# SINGLETON REPRESENTATIONS OF Osp($N$,4)

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We demonstrate the existence of singleton supermultiplets in Osp($N$,4) for all $N$. These represent extreme cases of multiplet shortening because they contain only particles of spin $s = 0$ and $s = \frac{1}{2}$. We also discuss some of their field theoretic aspects.

It is well known that the natural background geometry of gauged extended supergravity theories [1] is anti-de Sitter space (= AdS) with an SO(3,2) group of isometries. The bosonic symmetry group of these theories also contains an SO($N$) group and therefore the relevant superalgebra is the graded extension of SO(3,2)$\times$SO($N$) which is Osp($N$,4). It is clearly important to study the representations of Osp($N$,4) which describe particles in AdS and which are relevant for supersymmetric field theories in an AdS background. These must be unitary and hence are infinite-dimensional since SO(3,2) is non-compact. Positive energy representations of SO(3,2) and their relation to wave equations in AdS have been extensively studied by Fronsdal [2]. These representations are characterized by their lowest energy eigenvalue $E_0$ and the total angular momentum number $s$ of their lowest energy state and are denoted by $D(E_0, s)$ Unitarity requires that $E_0 \geq s + 1$ for $s = 1, \frac{3}{2}, \ldots,$ and $E_0 \geq s + \frac{1}{2}$ for $s = 0, \frac{1}{2}$. All of these representations become ordinary massless representations of the Poincaré group in the limit where the radius of AdS becomes infinite except those with $E_0 = s + \frac{1}{2}$ for $s = 0, \frac{1}{2}$ which are exceptional: the singleton representations $D(\frac{1}{2}, 0)$ and $D(1, \frac{1}{2})$ [3] possess no Poincaré limit. In this note, we demonstrate the existence of singleton supermultiplets in Osp($N$, 4) for arbitrary $N \geq 1$ and investigate some of their field theoretic properties. In the case $N = 1$, the existence of a singleton multiplet was already shown by Heidenreich [4] who constructed all particle representations of Osp(1, 4). Representations of Osp($N$, 4) for arbitrary $N$ were investigated in refs. [5,6]. In ref. [6], it was shown that there is a new type of multiplet shortening in these algebras for $N > 1$. The $N-$ extended singleton multiplets constitute the extreme case of multiplet shortening as their SO(3, 2) vacuum states contain only representations with $s = 0$ and $s = \frac{1}{2}$ for arbitrary $N$; this is to be contrasted with ordinary representations of both Poincaré sypersymmetry and Osp($N$, 4) which require particles of helicity $s \geq 1$ for $N \geq 3$.

For the convenience of the reader we now briefly summarize the essential properties of the Osp($N$, 4) algebra and refer to ref. [6], whose conventions and notations we follow in the first part of this paper, for further details and explanations. The even elements of Osp($N$, 4) are the 10 hermitean SO(3, 2) generators $M_{AB} = -M_{AB}$ where $A, B = 0, 1, \ldots, 4$ and $N(N-1)/2$ hermitean SO($N$) generators $T^{ij} = -T^{ij}$ where $i, j = 1, \ldots,$ N. The odd elements are given by $4N$ Majorana spinor charges

$Q^i_\alpha$, where $\alpha = 1, \ldots, 4$ is a Dirac index. The part of the algebra containing the spinor charges is characterized by the relations

$$\{ Q^i_\alpha, \bar{Q}^j_\beta \} = 2\delta^{ij} I^{AB}_{\alpha\beta} M_{AB} + \mathrm{i}\delta_{\alpha\beta} T^{ij}, \tag{1}$$

$$[ M_{AB}, Q^i_\alpha ] = -\mathrm{i}(I_{AB})_{\alpha\beta} Q^i_\beta, \tag{2}$$

$$[ T^{ij}, Q^k_\alpha ] = \mathrm{i}\delta^{ik} Q^j_\alpha - \mathrm{i}\delta^{jk} Q^i_\alpha, \tag{3}$$

with corresponding relations for the adjoint operators $Q^{-i}_\alpha$. The SO(3, 2) and SO($N$) structure relations are, respectively,

$$[M_{AB}, M_{CD}] = \mathrm{i}(\eta_{AD} M_{BC} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC} + \eta_{BC} M_{AD}), \tag{4}$$

