

## GEOMETRY OF COSET SPACES AND MASSLESS MODES OF THE SQUASHED SEVEN-SPHERE IN SUPERGRAVITY

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We derive some general results for Killing vectors on arbitrary coset manifolds and explicitly exhibit the squashed seven-sphere as the coset space  $Sp_4 \times Sp_2 / Sp_2 \times Sp_2$ . Using these results, we then analyze the zero-mass sector of supergravity on the squashed  $S^7$  and argue that it is not interpretable as a spontaneously broken version of  $N = 8$  supergravity. We also point out the existence of a new solution which combines squashing and torsion.

### 1. Introduction

It is by now well known that supergravity in eleven dimensions naturally compactifies to supergravity in four dimensions [1]. In this way, four-dimensional theories with or without breaking of supersymmetry can be obtained. So far the following compactifications on the seven-sphere  $S_7$  have been obtained:

- (i) round  $S_7$ : no supersymmetry breaking [2];
- (ii) round  $S_7$  with torsion [3]: all supersymmetries broken [4, 5];
- (iii) squashed  $S_7$ :  $N = 1$  supersymmetry remains [6].

In this paper we shall begin by presenting a new solution:

- (iv) squashed  $S_7$  with torsion: no supersymmetry left<sup>†</sup>.

It is known that the round  $S_7$  is a coset space, namely  $SO(8)/SO(7)$  [2]. On  $S_7$  with torsion, the vielbein and spin connection are still those of the maximally symmetric round  $S_7$ , but the torsion field has less symmetry, namely its symmetries form the

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<sup>†</sup> This solution was found independently in Ref. [7].

exceptional group  $G_2$  [3, 4]. In this paper we shall exhibit that the squashed sphere is the coset space

$$G/H = \text{Sp}(4) \times \text{Sp}(2)/\text{Sp}(2) \times \text{Sp}(2). \quad (1.1)$$

Since  $\text{Sp}_4 \sim \text{SO}_5$  and  $\text{Sp}_2 \sim \text{SO}_3$ , this group  $G$  is a subgroup of  $\text{SO}_8$ .

Having shown that the vacuum is described by the coset  $G/H$  in (1.1), we will then turn to the massless physical excitations. As expected, in addition to the graviton, there is one massless gravitino corresponding to the  $N = 1$  supersymmetry of the vacuum. Moreover, since  $\text{Sp}_4 \times \text{Sp}_2$  has  $10 + 3$  Killing vectors, there are 13 massless vectors. To demonstrate that there are also 13 massless fermionic partners, we must analyze the mass matrix of the spin- $\frac{1}{2}$  fields. In this way, one finds that the Killing vectors  $K_{(M)}^\alpha$  ( $M = 1, 13$  and  $\alpha = 1, 7$ ) must satisfy a remarkable identity

$$a_{\alpha\beta}{}^\gamma D_\gamma K_{(M)}^\beta = \delta_{\alpha\beta} K_{(M)}^\beta, \quad (1.2)$$

where  $a_{\alpha\beta}{}^\gamma$  are the octonionic structure constants.

It has been noted before that the octonions play a rôle in the solution of the squashed  $S_7$  with torsion. Specifically, the torsion field  $A_{\alpha\beta\gamma}(y)$  can be identified with the octonionic structure constants [7]. In this paper we de-emphasize the rôle of octonions somewhat in favour of coset spaces: we show that the octonionic structure constants are just the structure constants  $c_{\alpha\beta}{}^\gamma$  which appear in the commutators of the form  $[K_\alpha, K_\beta] \sim K_\gamma$  where the  $K$  are the coset generators of (1.1). In fact, we shall prove the following:

*Theorem:* the identity (1.2) with  $a_{\alpha\beta}{}^\gamma$  replaced by  $c_{\alpha\beta}{}^\gamma$  and  $\delta_{\alpha\beta}$  by  $c_{\alpha\gamma}{}^\delta c_{\beta\delta}{}^\gamma$  holds for any reductive Lie algebra whose Killing metric is block-diagonal on  $(H, K)$ .

