}{2})\psi+G_{-1/2}\phi^{(1/2)}+G_{-3/2}\phi^{(3/2)}\,, \tag{3.14}$$

$$\begin{aligned}
G_{1/2}\psi&=-2\phi^{(1/2)}+G_{-3/2}\zeta^{(2)}-G_{-1/2}\zeta^{(1)}\,,\\
G_{3/2}\psi&=-10\phi^{(3/2)}+G_{-3/2}\zeta^{(3)}-G_{-1/2}\zeta'^{(2)}+2G_{1/2}\zeta^{(1)}\,,
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
G_{1/2}\phi^{(1/2)}&=2G_{-1/2}\phi^{(3/2)}-(L_0+\tfrac{1}{2})\zeta^{(1)}\,,\\
G_{3/2}\phi^{(1/2)}&=-4(L_0+\tfrac{3}{2})\zeta^{(2)}-(L_0+\tfrac{3}{2})\zeta'^{(2)}\,,\\
G_{1/2}\phi^{(3/2)}&=(L_0+\tfrac{3}{2})\zeta^{(2)}\,,\\
G_{3/2}\phi^{(3/2)}&=(L_0+\tfrac{5}{2})\zeta^{(3)}\,.
\end{aligned} \tag{3.16}$$

These equations follow from an action very similar to the action of the R-sector. This is left as an exercise for the reader. The gauge transformations are given by

$$\delta\psi = G_{-1/2}\Lambda\,, \qquad \delta\phi^{(1/2)} = -L_0\Lambda\,, \qquad \delta\gamma^{(3/2)} = 0\,,$$
$$\delta\zeta^{(1)} = G_{1/2}\Lambda\,, \qquad \delta\zeta'^{(2)} = G_{3/2}\Lambda\,, \qquad \delta\zeta^{(2)} = \delta\zeta^{(3)} = 0\,, \tag{3.17}$$

("$G_{1/2}$ gauges") and

$$\delta'\psi = G_{-3/2}\Lambda'\,, \qquad \delta'\phi^{(1/2)} = 0\,, \qquad \delta'\phi^{(3/2)} = -(L_0+1)\Lambda'\,,$$
$$\delta'\zeta^{(1)} = -2G_{-1/2}\Lambda'\,, \qquad \delta'\zeta^{(2)} = -G_{1/2}\Lambda'\,,$$
$$\delta'\zeta'^{(2)} = 4G_{1/2}\Lambda'\,, \qquad \delta'\zeta^{(3)} = -G_{3/2}\Lambda'\,, \tag{3.18}$$

("$G_{3/2}$ gauges"). As before, one uses Λ and Λ' to gauge away $\phi^{(1/2)}$ and $\phi^{(3/2)}$; this gauge choice puts all fields on shell. In particular,

$$(L_0+\tfrac{1}{2})\zeta^{(1)} = (L_0+\tfrac{3}{2})(\zeta^{(2)} \text{ or } \zeta'^{(2)})$$
$$= (L_0+\tfrac{5}{2})\zeta^{(3)} = 0\,. \tag{3.19}$$

The residual on-shell gauge invariance can be used to set equal to zero all the remaining supplementary fields and we recover the familiar equations

$$(L_0-\tfrac{1}{2})\psi = 0\,, \qquad G_{1/2}\psi = G_{3/2}\psi = 0\,. \tag{3.20}$$

It should be emphasized that the value $D = 10$ for the critical dimension has played a crucial role in our considerations. Although it is not immediately evident what is so special about this value, one may convince oneself that the above construction with just six supplementary fields fails for $D \neq 10$.

As the number of supplementary fields is the same in the NS- and RS-sectors, one can argue that the supplementary spectrum is supersymmetric after the appropriate projections in each sector. This will however, have to wait for the explicit calculation of the fermion emission vertex and the supersymmetry generator.

4. Open bosonic string

We now return to the $D = 26$ open bosonic string which may also be "covariantized" by the introduction of just six supplementary fields $\phi^{(1)}$, $\phi^{(2)}$, $\zeta^{(1)}$, $\zeta^{(2)}$, $\zeta^{(3)}$ and $\zeta^{(4)}$. The trick which allows one to avoid the introduction of infinitely many supplementary fields is to go to a "first-order formalism" quite analogous to the treatment of NS- and R-sectors. In this formalism, the equations for motion are

$$(L_0-1)\psi + L_{-1}\phi^{(1)} + L_{-2}\phi^{(2)} = 0\,, \tag{4.1}$$

$$L_1\psi = -2\phi^{(1)} - L_{-1}\zeta^{(2)} - L_{-2}\zeta^{(3)} ,$$
$$L_2\psi = -4\phi^{(2)} - L_{-1}\rho'^{(3)} - L_{-2}\zeta^{(4)} - 3\zeta^{(2)} , \tag{4.2}$$

$$L_1\phi^{(1)} = (L_0+1)\zeta^{(2)} - 3\phi^{(2)} ,$$
$$L_2\phi^{(1)} = (L_0+2)\zeta'^{(3)} ,$$
$$L_1\phi^{(2)} = (L_0+2)\zeta^{(3)} ,$$
$$L_2\phi^{(2)} = (L_0+3)\zeta^{(4)} , \tag{4.3}$$