$$[T^{ij}, T^{kl}] = -\mathrm{i}(\delta^{jk} T^{il} - \delta^{ik} T^{jl} - \delta^{jl} T^{ik} + \delta^{il} T^{jk}), \tag{5}$$

where $\eta_{AB} = \mathrm{diag}(+, -, -, -, +)$. As in ref. [6], the Majorana spinor $Q^i_\alpha$ can be parametrized as

$$Q^i_\alpha = \begin{pmatrix} a^i_\alpha \\ \epsilon_{\alpha\beta} \bar{a}^i_\beta \end{pmatrix}, \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{6}$$

where the operators $a^i_\alpha$ and $\bar{a}^i_\alpha = (a^i_\alpha)^+$ are fermionic lowering and raising operators whose properties have been discussed in ref. [6]. We also recall the method employed in ref. [6] to construct positive energy representations of the algebra (1)–(5). One starts from a vacuum state $|\mathrm{vac}\rangle$ which is annihilated by the energy lowering operators $a^i_\alpha$

$$a^i_\alpha |\mathrm{vac}\rangle = 0. \tag{7}$$

As a consequence of the Osp($N$, 4) algebra, the vacuum state is also annihilated by the SO(3, 2) energy de-boost operators $M^-_a \equiv \mathrm{i} M_{0a} + M_{a4}$ ($a = 1, 2, 3$). The representation space is spanned by all vectors of the form

$$(M^+_1)^{n_1} (M^+_2)^{n_2} (M^+_3)^{n_3} \bar{B} |\mathrm{vac}\rangle. \tag{8}$$

Here, $M^+_a \equiv \mathrm{i} M_{0a} - M_{a4}$ are the SO(3, 2) boost operators, $n_i$ are non-negative integers and the set $\bar{B}$ consists of all combinations of products of the operators $\bar{a}^i_\alpha$ which are *antisymmetric* under interchange of the index pairs $\binom{i}{\alpha}$; the latter set contains $2^{2N}$ operators. A unitary representation of Osp($N$, 4) is defined on the span of (8) by imposing a Hilbert space structure. The vacuum state is assumed to be orthonormal. One then calculates the norms of all higher states (8) by using the structure relations (1)–(5). The requirement that the norms be non-negative, which is equivalent to the requirement of unitarity, imposes restrictions on the quantum numbers of the vacuum. In the limiting case where some norms vanish, the corresponding states are absent from this representation. This is the phenomenon of multiplet shortening.

We are here interested in singleton representations and will therefore assume from the outset that the vacuum state is characterized by the relations $E_0 = \frac{1}{2}$ and $s = 0$. Furthermore, the vacuum must belong to the $2^{[N/2]}$ dimensinal spinor representation of SO($N$), and we adopt the Gel'fand–Zeythin labelling [‡1]. Thus, (7) becomes

$$a^i_\alpha |\emptyset\rangle = a^i_\alpha \Big| (E_0 = \tfrac{1}{2}, s = 0) \Big( \underbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}_{[N/2]\mathrm{times}} \Big) \Big\rangle = 0, \tag{9}$$

$(\frac{1}{2}, \ldots, \frac{1}{2})$ denotes the highest weight. It is not difficult to label the $2^{[N/2]}$ components of the vacuum states, but for our purposes it suffices to consider only one label, $M$, which is the eigenvalue of $T^{12}$, i.e.,

$$T^{12} |(\tfrac{1}{2}, 0)(\tfrac{1}{2}, \ldots, \tfrac{1}{2}) M\rangle = M |(\tfrac{1}{2}, 0)(\tfrac{1}{2}, \ldots, \tfrac{1}{2}) M\rangle, \quad M = \pm\tfrac{1}{2}. \tag{10}$$

Acting on the vacuum state with $\bar{a}^i_1$, we obtain a new set of states with $E_0 = 1$ and $s = \frac{1}{2}$. Hence,

$$\bar{a}^i_\alpha |\emptyset\rangle = c_1 \Gamma^i |\chi\rangle + c_2 |\lambda^i\rangle, \tag{11}$$

where $c_1$ and $c_2$ are constants and the $\Gamma^i$'s form a representation of the SO($N$) Clifford algebra (the $\Gamma^i$'s play the role of Glebsch–Gordan coefficients here). Furthermore,

$$|\chi\rangle = |(1, \tfrac{1}{2})(\tfrac{1}{2}, \ldots, \tfrac{1}{2}, \pm\tfrac{1}{2})\rangle, \tag{12}$$