Let us begin by presenting our new solution, namely the squashed  $S_7$  with torsion. The field equations of eleven-dimensional supergravity are given by

$$R_{MN} = -\frac{1}{6} \left( F_{MPQR} F_N{}^{PQR} - \frac{1}{12} g_{MN} F_{PQRS}^2 \right), \quad (1.3a)$$

$$D_M F^{MNPQ} = \frac{i\sqrt{2}}{12 \times 96} \epsilon^{NPQR_1 \dots R_4 S_1 \dots S_4} F_{R_1 \dots R_4} F_{S_1 \dots S_4}, \quad (1.3b)$$

where we have put the fermionic fields equal to zero. We use the conventions of ref. [8]; in particular, the photon field strength  $F_{MNPQ}$  has strength 24 and  $M, N \dots$  are flat eleven-dimensional indices.

Spontaneous compactification means that eqs. (1.3a) and (1.3b) admit solutions whereby the eleven-dimensional manifold becomes a product of ordinary space-time and some compact internal space, at least locally. Following the ansatz of Freund and Rubin for  $F_{\mu\nu\rho\sigma}$  and that of Englert for  $F_{\alpha\beta\gamma\delta}$  we assume that the vielbein splits into a four- and a seven-dimensional part and put

$$F_{mnr s} = im \epsilon_{mnr s}, \quad F_{abcd} = a \bar{\eta} \Gamma_{abcd} \eta. \quad (1.4)$$

The indices  $m, n, \dots$  and  $a, b, \dots$  are flat four- and seven-dimensional indices, respectively, and  $m$  and  $a$  are free parameters. The  $\Gamma_a$  ( $a = 1, 7$ ) are  $8 \times 8$  Dirac matrices for seven-dimensional euclidean space, satisfying  $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$  and  $\eta$  is a covariantly constant spinor

$$D_\alpha \eta = c\Gamma_\alpha \eta, \tag{1.5}$$

where  $c$  is a parameter to be determined.

The ansatz (1.4) solves the Maxwell equations in (1.3b) if

$$8ac = i\sqrt{2} ma, \quad (\text{Maxwell}). \tag{1.6}$$

Substituting the ansatz in (1.4) into the Einstein equations in (1.3a) leads to two Einstein spaces

$$R_{\mu\nu} = \left(\frac{2}{3}m^2 + \frac{7}{3}a^2\right)g_{\mu\nu}, \quad R_{\alpha\beta} = -\left(\frac{1}{3}m^2 + \frac{5}{3}a^2\right)g_{\alpha\beta}. \tag{1.7}$$

The integrability condition of (1.5), obtained by differentiating (1.5) once more reads quite generally [7]

$$R_{\alpha\beta} = 24c^2g_{\alpha\beta}, \quad C_{\alpha\beta}{}^{\gamma\delta}\Gamma_{\gamma\delta}\eta = 0, \quad (\text{integrability}), \tag{1.8}$$

where  $C_{\alpha\beta}{}^{\gamma\delta}$  is the Weyl tensor. A last condition on the parameters is obtained when one requires that at least one supersymmetry is unbroken. In that case, not only the gravitino itself, but also its variation under supersymmetry must vanish. From the eleven-dimensional variation law

$$\delta\psi_M = D_M \epsilon + \frac{1}{288}\sqrt{2} \left(\Gamma_M{}^{PQRS} - 8\delta_M^P \Gamma^{QRS}\right)\epsilon F_{PQRS}, \tag{1.9}$$

one finds with  $\epsilon(x, y) = \epsilon(x)\eta(y)$  and using (1.4), two results

$$a = 0, \quad c = -\frac{1}{12}im\sqrt{2}, \quad (\text{supersymmetry left}). \tag{1.10}$$

Let us now consider the solutions which preserve supersymmetry. From (1.10) we see that in this case there is no torsion so that the Maxwell equations are automatically satisfied. Moreover, the value of  $c$  in (1.10) is compatible with the integrability condition and Einstein equations for  $R_{\alpha\beta}$ . From (1.10) we must find a solution of  $R_{\alpha\beta} = -\frac{1}{3}m^2g_{\alpha\beta}$  on a compact seven-dimensional space. There are two and only two solutions.