They are invariant under the local gauge transformations

$$\delta\psi = L_{-1}\Lambda , \qquad \delta\phi^{(1)} = -L_0\Lambda , \qquad \delta\phi^{(2)} = 0 ,$$
$$\phi\zeta^{(2)} = -L_1\Lambda , \qquad \delta\zeta^{(3)} = 0 ,$$
$$\delta\zeta'^{(3)} = -L_2\Lambda , \qquad \delta\zeta^{(4)} = 0 \tag{4.4}$$

and

$$\delta'\psi = L_{-2}\Lambda' , \qquad \delta'\phi^{(1)} = 0 , \qquad \delta'\phi^{(2)} = -(L_0+1)\Lambda' ,$$
$$\delta\zeta^{(2)} = -3\Lambda' , \qquad \delta\zeta^{(3)} = -L_1\Lambda' ,$$
$$\delta\zeta'^{(3)} = 0 , \qquad \delta\zeta^{(4)} = -L_2\Lambda' . \tag{4.5}$$

The invariant action from which the above equations follow is

$$\begin{aligned}&\tfrac{1}{2}(\psi, (L_0-1)\psi) + (L_1\psi, \phi^{(1)}) + (L_2\psi, \phi^{(2)})\\ &\quad + (\phi^{(1)}, \phi^{(1)}) + (L_1\phi^{(1)}, \zeta^{(2)}) + (L_2\phi^{(1)}, \zeta^{(3)})\\ &\quad + 2(\phi^{(2)}, \phi^{(2)}) + (L_1\phi^{(2)}, \zeta'^{(3)}) + (L_2\phi^{(2)}, \zeta^{(4)})\\ &\quad + 3(\phi^{(2)}, \zeta^{(2)}) - \tfrac{1}{2}(\zeta^{(2)}, (L_0+1)\zeta^{(2)})\\ &\quad - (\zeta^{(3)}, (L_0+2)\zeta'^{(3)}) - \tfrac{1}{2}(\zeta^{(4)}, (L_0+3)\zeta^{(4)}) .\end{aligned} \tag{4.6}$$

To show that the above answer is correct we must recover the well-known on-shell conditions which guarantee the correct spectrum. We first use Λ_1 and Λ_2 to gauge to zero $L_1\psi$ and $L_2\psi$, respectively. However, we can still make gauge transformations that preserve these conditions. Consider, in particular

$$\delta\psi = L_{-1}\frac{1}{L_0}\phi^{(1)} + L_{-2}\frac{1}{L_0+1}\phi^{(2)} . \tag{4.7}$$

Then one has

$$(L_0-1)(\psi+\delta\psi) = 0 . \tag{4.8}$$

Further, using the equations of motion, one verifies that $L_1\delta\psi = L_2\delta\psi = 0$. Finally, making the corresponding gauge transformations on the supplementary fields yields

$$0 = \phi^{(1)} + \delta\phi^{(1)} = \phi^{(2)} + \delta\phi^{(2)} , \qquad \text{etc.} \ldots . \tag{4.9}$$

The reader may wonder what is the relationship between the Veneziano model presented here and that given in ref. [4]. Since $L_1\psi$ and $L_2\psi$ are known, we can compute $L_3\psi$; we also observe that one can eliminate algebraically ϕ_1 and ϕ_2 from eqs. (4.1)-(4.3). One then obtains

$$(L_0-1-\tfrac{1}{2}L_{-1}L_1-\tfrac{1}{4}L_{-2}L_2)\psi-\tfrac{3}{4}L_{-2}\zeta_2-\tfrac{1}{2}L_{-1}^2\zeta_2-\tfrac{1}{2}L_{-1}L_{-2}\zeta_3-\tfrac{1}{4}L_{-2}^2\zeta_4=0\,, \quad (4.10)$$

$$L_3\psi=-5\zeta_3-4\zeta_3'-L_{-1}L_2\zeta_2+L_{-1}L_1\zeta_3'+L_{-2}L_1\zeta_4+L_{-1}\zeta_4\,, \quad (4.11)$$

$$L_1L_2\psi=-4(L_0+2)\zeta_3-2L_0\zeta_3'-L_{-1}L_1\zeta_3'-3L_{-1}\zeta_4-3L_1\zeta_2-L_{-2}L_1\zeta_4\,, \quad (4.12)$$

together equations of motion for ζ_2, ζ_3, ζ_3' and ζ_4. Setting $\phi_3=-\zeta_3'-2\zeta_3$, one then recovers, up to third level, the formulation of ref. [4] with the supplementary fields $\phi_2\equiv\zeta_2$, $\phi_3\equiv-\zeta_3'-2\zeta_3$. Introducing $\phi_4, \phi_5, \ldots$, one would recover the results of ref. [4] at higher levels.