‡1 See, e.g., ref. [7].

with + sign for odd $N$ and − sign for even $N$, and

$$|\lambda'\rangle = |(1, \tfrac{1}{2})(\tfrac{3}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2})\rangle, \tag{13}$$

with $\Gamma^i|\lambda_i\rangle = 0$. Again, it will not be necessary to exhibit a labelling system. We shall assume, however, that the vacuum state and $|\lambda_i\rangle$ are normalized to one. Specializing to $i = 1, 2$, we get from (11)

$$(\bar{a}_1^1 + \mathrm{i}\bar{a}_1^2)|\emptyset\rangle = c_1(\Gamma^1 + \mathrm{i}\Gamma^2)|\chi\rangle + c_2(|\lambda'\rangle = \mathrm{i}|\lambda^2\rangle). \tag{14}$$

Using the Osp($N$, 4) algebra, we can easily calculate the norm of both sides in this equation. The result is

$$\langle\emptyset|(M_{04} + J^3 - T^{12})|\emptyset\rangle = |c_1|^2\langle\chi|(1 + \mathrm{i}\Gamma^{12})|\chi\rangle + |c_2|^2\|(|\lambda'\rangle + \mathrm{i}|\lambda^2\rangle)\|^2, \tag{15}$$

where $\Gamma^{12} = \frac{1}{2}[\Gamma^1, \Gamma^2]$. Remembering that $J^3|\emptyset\rangle = 0$, $M_{04}|\emptyset\rangle = \frac{1}{2}|\emptyset\rangle$ from (9), and noting that $T^{12} = -(\mathrm{i}/2)\Gamma^{12}$, we find

$$\tfrac{1}{2} - M = (1 - 2M)|c_1|^2 + \tfrac{1}{2}|c_2|^2\|(|\lambda'\rangle + \mathrm{i}|\lambda^2\rangle)\|^2, \tag{16}$$

where we have made use of the assumed orthonormality as well as eq. (10). $M = \frac{1}{2}$ now implies $c_2 = 0$ while $M = -\frac{1}{2}$ yields $|c_1|^2 = \frac{1}{2}$. Choosing $c_1 = 1/\sqrt{2}$, we thus have

$$\bar{a}_1^i|\emptyset\rangle = (1/\sqrt{2})\Gamma^i|\chi\rangle. \tag{17}$$

Continuing in this manner, we now consider states obtained by applying an antisymmetrized product of two $\bar{a}_\alpha^i$'s to the vacuum state. There are only three operators of this type

$$A \equiv \epsilon_{\alpha\beta}\bar{a}_\alpha^i\bar{a}_\beta^i,$$
$$B \equiv \epsilon_{\alpha\beta}\left[\bar{a}_\alpha^i\bar{a}_\beta^j + \bar{a}_\alpha^j\bar{a}_\beta^i - (2/N)\delta^{ij}\bar{a}_\alpha^k\bar{a}_\beta^k\right], \tag{18}$$
$$C \equiv \bar{a}_{(\alpha}^i\bar{a}_{\beta)}^j - \bar{a}_{(\alpha}^j\bar{a}_{\beta)}^i.$$

Using the Osp($N$, 4) algebra again, we may straightforwardly calculate the norms of the states $A|\emptyset\rangle$ and $B|\emptyset\rangle$ with the results

$$\|A|\emptyset\rangle\|^2 = \langle\emptyset|(2NM_{04}(2M_{04} + N - 2) - 4T^{ij}T^{ij} - 4NJ^aJ^a)|\emptyset\rangle, \tag{19}$$

$$\|B|\emptyset\rangle\|^2 = \|A|\emptyset\rangle\|^2 + \langle\emptyset|(4N(N+1)M_{04}^2 - 4N^2M_{04} - 4N(N+1)J^aJ^a + 4T^{ij}T^{ij})|\emptyset\rangle. \tag{20}$$

Inserting the values $E_0 = \frac{1}{2}$, $s = 0$ and

$$\langle\emptyset|T^{ij}T^{ij}|\emptyset\rangle = \tfrac{1}{4}N(N-1). \tag{21}$$

one verifies that both states have zero norm for all $N$.