(i) *The round  $S_7$* : for which the Riemann tensor is maximally symmetric, i.e. it is proportional to  $g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}$  so that the Weyl tensor vanishes. There are two

covariantly constant spinors, satisfying

$$D_\alpha \eta_+ = \frac{1}{12} im\sqrt{2} \Gamma_\alpha \eta_+, \quad D_\alpha \eta_- = -\frac{1}{12} im\sqrt{2} \Gamma_\alpha \eta_-, \quad (1.11)$$

which can be written as  $(\exp y^\alpha \Gamma_\alpha) \eta(0)$  [4], but only  $\eta_-(y)$  can be used to parametrize the supersymmetry parameter  $\epsilon(x, y)$  because only  $\eta_-$  satisfies (1.10).

(ii) *The squashed  $S_7$* : in this case the Ricci tensor is the same as for the round  $S_7$  but the Riemann tensor contains a non-vanishing Weyl tensor satisfying the integrability condition in (1.8). There is only one covariantly constant spinor, not two, because the matrices  $C_{\alpha\beta}{}^{\gamma\delta} \Gamma_{\gamma\delta}$  generate the spinor representation of  $G_2$  [6] which can leave only one spinor fixed. This spinor still satisfies the criterion in (1.10) that supersymmetry is preserved. Actually, this spinor  $\eta$  is really constant [7]. This can be shown by using the general relation

$$c_{ab}{}^c \omega_\alpha{}^b{}_c = \frac{1}{2} (c_{ab}{}^c c_{dc}{}^b) e_\alpha{}^d, \quad (1.12)$$

valid for reductive algebras whose Killing metric is block diagonal on  $(H, K)$  (see sect. 2) and noting that in a suitable representation  $c_{ab}{}^c \sim (\Gamma_{ab}\eta)^c$  and  $(\Gamma_a\eta)^b \sim \delta_a^b$ , so that the two- and one-gamma terms in eq. (1.5) defining the covariantly constant spinor, cancel.

Let us now present our new solution for the squashed  $S_7$  with torsion. Since  $a \neq 0$ , the Maxwell equation in (1.6) fixes  $c$ , and this value of  $c$  clashes with (1.10) so that all supersymmetries are broken. Comparing the Einstein equations with the integrability condition we have

$$c = \frac{1}{8} i\sqrt{2} m, \quad a^2 = \frac{1}{4} m^2. \quad (1.13)$$

Thus the solution of the squashed  $S_7$  with torsion is given by

$$F_{mnrs} = im \varepsilon_{mnrs}, \quad F_{abcd} = \pm \frac{1}{2} m \bar{\eta} \Gamma_{abcd} \eta, \quad (1.14)$$

while the vielbein is that of the squashed  $S_7$  without torsion but rescaled such that

$$R_{\mu\nu} = \left(\frac{2}{3} m^2 + \frac{7}{12} m^2\right) g_{\mu\nu}, \quad R_{\alpha\beta} = -\left(\frac{1}{3} m^2 + \frac{5}{12} m^2\right) g_{\alpha\beta}. \quad (1.15)$$

The Maxwell equations are satisfied because  $c$  is given by (1.13), and the integrability conditions are satisfied by fixing  $a$  as in (1.13). The Einstein equations are then satisfied by taking the vielbeins of the squashed  $S_7$  without torsion, rescaled according to (1.15).

The spinor  $\eta$  is covariantly constant and equal to the covariantly constant spinor of the squashed  $S_7$  without torsion, but again with rescaled  $m$ . Although the metric and  $\eta$  are the same as those of the squashed  $S_7$  (up to rescalings) supersymmetry is broken, because in  $\delta\psi_M$  new terms appear proportional to torsion.

## 2. Geometry of coset spaces

In this section we will first summarize some elements of the theory of coset spaces\* and then use them to derive a general differential equation for Killing vectors  $K_{(M)}^\alpha(y)$ ,

$$c_{ab}{}^c D_c K_{(M)}^b = \frac{1}{2} (c_{ab}{}^c c_{dc}{}^b) K_{(M)}^d. \tag{2.1}$$

This relation holds for *any* reductive Lie algebra whose Killing form is block-diagonal on the subgroup  $\mathbb{H}$  and its complement  $\mathbb{K}$ .