5. Closed bosonic string

The construction of a gauge covariant action for the closed bosonic string proceeds in similar fashion to that of the open strings; it is just made somewhat more lengthy by the fact there are both left- and right-moving modes; the gauge transformations for the original string field are

$$\delta\psi=L_{-1}\bar{\Lambda}\,, \qquad \bar{\delta}\psi=\bar{L}_{-1}\Lambda\,,$$
$$\delta'\psi=L_{-2}\bar{\Lambda}\,, \qquad \bar{\delta}'\psi=\bar{L}_{-2}\Lambda\,. \quad (5.1)$$

By analogy with the previous sections, we introduce four supplementary fields, ϕ_1, $\bar{\phi}_1$, ϕ_2, $\bar{\phi}_2$, related to the action of L_1, $\bar{L}_1$, L_2, $\bar{L}_2$ on ψ, and sixteen more, analogous to ζ_2, ζ_3, ζ_3', ζ_4 of the previous section, related to the action of L_1, $\bar{L}_1$, L_2, $\bar{L}_2$ on each of the supplementary fields ϕ_1, $\bar{\phi}_1$, ϕ_2, $\bar{\phi}_2$. The non-vanishing gauge transformations of the supplementary fields are:

$$\delta\phi_1=-2L_0\Lambda\,, \qquad \delta\zeta_2=-L_1\bar{\Lambda}\,, \qquad \delta\bar{\zeta}_2=-\bar{L}_1\bar{\Lambda}\,,$$
$$\delta\chi_3=-L_2\bar{\Lambda}\,, \qquad \delta\bar{\chi}_3=-\bar{L}_2\bar{\Lambda}\,, \qquad \delta'\phi_2=-2(L_0+1)\bar{\Lambda}'\,,$$
$$\delta'\zeta_2=-3\bar{\Lambda}'\,, \qquad \delta'\zeta_3=-L_1\bar{\Lambda}'\,, \qquad \delta'\bar{\zeta}_3=-\bar{L}_1\bar{\Lambda}'\,,$$
$$\delta'\chi_4=-L_2\bar{\Lambda}'\,, \qquad \delta'\bar{\chi}_4=-\bar{L}_2\bar{\Lambda}'\,. \quad (5.2)$$

It is then a trivial matter to verify the invariance of the action

$$\begin{aligned}S=\;&\tfrac{1}{2}\psi(L_0+\bar{L}_0-2)\psi+\psi L_{-1}\phi_1+\psi\bar{L}_{-1}\bar{\phi}_1+\psi L_{-2}\phi_2+\psi\bar{L}_{-2}\bar{\phi}_2+\tfrac{1}{2}\phi_1^2\\
&+\phi_1L_{-1}\zeta_2+\phi_1\bar{L}_{-1}\zeta_2'+\phi_1L_{-2}\zeta_3+\phi_1\bar{L}_{-2}\zeta_3'+\tfrac{1}{2}\bar{\phi}_1^2+\bar{\phi}_1L_{-1}\bar{\zeta}_2+\bar{\phi}_1\bar{L}_{-1}\bar{\zeta}_2'\\
&+\bar{\phi}_1L_{-2}\bar{\zeta}_3+\bar{\phi}_1\bar{L}_{-2}\bar{\zeta}_3'+\phi_2^2+\phi_2L_{-1}\chi_3+\phi_3\bar{L}_{-1}\chi_3'+\phi_2L_{-2}\chi_4\\
&+\phi_2\bar{L}_{-2}\chi_4'+3\phi_2\zeta_2+\bar{\phi}_2^2+\bar{\phi}_2L_{-1}\bar{\chi}+\bar{\phi}_2\bar{L}_{-1}\bar{\chi}_3'+\bar{\phi}_2L_{-2}\bar{\chi}_4+\bar{\phi}_2\bar{L}_{-2}\bar{\chi}_4'\\
&+3\bar{\phi}_2\bar{\zeta}_2'-\zeta_2(L_0+1)\zeta_2-2\zeta_2'(L_0+2)\zeta_2'-2\zeta_3(L_0+2)\chi_3-2\bar{\zeta}_2L_0\zeta_2'-\bar{\zeta}_2'(L_0+1)\bar{\zeta}_2'\\
&-2\bar{\zeta}_3(L_0+1)\chi_3'-2\chi_3(L_0+2)\zeta_3-\chi_4(L_0+3)\chi_4-2\chi_4'(L_0+3)\chi_4'\,.\end{aligned} \quad (5.3)$$

In checking the gauge invariance of S in eq. (5.3), one must bear in mind the algebraic constraints of sect. 1,

$$(L_0-\bar{L}_0)\psi=(L_0-\bar{L}_0+1)\phi_1=(L_0-\bar{L}_0-1)\bar{\phi}_1$$
$$=(L_0-\bar{L}_0+2)\zeta_2\cdots=0\,. \tag{5.4}$$

It is a straightforward exercise to gauge away on-shell all supplementary fields and recover the usual equation of motion and constraints

$$(L_0+\bar{L}_0-2)\psi=0\,,\qquad L_1\psi=\bar{L}_1\psi=L_2\psi=\bar{L}_2\psi=0 \tag{5.5}$$

by the same procedure as described in the previous section.