The calculation is slightly different for the state $C|\phi\rangle$ because of boost admixtures. We have

$$\bar{a}_{(\alpha}^{[i}\bar{a}_{\beta)}^{j]}|\emptyset\rangle = c_3\bar{a}_{(\alpha}^k\bar{a}_{\beta)}^k T^{ij}|\emptyset\rangle + |{}^{ij}_{\alpha\beta}\rangle, \tag{22}$$

where $c_3$ is a constant. The first term on the right-hand side of (22) is not genuinely new because $\bar{a}_{(\alpha}^k\bar{a}_{\beta)}^k$, being a symmetric combination, is proportional to the SO(3, 2) boost operators $M_a^+$ and therefore belongs to the Regge trajectory of the vacuum state to which the state $|{}^{ij}_{\alpha\beta}\rangle$ is orthogonal by construction. To determine $c_3$, we observe that $|{}^{ij}_{\alpha\beta}\rangle$ is annihilated by the SO(3, 2) de-boosts as it is a new state in the supermultiplet. Applying the de-boost to both sides of (22), we obtain, after a little calculation,

$$\mathrm{i}(\sigma_2\sigma_a)_{\alpha\beta}T^{ij}|\emptyset\rangle = c_3N(\sigma_2\sigma_a)_{\alpha\beta} \times (M_{04} + \sigma^aJ^a)T^{ij}|\emptyset\rangle, \tag{23}$$

from which we read off that $c_3 = \mathrm{i}/NE_0$. Substituting this value and $E_0 = \frac{1}{2}$ into (22) and calculating the norms on both sides, we find that,

$$\langle{}^{ij}_{\alpha\beta}|{}^{ij}_{\alpha\beta}\rangle = 0, \quad \text{for all } N. \tag{24}$$

Thus, we have established that no new states are obtained by applying products of two fermion creation operators to the vacuum since $A|\emptyset\rangle$ and $B|\emptyset\rangle$ have vanishing norm, and $C|\emptyset\rangle$ is an element of the space belonging to the Regge trajectory of the vacuum. Applying products of three and more fermionic operators will then lead to no new states. Altogether, we have thus shown that the states $|\emptyset\rangle$ and $|\chi\rangle$ already form a supermultiplet by themselves for arbitrary $N$. For $N = 1$, one recovers the singleton multiplet discussed in ref. [4]; for $N = 2$, the $s = 0$ state has charge $+\frac{1}{2}$ whereas the $s = \frac{1}{2}$ state has charge $q = -\frac{1}{2}$ while for $N = 3$, both states belong to the $J = \frac{1}{2}$ representation of SO(3), etc. For $N = 8$, the supercharge, and the two singleton states belong to the three inequivalent eight-dimensional representations of SO(8). The weight diagram of the $N = 8$ singleton representation has already been constructed by Günaydin [5]. It is obvious from our construction that singleton representations also exists for all $N > 8$.

Having established the existence of the singleton representations of Osp($N$, 4), we now consider the problem of building a lagrangian field theory for them. For $N = 1$, Fronsdal has already shown that [3] no physical information is lost if one writes the singleton action as a surface integral over spatial infinity of AdS, which is $S^2 \times S^1$. This is not surprising, since in $d = 3$, a Majorana fermion has one physical degree of freedom, which matches the single bosonic degree of freedom in three dimensions. The AdS group SO(3, 2) acts as the conformal group on $S^2 \times S^1$. Accordingly, one can introduce only interaction terms that are consistent with this conformal invariance [3].