Let  $\mathbb{G}$  be a Lie algebra,  $\mathbb{H}$  a subalgebra and  $\mathbb{K}$  a complement, so that  $\mathbb{G} = \mathbb{H} + \mathbb{K}$ . We restrict ourselves to *reductive* algebras, meaning that  $[\mathbb{H}, \mathbb{K}] \subset \mathbb{K}$  but we do not assume that  $\mathbb{G}$  is symmetric (a symmetric algebra satisfies not only  $[\mathbb{H}, \mathbb{K}] \subset \mathbb{K}$  but also  $[\mathbb{K}, \mathbb{K}] \subset \mathbb{H}$ ). Thus the coset transforms in a (reducible or irreducible) representation of  $\mathbb{H}$ . We will also assume that the Killing metric

$$\gamma_{MN} = c_{MR}{}^S c_{NS}{}^R \tag{2.2}$$

is block-diagonal:  $\gamma_{MN}$  with  $M$  in  $\mathbb{H}$  and  $N$  in  $\mathbb{K}$  vanishes.

Our notation will be as follows:  $M, N, \dots$  run over all generators of  $\mathbb{G}$ ,  $a, b, c, d, \dots$  will be flat indices in  $\mathbb{K}$ ,  $\alpha, \beta, \gamma, \delta$  will be curved  $\mathbb{K}$  indices and  $i, j, k, l$  will be indices in  $\mathbb{H}$ .

Near the identity element of the group, we write an arbitrary group element as

$$g = \exp(k \cdot K) \exp(h \cdot H), \tag{2.3}$$

where  $k^a$  and  $h^i$  are arbitrary numbers, and  $K_a$  and  $H_i$  a basis for the generators in  $\mathbb{K}$  and  $\mathbb{H}$ , respectively. We define vielbeins  $e_\alpha^a(y)$  and H connections  $\omega_\alpha^i(y)$  by

$$e^{y \cdot K} e^{d \cdot y \cdot K} = \exp[(y^\alpha + d y^\alpha e_\alpha^a(y)) K_a] \exp(-d y^\alpha e_\alpha^a(y) \omega_\alpha^i(y) H_i). \tag{2.4}$$

These vielbeins, and H connections satisfy Cartan-Maurer equations:

$$\partial_\beta e_\gamma^a - \partial_\gamma e_\beta^a + c_{bc}{}^a e_\beta^b e_\gamma^c + c_{ib}{}^a (\omega_\beta^j e_\gamma^b - \omega_\gamma^j e_\beta^b) = 0, \tag{2.5}$$

$$\partial_\alpha \omega_\beta^i - \partial_\beta \omega_\alpha^i + c_{jk}{}^i \omega_\alpha^j \omega_\beta^k + c_{ab}{}^i e_\alpha^a e_\beta^b = 0. \tag{2.6}$$

Points on the coset manifold, represented by  $\exp(y \cdot K)$ , can be swept over the coset manifold by left multiplication by an arbitrary group element  $g$  as in (2.3). Suppose one has given a connection on the coset manifold  $\omega_{\alpha c}{}^b(y)$  which defines parallel transport. If the operations of parallel transport and sweeping out commute,

\* For a complete discussion, see a forthcoming book by B.S. DeWitt, P. van Nieuwenhuizen and P.C. West.

this connection is called invariant. The most general invariant connection is given by

$$\omega_\alpha^b{}_c(y) = \omega_\alpha^i c_{ic}{}^b + e_\alpha^a \omega_a^b{}_c(0), \tag{2.7}$$

where  $\omega_a^b{}_c(0)$  is any invariant tensor of the adjoint representation of  $\mathbb{H}$  restricted to the coset. A natural set of invariant connections are the structure constants

$$\omega_a^b{}_c(0) = \lambda c_{ac}{}^b. \tag{2.8}$$

The relation (2.7) between the  $\mathbb{H}$  connection  $\omega_\alpha^i(y)$  and the coset connection  $\omega_\alpha^a{}_b(y)$  forms the bridge between the group-theoretic approach with the  $\mathbb{H}$  connection and the differential geometric approach with coset connection. For symmetric algebras, using (2.8), both connections coincide but for general reductive algebras they differ.