6. Closed superstrings

Closed superstrings come in three different sectors, according to whether the fermionic right- and left-moving modes are in the same sector, Neveu-Schwarz or Ramond, or in different sectors. The corresponding string fields are $\psi_{\mathrm{NS},\overline{\mathrm{NS}}}$, $\psi_{\mathrm{R},\overline{\mathrm{NS}}}$ and $\psi_{\mathrm{R},\bar{\mathrm{R}}}$. The NS, $\overline{\mathrm{NS}}$ sector proceeds in exactly the same fashion as the bosonic closed string, requiring twenty auxiliary fields, and the action is found to be

$$\begin{aligned}
&\tfrac{1}{2}+(L_0+\bar{L}_0-1)\psi+\psi G_{-1/2}\varphi_{1/2}+\psi\bar{G}_{-1/2}\bar{\varphi}_{1/2}+\psi G_{-3/2}\varphi_{3/2}+\psi\bar{G}_{-3/2}\bar{\varphi}_{3/2}\\
&\quad+\tfrac{1}{2}\varphi_{1/2}^2+\varphi_{1/2}G_{-1/2}\zeta_1+\varphi_{1/2}G_{-3/2}\zeta_2+\varphi_{1/2}\bar{G}_{-1/2}\chi_1+\varphi_{1/2}\bar{G}_{-3/2}\chi_2\\
&\quad+\tfrac{5}{2}\varphi_{3/2}^2+\varphi_{3/2}G_{-1/2}\zeta_2'+\varphi_{3/2}G_{-3/2}\zeta_3-2\varphi_{3/2}G_{1/2}\zeta_1+\varphi_{3/2}\bar{G}_{-1/2}\bar{\lambda}_2\\
&\quad+\varphi_{3/2}\bar{G}_{-3/2}\bar{\lambda}_3+\tfrac{1}{2}\bar{\varphi}_{1/2}^2+\bar{\varphi}_{1/2}\bar{G}_{-1/2}\bar{\zeta}_1+\bar{\varphi}_{1/2}\bar{G}_{-3/2}\,\bar{\zeta}_2+\bar{\varphi}_{1/2}G_{-1/2}\bar{\chi}_1\\
&\quad+\bar{\varphi}_{1/2}G_{-3/2}\chi_2+\tfrac{5}{2}\bar{\varphi}_{3/2}^2+\bar{\varphi}_{3/2}\bar{G}_{-1/2}\bar{\zeta}_2'+\bar{\varphi}_{3/2}\bar{G}_{-3/2}\bar{\zeta}_3-2\bar{\varphi}_{3/2}\bar{G}_{1/2}\zeta_1\\
&\quad+\bar{\varphi}_{3/2}\bar{G}_{-1/2}\lambda_2+\bar{\varphi}_{3/2}G_{-3/2}\lambda_3+\zeta_1(L_0+\tfrac{1}{2})\zeta_1-\zeta_2(L_0+\tfrac{3}{2})\zeta_2\\
&\quad+\zeta_2(L_0+\tfrac{3}{2})\zeta_2'+\chi_1L_0\chi_1+\chi_2(\bar{L}_0+1)\lambda_2+\bar{\zeta}_1(\bar{L}_0+\tfrac{1}{2})\bar{\zeta}_1\\
&\quad-4(\bar{\zeta}_2(\bar{L}_0+\tfrac{3}{2})\bar{\zeta}_2+2\bar{\zeta}_2(L_0+\tfrac{3}{2})\bar{\zeta}_2'-\bar{\chi}_1\bar{L}_0\bar{\chi}_1+2\bar{\chi}_2(L_0+1)\bar{\lambda}_2\\
&\quad+2\lambda_3(L_0+1)\bar{\lambda}_3+2\bar{\zeta}_2'(\bar{L}_0+\tfrac{3}{2})\bar{\zeta}_2+\bar{\zeta}_3(\bar{L}_0+\tfrac{5}{2})\bar{\zeta}_3\\
&\quad+\zeta_3(L_0+\tfrac{5}{2})\zeta_3\,.
\end{aligned} \tag{6.1}$$

The corresponding gauge invariances are easily deduced.