One way to derive an action for singletons is the following. Consider the free action of a massive Osp(1, 4) multiplet in AdS, which has been given by Breitenlohner and Freedman and which reads [8]

$$S = \frac{1}{2}\int d^4x\sqrt{-\bar{g}}\,\big[\bar{g}^{\mu\nu}\partial_\mu A\partial_\nu A + \bar{g}^{\mu\nu}\partial_\mu B\partial_\nu B + i\bar{\chi}\gamma^\mu\bar{D}_\mu\chi + (2a^2 + am - m^2)A^2 + (2a^2 - am - m^2)B^2 - m\bar{\chi}\chi\big], \tag{25}$$

where $a$ is the inverse AdS radius and $M$ an arbitrary mass parameter. $\bar{g}^{\mu\nu}$ is the AdS background metric and $\bar{D}_\mu$ denotes the AdS-covariant derivative [in what follows, we will rely on the results, conventions and notation of ref. [8]. The action (25) is invariant under

$$\delta A = (1/\sqrt{2})\bar{\epsilon}\chi, \quad \delta B = (i/\sqrt{2})\bar{\epsilon}\gamma^5\chi,$$
$$\delta_\chi = -(1/\sqrt{2})\big[i\gamma^\mu\partial_\mu(A + i\gamma^5 B) + a(A - i\gamma^5 B) + m(A + i\gamma^5 B)\big]\epsilon. \tag{26}$$

Here, $\epsilon(x)$ is a Killing spinor in AdS which obeys $\tilde{D}_\mu\epsilon = 0$ and can be written as

$$\epsilon(x) = S(x)\xi, \tag{27}$$

with constant $\xi$ and

$$S(x) = (\cos\rho)^{-1/2}(\cos\tfrac{1}{2}\rho + i\gamma^i\hat{x}^i\sin\tfrac{1}{2}\rho) \times(\cos\tfrac{1}{2}t - i\gamma^0\sin\tfrac{1}{2}t), \tag{28}$$

where we have introduced polar coordinates $x^1 = \rho\sin\theta\cos\phi$, $x^2 = \sin\theta\sin\phi$, $x^3 = \rho\cos\theta$ and the radial unit vector $\hat{x}^i \equiv x^i/\rho$. In these coordinates, the AdS metric is given by

$$ds^2 = (a\cos\rho)^{-2}\big[dt^2 - d\rho^2 - \sin^2\rho(d\theta^2 + \sin^2\theta\, d\phi^2)\big]. \tag{29}$$

The angular variables $\theta$ and $\phi$ satisfy the usual constraints, whereas $0 \leqslant \rho \leqslant \pi/2$ and spatial infinity corresponds to $\rho = \pi/2$. The time coordinate $t$ is also periodic with period $2\pi$, but from the dependence of $S(x)$ on $t$ in (28), it is obvious that one should formulate the theory on the *double* covering of AdS, i.e., identify $t$ with $t + 4\pi$. The results of ref. [8] guarantee that the commutator algebra following from (26) coincides with the algebra of Osp(1, 4), i.e., eqs. (1)–(5) for $N = 1$. We next observe that, in the limit $\rho \to \pi/2$, the matrix $S$ tends to

$$S(x) \underset{\rho\to\pi/2}{\to} \sqrt{2}(\cos\rho)^{-1/2}P_+M(t), \tag{30}$$

where $M(t) \equiv \cos(t/2) - i\gamma^0\sin(t/2)$. Furthermore,

$$P_\pm \equiv \tfrac{1}{2}(1 \pm i\gamma^i\hat{x}^i),$$
$$P_\pm^2 = P_\pm, \quad P_+P_- = P_-P_+ = 0, \tag{31}$$
$$P_+ + P_- = 1,$$

are two projection operators. Their existence enables us to truncate the action (25) by imposing a "chirality condition" at spatial infinity with respect to the operator $P_+$ (or, equivalently, $P_-$); it is important that such a condition does *not* interfere with the Majorana properties of the spinors unlike the usual chirality condition because here $\gamma^5$ is replaced by $i\gamma^i\hat{x}^i$. Since the "chirality" of $\epsilon(x)$ at $\rho = \pi/2$ is already fixed by (27), we see that a possible truncation is

$$B = 0, \chi_+ \equiv P_+\chi = 0, \quad \epsilon_- \equiv P_-\epsilon = 0. \tag{32}$$

Consistency then requires that $\delta B = \delta\chi_+ = 0$. Evidently, (26) and (32) imply $\delta B = 0$, but for $\delta\chi$, the argument is more subtle. Noting that $\gamma^\mu = \gamma^a V_a^\mu$, where $V_a^\mu$ is given in eq. (3.13) of ref. [8], and applying the projector $P_+$ to $\delta\chi$ in (26), we obtain

$$\delta\chi_+ = -(1/\sqrt{2})[a \cos\rho\, \partial A/\partial\rho + (m+a)A]\epsilon_+ . \tag{33}$$