For  $\lambda = \frac{1}{2}$ , we can rewrite the Cartan-Maurer equation in (2.5) such that there is no torsion

$$\partial_\alpha e_\beta^a + \omega_\alpha^a{}_b e_\beta^b - \alpha \leftrightarrow \beta = 0, \tag{2.9}$$

$$\omega_\alpha^a{}_b(y) = \omega_\alpha^i(y) c_{ib}{}^a + \frac{1}{2} e_\alpha^c(y) c_{cb}{}^a. \tag{2.10}$$

We can now find the curvature tensor for the torsionless connection in (2.10) by taking a linear combination of (2.5) and (2.6)

$$R_{\alpha\beta}{}^a{}_b \equiv \partial_\alpha \omega_\beta^a{}_b + \omega_\alpha^a{}_c \omega_\beta^c{}_b - \alpha \leftrightarrow \beta, \tag{2.11}$$

$$R_{\alpha\beta}{}^a{}_b = \left( \frac{1}{4} c_{bc}{}^a c_{de}{}^c + \frac{1}{2} c_{bi}{}^a c_{de}{}^i + \frac{1}{4} c_{dc}{}^a c_{eb}{}^c \right) \left( e_\alpha^d e_\beta^e - e_\beta^d e_\alpha^e \right). \tag{2.12}$$

Killing vectors  $K_{(M)}^\alpha(y)$  are defined by the reverse of the definition of vielbeins

$$e^{dg \cdot M} e^{y \cdot K} = \exp\left[ (y^\alpha + dg^M K_{(M)}^\alpha(y)) K_\alpha \right] \exp(-dg^M \Omega_M^i(y) H_i). \tag{2.13}$$

Note that  $M$  runs over all generators of  $\mathbb{G}$ , hence  $K_{(M)}^\alpha(y)$  cannot be inverted. The inverse of  $e_a^\alpha(y)$  will be denoted by  $e_\alpha^a(y)$  as usual.

By evaluating in the following expression first the terms between curly brackets and equating the result to what one gets if one first evaluates the terms within parentheses

$$\left\{ e^{(x^\alpha + dg \cdot K^\alpha) K_\alpha} e^{-dg \cdot \Omega \cdot H} \left( e^{-y \cdot K} \right) e^{(y + \Delta y) \cdot K} \right\}, \tag{2.14}$$

one finds, for the coefficients of the  $\mathbb{K}$  and  $\mathbb{H}$  generators, respectively,

$$K_{(M)}^\beta \partial_\beta e_a^\alpha - e_a^\beta \partial_\beta K_{(M)}^\alpha = \Omega_M^i c_{ia}{}^{\alpha'} e_{\alpha'}^\alpha, \tag{2.15}$$

$$K_{(M)}^\beta \partial_\beta (e_a \cdot \omega^i) - e_a^\alpha \partial_\alpha \Omega_M^i = \Omega_M^j \left( c_{ja}{}^{\alpha'} e_{\alpha'} \cdot \omega^i - c_{ji}{}^{\alpha'} e_a \cdot \omega^{\alpha'} \right). \tag{2.16}$$

This result tells us that the tangent vectors  $K_{(M)} \equiv K_{(M)}^\alpha \partial_\alpha$  and  $e_a \equiv e_a^\alpha \partial_\alpha$  no longer commute on coset manifolds, contrary to what happens on group manifolds, but that their commutator  $[K_{(M)}, e_a]$  is equal to an H-gauge transformation. In terms of Lie derivatives

$$\mathcal{L}_{K_{(M)}} e_a^\alpha = \delta_{\mathbb{H}}(\Omega_M^i) e_a^\alpha, \tag{2.17}$$

$$\mathcal{L}_{K_{(M)}} \omega_\alpha^i = D_\alpha \Omega_M^i = \delta_{\mathbb{H}}(\Omega_M^i) \omega_\alpha^i. \tag{2.18}$$

Thus the vielbein and connection fields  $e_a^\alpha$  and  $\omega_\alpha^i$ , and a fortiori  $\omega_{\alpha b}^a$ , are physically invariant [9, 4]: if one moves on the coset manifold along  $K_{(M)}^\alpha(y)$ , they change only by an H-gauge transformation with parameter  $\Omega_M^i(y)$ .