The R, $\overline{\mathrm{NS}}$ (or $\bar{\mathrm{R}}$, NS) sector naturally exhibits mixed features of R and $\overline{\mathrm{NS}}$ open strings. The $\psi_{\mathrm{R},\overline{\mathrm{NS}}}$ string field satisfies the algebraic constraint

$$(L_0-\bar{L}_0+\tfrac{1}{2})\psi_{\mathrm{R},\overline{\mathrm{NS}}}=0\,. \tag{6.2}$$

In this sector also, one needs twenty auxiliary fields, and the action writes

$$\begin{aligned}
SS_{\mathrm{R,N\bar{S}}} = {} & \tfrac{1}{2}\psi F_0\psi + \psi F_{-1}\varphi_1 + \psi L_{-1}\chi_1 + \psi\bar{G}_{-1/2}\bar{\varphi}_{1/2} + \psi\bar{G}_{-3/2}\bar{\varphi}_{3/2} - \varphi_1 F_0\varphi_1 + \tfrac{5}{2}\varphi_1\chi_1 \\
& + \varphi_1 L_{-2}\varphi_2' + \varphi_1 F_{-1}\varphi_2 + \varphi_1\bar{G}_{-1/2}\xi_{3/2} + \varphi_1\bar{G}_{3/2}\xi_{5/2} + \chi_1 F_0\chi_1 \\
& + \chi_1 L_{-1}\chi_2'\chi_1 F_{-1}\chi_2 + \chi_1\bar{G}_{-1/2}\eta_{3/2} + \chi_1\bar{G}_{-3/2}\eta_{5/2} + \tfrac{1}{2}\varphi_2 F_0\varphi_2 - \tfrac{1}{2}\varphi_2\varphi_2' \\
& + 2\varphi_2\chi_2 + \chi_2 F_0\varphi_2' - 2\chi_2\chi_2' - \tfrac{1}{2}\varphi_2'\chi_2' - \tfrac{1}{2}\chi_2' F_0\chi_2' + \tau_{3/2}F_1\bar{\varphi}_{1/2} + 2\tau_{3/2}\eta_{3/2} \\
& + \tau_{3/2}F_0\xi_{3/2} + \tau_{5/2}F_1\bar{\varphi}_{3/2} + 2\tau_{5/2}\eta_{5/2} + \tau_{3/2}F_0\xi_{5/2} + \sigma_{3/2}L_1\bar{\varphi}_{1/2} \\
& - \tfrac{1}{2}\sigma_{3/2}\xi_{3/2} + \sigma_{3/2}F_0\eta_{3/2} - \tfrac{1}{2}\sigma_{5/2}\xi_{5/2} + \sigma_{5/2}F_0\eta_{5/2} + \sigma_{5/2}L_1\bar{\varphi}_{3/2} \\
& + \bar{\varphi}_{3/2}F_0\bar{\varphi}_{1/2} - \bar{\varphi}_{1/2}\bar{G}_{-3/2}\bar{\zeta}_2 + \bar{\varphi}_{1/2}\bar{G}_{-1/2}\bar{\zeta}_1 + \bar{\varphi}_{1/2}F_{-1}\tau_{3/}2 \\
& + \bar{\varphi}_{1/2}L_{-1}\sigma_{3/2} - 2\bar{\zeta}_1\bar{G}_{-1/2}\bar{\varphi}_{3/2} + \tfrac{1}{2}\bar{\zeta}_1 F_0\bar{\zeta}_1 + \bar{\zeta}_2'\bar{G}_{1/2}\bar{\varphi}_{3/2} - \bar{\zeta}_2' F_0\bar{\zeta}_2 \\
& + \xi_{3/2}\bar{G}_{1/2}\varphi_1 - 5\bar{\varphi}_{3/2}F_0\bar{\varphi}_{3/2} - \bar{\varphi}_{3/2}\bar{G}_{-3/2}\bar{\zeta}_3 + 2\bar{\zeta}_2 F_0\bar{\zeta}_2 - \tfrac{1}{2}\bar{\zeta}_3 F_0\bar{\zeta}_3 \,.
\end{aligned} \tag{6.3}$$

The gauge transformations of the various auxiliary fields can be found by inspection, by analogy with those of the Ramond and Neveu-Schwarz open strings, and we do not write them out here.

Finally, the $\psi_{\mathrm{R,\bar{R}}}$ sector exhibits some special features. These are due to the fact that it is a bispinor. Its massless fields are bosons which are described by a Kemmer-Duffin equation:

$$(F_0 + \bar{F}_0)\psi_{\mathrm{R,\bar{R}}} = 0\,. \tag{6.4}$$

The solutions of eq. (6.4), together with the massless modes of the NS, $\overline{\mathrm{NS}}$ sector correctly reproduce, after the appropriate chiral truncations, the bosonic spectrum of massless bosonic states of closed superstrings previously worked out in the light-cone formulation [12]. Hence, it is the correct starting point for the string field equations of motion. One then proceeds by analogy with the other sectors, looking for the supplementary fields, and an action invariant under the gauge transformations

$$\delta\psi = F_{-1}\Lambda\,, \qquad \delta'\psi = L_{-1}\Lambda'\,, \qquad \bar{\delta}\psi = \bar{F}_{-1}\bar{\Lambda}\,, \qquad \bar{\delta}'\psi = \bar{L}_{-1}\bar{\Lambda}'\,. \tag{6.5}$$