It has been shown in ref. [3] that the singleton action can be written as a surface integral over $S^2 \times S^1$. In order to yield a non-trivial contribution at spatial infinity, the singleton lagrangian must scale as $(\cos\rho)^3$. To determine the scaling behaviour of $A$, it is sufficient to look at the kinetic term. This gives

$$A(\rho, \theta, \varphi, t) \to (a \cos\rho)^{1/2} \phi(\theta, \varphi, t), \tag{34}$$

as $\rho \to \pi/2$. Inserting this into (33), we see that (33) will only vanish for the special mass value

$$m = -\tfrac{1}{2}a. \tag{35}$$

Remarkably, this is precisely the value $m$ must have in a singleton representation, because the field equation for $A$ reads

$$[E_0(E_0-3)a^2 + 2a^2 + am - m^2]A = 0, \tag{36}$$

since the eigenvalue of the SO(3,2) Casimir operator for a spinless particle is just $E_0(E_0-3)a^2$. Substitution of $E_0 = \frac{1}{2}$ into (36) immediately yields (35) (the alternative value $m = \frac{3}{2}a$ is obtained if, instead of (32), we impose $A = \chi_- = 0$).

One can further clarify the structure by rewritting the action and transformation rules in a manifestly $S^2 \times S^1$ covariant form. To this end, we express $\chi = \chi_-$ in the form

$$\chi_- = a \cos\rho \begin{pmatrix} \lambda \\ i\sigma^i \hat{x}^i \lambda \end{pmatrix}, \tag{37}$$

where $\lambda$ is a two component Majorana-spinor on $S^2 \times S^1$. The scaling behaviour in (37) is dictated by the same requirement as in (34). The supersymmetry transformation parameter $\epsilon = \epsilon_+$ may be re-expressed in an analogous fashion

$$\epsilon_+ = \sqrt{2}(a\cos\rho)^{-1/2} \begin{pmatrix} \eta \\ -i\sigma^i\hat{x}^i\eta \end{pmatrix}. \tag{38}$$

From (26), (32), (34), (37) and (38), it now follows that

$$\delta\phi = 2\bar{\eta}\lambda \tag{39}$$

and

$$\delta\lambda = -i\partial_0\phi\eta - i\epsilon^{ijk}\hat{x}^j\sigma^k \times (\rho\partial\phi/\partial x^i - x^i\hat{x}^j\partial\phi/\partial x^j)\eta. \tag{40}$$

The second term in (40) can be further simplified by noting that it is actually $\rho$-independent and therefore contains only derivatives with respect to the angular variables $\theta$ and $\phi$. Thus, (40) can be rewritten in the form

$$\delta\lambda = -i\partial_0\phi\cdot\eta - i\tilde{\gamma}^\alpha\partial_\alpha\phi\cdot\eta, \tag{41}$$

where $\alpha = 1, 2$ labels the coordinates on $S^2$ and the two-by-two matrices $\tilde{\gamma}^\alpha$ generate the Clifford algebra in two dimensions, i.e., $\{\tilde{\gamma}^\alpha, \tilde{\gamma}^\beta\} = 2\tilde{g}^{\alpha\beta}$ with the $S^2$-metric $\tilde{g}^{\alpha\beta}$. Applying the same procedure to (25), one can obtain the singleton action as an integral over $S^2 \times S^1$. There is, however, a further subtlety. In proving the supersymmetry of the action (25), partial integrations were carried out at liberty and surface terms were discarded. If, however, the fields scale as (34) and (37), surface terms cannot be neglected. Substituting the value (35) into (25) and using (34), one finds that the sum of the mass term and the part of the kinetic term containing derivatives with respect to the radial variable $\rho$ have the wrong scaling behaviour. It turns out that this term is precisely cancelled by the corresponding term with the wrong scaling behaviour in

$$\frac{1}{2}\int d^4x\sqrt{-\bar{g}}\,\partial_\mu(\bar{g}^{\mu\nu}A\partial_\nu A). \tag{42}$$