Let us now consider the relation (2.15) in more detail. Multiplying by  $e_\gamma^a$  and replacing  $e_\gamma^a \partial_\beta e_a^\alpha$  by  $-e_a^\alpha \partial_\beta e_\gamma^a$  we find

$$\partial_\gamma (K_{(M)}^\alpha e_a^\alpha) + (\partial_\beta e_\gamma^a - \partial_\gamma e_\beta^a) K_{(M)}^\beta = -\Omega_M^i c_{ic}^a e_\gamma^c. \tag{2.19}$$

Using the Cartan-Maurer equations for  $e_\gamma^a$  to eliminate the derivatives of the vielbeins, we get

$$D_c K_{(M)}^a = K_{(M)}^b \omega_b^a c^c - \Omega_M^i c_{ic}^a. \tag{2.20}$$

Substituting (2.10) and contracting with  $c_{da}^c$ , the terms involving  $\omega_\alpha^i$  and  $\Omega_M^i$  cancel because we have assumed that

$$\gamma_{di} = c_{dM}^N c_{iN}^M = c_{da}^c c_{ic}^a = 0. \tag{2.21}$$

Thus

$$c_{da}^c D_c K_{(M)}^a = \frac{1}{2} (c_{da}^c c_{ec}^a) K_{(M)}^e. \tag{2.22}$$

Note that  $c_{da}^c c_{ec}^a$  is *not* the Killing metric  $\gamma_{de}$  because for that one also needs the terms  $c_{di}^c c_{ec}^i$  and  $c_{dc}^i c_{ei}^c$ .

We now discuss up to what point  $\mathbb{K}$  is unique, once  $\mathbb{H}$  and  $\mathbb{G}$  are given. The most general coset generators are

$$\hat{K}_a = K_a + d_a^j H_j; \tag{2.23}$$

but imposing reductivity  $[\hat{\mathbb{K}}, \mathbb{H}] \subset \hat{\mathbb{K}}$  shows that the  $d_a^i$  must be invariant tensors of the adjoint representation of  $\mathbb{H}$

$$d_a^j c_{ij}^j - d_a^j c_{ia}^{a'} = 0. \tag{2.24}$$

Thus, the freedom in the choice of the generators of  $K$  can be found as follows. One decomposes the coset representation of  $H$  on  $K$

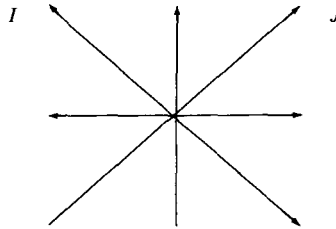
$$dv^a = -c_{ia'}{}^a v^{a'}, \tag{2.25}$$

into irreducible representations of  $H$ , and determines how many scalars there are in the product  $v^m y_j$  where  $\delta y_j = c_{ij}{}^k y_k$ . In general there is little freedom in  $K$ , and in the case we study in sect. 3, it can be completely fixed by requiring block-diagonality of the Killing form.

### 3. The squashed seven-sphere as the coset space $Sp_4 \times Sp_2 / Sp_2 \times Sp_2$

In this section we show explicitly that the coset manifold  $G/H$ , where  $G = Sp_4 \times Sp_2^L$  and  $H = Sp_2^I \times Sp_2^{J+L}$ , corresponds to the squashed  $S^7$ . (We shall explain the superscripts  $I, J$  and  $L$  below.) We will do so by computing the factors involving the structure constants which appear in the fundamental expression (2.12) of the curvature in terms of the vielbeins. This necessitates a rather detailed discussion of the precise structure of the algebras  $\mathfrak{G}$  and  $\mathfrak{H}$ , as well as  $\mathfrak{K}$ , the complement of  $\mathfrak{H}$  in  $\mathfrak{G}$ .

The semisimple algebra  $\mathfrak{G} = Sp_4 + Sp_2^L$  can be naturally thought of as the maximal  $SO_5 + SO_3^L$  subalgebra of  $SO_8$  (we recall that on the algebraic level one has the isomorphism  $SO_5 \simeq Sp_4$  and  $SO_3 \simeq Sp_2$ ). It is instructive to first consider the root diagram of  $Sp_4 \simeq SO_5$ :



The algebras  $Sp_2^I$  and  $Sp_2^J$  are the regular  $Sp_2$  subalgebras corresponding to the orthogonal *long* roots. These two algebras naturally commute and their sum generates the  $SO_4$  ( $\simeq Sp_2 \times Sp_2$ ) subgroup. Note that the two regular subalgebras associated with the short roots do *not* form a subalgebra; in fact, their commutators produce the whole  $\mathfrak{G}$ . As is obvious from the root diagram, the adjoint representation of  $Sp_4$  decomposes under the  $Sp_2^I \times Sp_2^J$  subgroup into

$$\mathbf{10} \rightarrow (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{2}). \tag{3.1}$$

The roots belonging to  $(\mathbf{2}, \mathbf{2})$  form a diamond in the figure. Furthermore, this decomposition of  $Sp_4$  into  $Sp_2 + Sp_2$  and its complement is symmetric, i.e. the commutator of two generators belonging to  $(\mathbf{2}, \mathbf{2})$  is in  $Sp_2^I + Sp_2^J$ .