However, one finds at the first excited level that four more supplementary fields are needed, ζ_1, ξ_1, $\bar{\zeta}_1$, $\bar{\xi}_1$, with the following non-zero gauge transformations

$$\begin{aligned}
\delta\zeta_1 &= \bar{F}_0\Lambda\,, & \delta'\xi_1 &= \bar{F}_0\Lambda'\,, \\
\delta\bar{\zeta}_1 &= F_0\bar{\Lambda}\,, & \delta'\bar{\xi}_1 &= F_0\bar{\Lambda}'\,,
\end{aligned} \tag{6.6}$$

together with the supplementary fields $\bar{\phi}_1$, ϕ_1, χ_1, $\bar{\chi}_1$ of sect. 3 and their transformation laws. Further when constructing the invariant action, one finds that it depends on one free parameter. This parameter, however, is fixed when one requires that by choosing the gauge

$$F_1\psi=\bar{F}_1\psi=L_1\psi=\bar{L}_1\psi=0 \tag{6.7}$$

all the supplementary fields are gauged away, and one has the equation of motion (6.4). For higher levels, one then finds that sixteen other supplementary fields must be introduced, as for the other sectors of closed strings. Since this case is slightly special, we write out the complete action to all levels and its gauge invariances:

$$\begin{aligned}S=&\tfrac{1}{2}\psi(F_0+\bar{F}_0)\psi+\psi F_1\varphi_1+\psi L_{-1}\chi_1+\tfrac{5}{2}\varphi_1\chi_1-\varphi_1F_0\varphi_1\\&+\psi F_{-1}\zeta_1-\psi L_{-1}\xi_1-\zeta_1F_0\varphi_1-\tfrac{1}{2}\varphi_1\bar{F}_0\varphi_1-\tfrac{1}{2}\varphi_1\xi_1-\xi_1F_0\chi_1+\tfrac{1}{2}\chi_1\bar{F}_0\chi_1\\&+2\chi_1\zeta_1-\tfrac{1}{2}\xi_1\bar{F}_0\xi_1+\tfrac{1}{2}\zeta_1\bar{F}_0\zeta_1+\zeta_1F_{-1}\varphi_2+\tfrac{1}{2}\varphi_2\bar{F}_0\varphi_2+\zeta_1L_{-1}\chi_2'+\chi_2'\bar{F}_0\chi_2\\&-\xi_1F_{-1}\chi_2-\xi_1L_{-1}\varphi_2'-\tfrac{1}{2}\varphi_2'\bar{F}_0\varphi_2'+\varphi_1\bar{F}_{-1}\lambda_2-\varphi_1\bar{L}_{-1}\mu_2-\chi_1\bar{F}_{-1}\mu_2'\\&-\chi_1\bar{L}_{-1}\lambda_2'+\zeta_1\bar{F}_{-1}\lambda_2-\zeta_1\bar{L}_{-1}\mu_2+\xi_1\bar{F}_{-1}\mu_2'+\xi_1\bar{L}_1\lambda_2'+\lambda_2F\bar{\lambda}_2+\tfrac{1}{2}\lambda_22\\&-2\mu_2'\bar{\lambda}_2-\lambda_2'F_0\bar{\lambda}_2'-\tfrac{1}{2}\mu_2\bar{\lambda}_2'+\mu_2F_0\bar{\mu}_2'+\mu_2'F_0\bar{\mu}_2-2\lambda_2'\bar{\mu}_2'+\varphi_1L_{-1}\chi_2'\\&+\varphi_1F_{-1}\varphi_2+\chi_1F_{-1}\chi_2+\chi_1L_{-1}\varphi_2'+\chi_1F_0\chi_1+\tfrac{1}{2}\varphi_2F_0\varphi_2-\tfrac{1}{2}\varphi_2\chi_2'+\\&2\varphi_2\chi_2-\tfrac{1}{2}\varphi_2'F_0\varphi_2'-\tfrac{1}{2}\varphi_2'\chi_2'-2\varphi_2'\chi_2+\chi_2'F_0\chi_2\,.\end{aligned} \tag{6.8}$$

