$C^{1,1}$-REGULARITY OF SOLUTIONS TO VARIATIONAL INEQUALITIES

Gerhard Huisken

INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open bounded set with smooth boundary and let u be a solution of a variational inequality of the form

(1) $$u \in K_1 \; ; \; \langle Au+H(x,u,Du), v-u\rangle \geq 0 \qquad \forall v \in K_1$$

where A is a quasilinear elliptic operator in divergence form

(2) $$Au = -D_i(a^i(x,u,Du)) \quad {}^{1)}$$

and the convex set K_1 is given by

(3) $$K_1 = \{v \in H^{1,\infty}(\Omega) \,|\, v \geq \psi, v|_{\partial\Omega} = \phi\} \; .$$

We also consider the case of Neumann boundary conditions, i.e. solutions of

(4) $$u \in K_2 \; ; \; \langle\langle Au + H(x,u,Du) \; , \; v-u\rangle\rangle \geq 0 \qquad \forall v \in K_2$$

(5) $$K_2 = \{v \in H^{1,\infty}(\Omega) \,|\, v \geq \psi\}$$

where

(6) $$\langle\langle Au,\eta\rangle\rangle = \int_\Omega a^i(x,u,Du) \cdot D_i\eta dx + \int_{\partial\Omega} \beta \cdot \eta dH_{n-1}$$

for some function β on $\partial\Omega$.

1) Here and in the following we sum over repeated indices.

In particular a capillary surface with contact angle β , lying above an obstacle ψ is included in this setting.

The solution u of (1) or (4) will be of class $H^{2,p}(\Omega)$ for any finite p , provided the data are sufficiently smooth. Moreover it is well-known that the second derivatives of u could at most be bounded.

In the interior, this borderline regularity result has been established first by Frehse [2,3] for linear elliptic operators. General quasilinear elliptic operators have been studied by Gerhardt [4] and independently Brézis-Kinderlehrer [1]. Jensen [7] obtained a first global result for linear operators and Dirichlet boundary conditions.

Here we want to survey recent results on nonlinear elliptic operators A in the case of Dirichlet and Neumann boundary conditions.

GLOBAL $C^{1,1}$-REGULARITY

In the case of Dirichlet boundary conditions, Gerhardt [5] proved

THEOREM 1 *Let* $\partial\Omega$ *be of class* $C^{3,\alpha}$, $\phi \in C^3(\bar{\Omega})$, $\psi \in C^2(\bar{\Omega})$, *and assume that the* a^i*'s are of class* C^2 *in* x *and* u *and of class* C^3 *in the* p-*variable, and that* H *is of class* C^1 *in all its arguments. Then, any solution of the variational inequality* (1) *is in* $H^{2,\infty}(\Omega)$.

In the Neumann case we have [6]:

THEOREM 2 *Let* $\partial\Omega$, H , ψ *and the* a^i*'s be as above. Furthermore assume that* $\beta \in C^2(\partial\Omega)$ *satisfies*

$$-a^i(x,\psi,D\psi)\cdot\gamma_i \geq \beta \qquad \forall x \in \partial\Omega \tag{7}$$

where γ *is the outer unit normal at* $\partial\Omega$. *Then any solution of the variational inequality* (4) *is in* $H^{2,\infty}(\Omega)$.

In both theorems we can replace C^m by $C^{m-1,1}$.

We give a short outline of the proofs, which depend on a priori estimates for the solutions of approximating problems.

Let u_0 be a solution of (1) or (4), which is known to be in $H^{2,p}(\Omega)$ and therefore satisfies

$$(8) \qquad 1 + |u_0|_\infty + |Du_0|_\infty \leq M$$

for some $M > 0$. Then we can change the operators A and H into operators $\tilde{A}$ and $\tilde{H}$ with suitable growth properties such that $\tilde{A}$ and $\tilde{H}$ coincide with A and H on all functions v satisfying

$$(9) \qquad 1 + |v|_\infty + |Dv|_\infty \leq 2M .$$

In particular we have

$$(10) \qquad \tilde{A}u_0 + \tilde{H}u_0 = Au_0 + Hu_0$$

and the approximating boundary value problems

$$(11) \qquad \begin{aligned} &\tilde{A}u_\mu + \tilde{H}u_\mu + \gamma u_\mu + \mu\theta(u_\mu - \psi) = \gamma u_0 \quad \text{in } \Omega \\ &u_\mu|_{\partial\Omega} = \phi \quad \text{on } \partial\Omega \\ &(\text{resp. } -\tilde{a}^i(x,u_\mu,Du_\mu)\cdot\gamma_i = \beta) \end{aligned}$$

have a smooth solution u_μ for any $\mu > 0$, if γ is large enough. Here we use the penalization

$$(12) \qquad \theta(t) = \begin{cases} 0 & t \geq 0 \\ -t^2 & t \leq 0 \end{cases} .$$

Our aim is to bound the $H^{2,\infty}(\Omega)$-norm of u_μ independently of μ. Then we can easily conclude that in the limit $\mu \to \infty$ a subsequence of the u_μ will converge to u_0, which therefore lies in $H^{2,\infty}(\Omega)$.

In order to establish the *a priori* estimate, we have to control the penalization term in (11). Using the maximum principle one can show

LEMMA *Let* u_μ *be a solution of* (11). *Then* $u_\mu - \psi \geq -c\,\mu^{-\frac{1}{2}}$ *and*

$$\mu|\theta(u-\psi)| \leq c^2 \tag{13}$$

where

$$c^2 = \sup_\Omega |A\psi + H\psi| \quad , \quad c > 0 \ .$$

We remark that in the case of Neumann boundary conditions the assumption (7) is absolutely essential for the validity of the Lemma.

It was an idea of Jensen [7], to estimate the quantity

$$W = \lambda \cdot a^{k\ell} D_k D_\ell u \pm D_r D_s u \tag{14}$$

from below, first for $1 \leq r,s \leq n-1$, then for $1 \leq r \leq n-1$, $s = n$. Here we have

$$a^{k\ell} = \frac{\partial a^k}{\partial x^\ell} \tag{15}$$

and $\lambda > 0$ is large. This yields an estimate for $D_r D_s u$ since $a^{k\ell} D_k D_\ell u$ is bounded. The remaining derivative $D_n D_n u$ can then be estimated in view of the ellipticity of A.

It turns out that in the case of Dirichlet boundary conditions ([5]) the tangential derivatives $D_r D_s u$ are easy to handle in view of the boundary condition, whereas the normal derivative $D_n D_r u$ causes some difficulties. Vice versa, in the Neumann case ([6]) the hard part of the proof is to obtain an estimate on the tangential derivatives, whereas the normal derivative can then be easily estimated by differentiating the boundary condition.

Finally, using the Stampacchia-iteration approach it is possible to obtain the estimate

$$\|u_\mu\|_{2,\infty,\Omega} \leq C_\varepsilon \cdot (1+\|u_\mu\|_{2,\infty,\Omega})^\varepsilon \tag{16}$$

for any $0 < \varepsilon < 1$, where C_ε does not depend on μ. This yields an estimate for $\|u_\mu\|_{2,\infty}$.

We remark that this method unfortunately does not yield a local estimate near the boundary.

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Centre for Mathematical Analysis
Australian National University
Canberra ACT 2601
AUSTRALIA