Therefore, the kinetic term in (25) should read

$$-\frac{1}{2}\int d^4x\sqrt{-\bar{g}}\,\bar{g}^{\mu\nu}A\bar{D}_\mu\partial_\nu A, \tag{43}$$

instead. The final lagrangian on $S^2 \times S^1$ now reads

$$\mathscr{L}(\text{singleton}) = (\partial_0\phi^2) = \tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + 2i\bar{\lambda}\partial^0\lambda + 2i\bar{\lambda}\tilde{\gamma}^\alpha\partial_\alpha\lambda. \tag{44}$$

Note that there is no need for covariantizing the $\partial_\alpha$-derivative, which acts on $\lambda$, because a possible connection term drops out as $\lambda$ is a Majorana spinor. In accordance with the remark after (29), the singleton lagrangian (44) has to be integrated over the double covering of $S^2 \times S^1$, i.e.,

$$S[\text{singleton}] = \frac{1}{2}\int_0^{4\pi} \mathrm{d}t \int_{S^2} \mathrm{d}\Omega\, \mathscr{L}(\theta, \varphi, t). \qquad (45)$$

We now briefly comment on the question of self-interactions. Fronsdal has shown [3] how to extend (39), (41) and (44) by including supersymmetric interactions. The only terms with the correct scaling behaviour are $\phi^6$ and $\phi^2\bar{\chi}\chi$. To see whether such interactions could also be present for higher $N > 1$, we consider the case $N = 8$ and utilize some results of Marcus and Schwarz [9] who have classified three-dimensional supergravity theories. Adopting their notation, we assign the supersymmetry transformation parameter $\eta^i$ and the fields $\phi^A$ and $\lambda^{\dot{A}}$ to the three inequivalent representations of SO(8). Eqs. (39) and (41) are consequently replaced by

$$\begin{aligned} &\delta\phi^A = 2\Gamma^i{}_{A\dot{A}}\bar{\eta}^i\lambda^{\dot{A}}, \\ &\delta\lambda^{\dot{A}} = \left(-\mathrm{i}\partial_0\phi^A - \mathrm{i}\tilde{\gamma}\alpha\partial_\alpha\phi^A\right)\Gamma^i{}_{A\dot{A}}\eta^i. \end{aligned} \qquad (46)$$

To include interactions, a further term must be added to $\delta\lambda^{\dot{A}}$, namely

$$\delta'\lambda^{\dot{A}} = \delta\lambda^{\dot{A}} + \mathrm{f}(\phi)\Gamma^i{}_{A\dot{A}}\phi^A\eta^i, \qquad (47)$$

with $f(\phi)$ an arbitrary scalar function of $\phi$. In the commutator $\phi^A$, this leads to an additional term

$$\left[\delta_1', \delta_2'\right]\phi^A = \ldots + 2f(\phi)\bar{\eta}_1^i\eta_2^j\left(\Gamma^i\Gamma^{jT} - \Gamma^j\Gamma^{iT}\right)_{AB}\phi^B. \qquad (48)$$

For $f(\phi) = 1$, the additional term is just the SO(8) rotation in the commutator (1). However, proper scaling behaviour requires $f(\phi) \propto \phi^2$ and there appears to be no way to reconcile the required scaling with the closure of the algebra. This argument is easily seen to apply to all $N > 1$. We therefore conclude that, for $N > 1$, there is no supersymmetric self-interaction for singletons.

One may, however, enquire whether $N > 1$ singletons could be made to interact with other fields such as gravity. This would require a formulation of singleton lagrangians inside AdS and not just at its boundary. A natural setting where such couplings might occur is Kaluza-Klein supergravity. Indeed, inspection of the recently determined mass spectrum of 11-dimensional supergravity on $S^7$ [10,7] [‡2] reveals the possible existence of two states with the quantum numbers of the $N = 8$ singleton multiplet in the "basement" of the infinite tower of states on $S^7$. Unfortunately, the associated modes vanish identically and therefore do not appear at the linearized level in the compactification. Whether they could appear at the non-linear level and interact with the other states that arise in compactification or whether they appear in other compactifications such as a squashed $S^7$ [12], remains to be investigated.

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‡2 The spectrum of the fermions has also been given in ref. [11].

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