To the above terms one should add all those obtained by exchanging barred and unbarred symbols [except for the first one, $\frac{1}{2}\psi(F_0+\bar{F}_0)\psi$, of course). The gauge transformations which leave this action invariant are:

$$\begin{aligned}&\delta\varphi_1=F_0\Lambda\,,\qquad \delta\chi_1=-2\Lambda\,,\qquad \delta\zeta_1=\bar{F}_0\Lambda\,,\qquad \delta'\xi_1=\bar{F}_0\Lambda'\,,\\&\delta\bar{\lambda}_2=\bar{F}_1\Lambda\,,\qquad \delta'\lambda_2'=\bar{L}_1\Lambda'\,,\qquad \delta'\varphi_1=-\tfrac{1}{2}\Lambda'\,,\qquad \delta'\chi_1=-F_0\Lambda'\,,\\&\delta\bar{\mu}_2'=\bar{L}_1\Lambda\,,\qquad \delta\bar{\mu}_2=\bar{F}_1\Lambda'\end{aligned} \tag{6.9}$$

together with those in which all barred and unbarred symbols are interchanged. It is left for the reader to check that one can gauge away to zero all twenty-four supplementary fields, and end up with

$$(F_0+\bar{F}_0)\psi=F_1\psi=L_1\psi=\bar{F}_1\psi=\bar{L}_1\psi=0\,. \tag{6.10}$$

7. Heterotic string

The Lorentz covariant formulation of the heterotic string involves the co-ordinates x^μ, $\mu=1$ to 10, x^I, $I=11$ to 26 and $\psi^{\mu\alpha}$, $\mu=1$ to 10 and $\alpha=1$ to 2, where the index α is a two-dimensional spinor index. This means that it has α^μ, b^μ and d^μ right moving oscillators and $\bar{\alpha}^\mu$, $\bar{\alpha}^I$ left-moving oscillators. In this formulation, there are two string fields $\psi_{\mathrm{NS},\bar{\mathrm{V}}}$ for the bosons and $\psi_{\mathrm{R},\bar{\mathrm{V}}}$ for the fermions. The result is found in a similar way to the previous closed strings and reflects a mixture of the features found in the open superstring and open bosonic string. One requires twenty supplementary fields in each sector. The result for the action for all levels is, in the NS, $\bar{\mathrm{V}}$ sector:

$$\begin{aligned}
S_{\mathrm{NS},\bar{\mathrm{V}}} = {}& \tfrac{1}{2}\psi(L_0+\bar{L}_0-\tfrac{3}{2})\psi+\psi G_{-1/2}\varphi_{1/2}+\psi G_{-3/2}\varphi_{3/2}+\psi\bar{L}_{-1}\varphi_1+\psi\bar{L}_{-2}\varphi_2 \\
&+\tfrac{1}{2}\varphi_{1/2}\varphi_{1/2}+\varphi_{1/2}G_{-1/2}\zeta_1+\varphi_{1/2}G_{-3/2}\zeta_2+\varphi_{1/2}\bar{L}_{-1}\chi_{3/2}+\varphi_{1/2}\bar{L}_{-2}\chi_{5/2} \\
&+\tfrac{5}{2}\varphi_{3/2}^2+\varphi_{3/2}G_{-1/2}\zeta_2'+\varphi_{3/2}G_{-3/2}\zeta_3+\varphi_{3/2}\bar{L}_{-1}\chi_{5/2}' \\
&+\varphi_{3/2}\bar{L}_{-2}\chi_{7/2}-2\varphi_{3/2}G_{1/2}\zeta_1+\zeta_1(L_0+\tfrac{1}{2})\zeta_1-4\zeta_2(L_0+\tfrac{3}{2})\zeta_2-2\zeta_2(L_0+\tfrac{3}{2})\zeta_2' \\
&-2\chi_{3/2}\bar{L}_0\kappa_{3/2}-2\chi_{5/2}(\bar{L}_0+1)\kappa_{5/2}'-\lambda_2(\bar{L}_0+1)\lambda_2+\zeta_3(L_0+\tfrac{5}{2})\zeta_3-2\chi_{5/2}'\bar{L}_0\kappa_{5/2} \\
&-2\chi_{7/2}(\bar{L}_0+1)\kappa_{7/2}-2\lambda_3'(L_0+2)\lambda_3-\lambda_4(\bar{L}_0+3)\lambda_4+\varphi_1\bar{L}_1\psi+\tfrac{1}{2}\varphi_1^2 \\
&+3\varphi_2\lambda_2+\varphi_1\bar{L}_{-1}\lambda_2+\varphi_1\bar{L}_{-2}\lambda_3+\varphi_1 G_{-1/2}\kappa_{3/2}+\varphi_1 G_{-3/2}\kappa_{5/2}+\varphi_2^2+\varphi_2\bar{L}_{-1}\lambda_3' \\
&+\varphi_2\bar{L}_{-2}\lambda_4+\varphi_2 G_{-1/2}\kappa_{5/2}'+\varphi_2 G_{-3/2}\kappa_{7/2}
\end{aligned} \tag{7.1}$$

and, in the R, $\bar{\mathrm{V}}$ sector:

$$\begin{aligned}
S_{\mathrm{R},\bar{\mathrm{V}}} = {}& \tfrac{1}{2}\psi F_0\psi+\psi F_{-1}\varphi_1+\psi L_{-1}\chi_1+\chi\bar{L}_{-1}\zeta_1+\psi L_{-2}\zeta_2-\varphi_1 F_0\varphi_1+\tfrac{5}{2}\varphi_1\chi_1+\varphi_1 L_{-1}\bar{\varphi}_2 \\
&+\varphi_1 F_{-1}\varphi_2+\varphi_1\bar{L}_{-1}\lambda_2+\varphi_1\bar{L}_{-2}\lambda_3+\chi_1\kappa_0\chi_1+\chi_1 L_{-1}\bar{\chi}_2+\chi_1 F_{-1}\chi_2 \\
&+\chi_1\bar{L}_{-1}\lambda_2'+\chi_1\bar{L}_{-2}\lambda_3'+\zeta_1 F_{-1}\kappa_2+\zeta_1 L_{-1}\kappa_2'+\zeta_1\bar{L}_{-1}\mu_2+\zeta_1\bar{L}_{-2}\mu_3 \\
&+\zeta_1 F_0\zeta_1+\zeta_2 F_{-1}\kappa_3+\zeta_2 L_{-1}\kappa_3'+\zeta_2\bar{L}_{-1}\mu_3'+\zeta_2\bar{L}_{-2}\mu_4+\zeta_2 F_0\zeta_2+3\zeta_2\mu_2 \\
&+\tfrac{1}{2}\varphi_2 F_0\varphi_2-\tfrac{1}{2}\varphi_2\bar{\varphi}_2+2\varphi_2\chi_2+\chi_2 F_0\bar{\varphi}_2-\chi_2\bar{\chi}_2-\tfrac{1}{2}\bar{\varphi}_2\bar{\chi}_2-\tfrac{1}{2}-\tfrac{1}{2}\bar{\chi}_2 F_0\bar{\chi}_2 \\
&+\lambda_2 F_0\kappa_2-\tfrac{1}{2}\lambda_2\kappa_2'+\lambda_3 F_0\kappa_3+\lambda_3\kappa_3'-2\lambda_2'\kappa_3-\lambda_2' F_0\kappa_2'-\lambda_3' F_0\kappa_3'-2\lambda_3'\kappa_3 \\
&-\tfrac{1}{2}\mu_2 F_0\mu_2-\mu_3 F_0\mu_3'-\mu_4 F_0\mu_3\,.
\end{aligned} \tag{7.2}$$

The gauge transformations of the fields and the gauge invariances of $S_{\mathrm{NS},\bar{\mathrm{V}}}$ and $S_{\mathrm{R},\bar{\mathrm{V}}}$ are easily deduced.

8. Conclusions

We have displayed the linearized gauge invariance and corresponding free actions for all known string theories. In ref. [4] we sketched the method to introduce interactions. The interactions and the corresponding non-abelian vertex algebras will be reported elsewhere. Knowing the algebra and transformations, the geometric principle from which they can be derived can be more easily found.

One could also examine the equations of motion found in this paper in terms of the infinite set of component fields. One would find that it describes particles of arbitrarily high spin in a causal, covariant and unitary way. It is interesting that it involves a kind of first-order formalism.

References

[1] Dual Theory, Phys. Reports reprint vol. I, ed. M. Jacob (North-Holland, Amsterdam, 1974); J. Scherk, Rev. Mod. Phys. 47 (1975) 123

[2] E. Cremmer and J.-L. Gervais, Nucl. Phys. B76 (1974) 209; M. Kaku and K. Kikkawa, Phys. Rev. D10 (1974) 1110, 1823; E. Cremmer and J.-L. Gervais, Nucl. Phys. B90 (1975) 410

[3] W. Siegel, Phys. Lett. 148B (1984) 556, 149B (1984) 157, 162

[4] A. Neveu and P.C. West, CERN preprint TH. 4200 (1985), Nucl. Phys. B, to appear

[5] C. Marshall and P. Ramond, Nucl. Phys. B85 (1975) 375

[6] T. Banks and M. Peskin, Proc. of the Symp. on Anomalies, geometry and topology, Argonne National Laboratory (March 1985); M. Kaku and J. Lynkken, ibid

[7] D. Friedman, Univ. of Chicago preprint EFI 85-27 (April 1985)

[8] R.C. Brower and C.B. Thorn, Nucl. Phys. B31 (1971) 163

[9] P. Ramond, Phys. Rev. D3 (1971) 2415

[10] A. Neveu and J.H. Schwarz, Nucl. Phys. B31 (1971) 86

[11] J.H. Schwarz, Phys. Reports 89C (1982) 223

[12] M.B. Green and J.H. Schwarz, Phys. Lett. 136B (1984) 367

[13] P. Goddard and D. Olive, Univ. of Cambridge preprint, DAMTP 83/22 (1983), to be published in Proc. of the MSRI Workshop on Vertex operators, ed. J. Lepowsky (Springer, New York)

[14] I.B. Frenkel and V. Kac, Inv. Math. 62 (1980) 23