The subgroup  $\mathbf{H} = \text{Sp}_2^I \times \text{Sp}_2^{J+L}$  is determined by the following considerations. (By  $\text{Sp}_2^{J+L}$  we mean the diagonal  $\text{Sp}_2$  subgroup of  $\text{Sp}_2^J \times \text{Sp}_2^L$  generated by  $(J + L)_i = J_i + L_i$ .) As we are interested in a seven-dimensional space and  $\dim(\mathbf{G}) = 13$ , we need  $\dim(\mathbf{H}) = 6$ , which implies that the only possibility is in fact some  $\text{Sp}_2 \times \text{Sp}_2$  subgroup of  $\mathbf{G}$ . Clearly  $\text{Sp}_2^L$  cannot be one of the factors in  $\mathbf{H}$  because it would reduce  $\mathbf{G}$  effectively to  $\text{Sp}_4$ . On the other hand,  $\text{Sp}_2^I \times \text{Sp}_2^J$  would make  $\mathbf{G}/\mathbf{H}$  a direct product space. It follows that the only possibility is of the form  $\text{Sp}_2^I \times \text{Sp}_2^{J+L}$  (or, equivalently, with  $I$  and  $J$  interchanged). It should be clear that the diagonal subgroup of  $\text{Sp}_2^I$  with any other  $\text{Sp}_2$  subgroup of  $\text{Sp}_4$  cannot enter as a factor in  $\mathbf{H}$  for the simple reason that there would be no other  $\text{Sp}_2$  subgroup of  $\mathbf{G}$  which commutes with it. This completes the determination of  $\mathbf{H}$ : it is unique.

We now turn to the complement  $\mathbf{K}$  of  $\mathbf{H}$  in the decomposition

$$\mathbf{G} = \mathbf{H} + \mathbf{K} . \tag{3.2}$$

As such the generators in  $\mathbf{K}$  are not fixed as we could add always an admixture of  $\mathbf{H}$  generators to them. However, as we will show in the following, the choice of  $\mathbf{K}$  generators is entirely fixed (up to some irrelevant scales) by imposing the two conditions that:

(i)  $\mathbf{H} + \mathbf{K}$  be a reductive decomposition of  $\mathbf{G}$ , i.e. we demand  $[\mathbf{H}, \mathbf{K}] \subset \mathbf{K}$  (but not  $[\mathbf{K}, \mathbf{K}] \subset \mathbf{H}$ );

(ii)  $\mathbf{K}$  and  $\mathbf{H}$  are orthogonal with respect to the Killing metric in (2.2) of  $\mathbf{G}$  i.e. we demand  $\gamma_{MN}$  to be block diagonal in  $\mathbf{H}$  and  $\mathbf{K}$  indices.

At this point, it is convenient to further decompose  $\mathbf{K}$  as

$$\mathbf{K} = \mathbf{Q} + \mathbf{T} , \tag{3.3}$$

where  $\mathbf{Q}$  stands for the quartet of four generators forming the  $(2, 2)$  representation in (3.1). Concerning  $\mathbf{Q}$  we make the observation that the decomposition

$$\mathbf{G} = (\text{Sp}_2^I + \text{Sp}_2^J + \text{Sp}_2^L) + \mathbf{Q} , \tag{3.4}$$

is not just reductive but in fact symmetric. This not only implies that  $[\mathbf{H}, \mathbf{Q}] = \mathbf{Q}$ , but because of the symmetry property also that the Killing form of  $\mathbf{G}$  is block diagonal in  $\text{Sp}_2^I + \text{Sp}_2^J + \text{Sp}_2^L$  and  $\mathbf{Q}$ , hence in  $\mathbf{H}$  and  $\mathbf{Q}$ . The remaining three generators in  $\mathbf{T}$  may in general correspond to some linear combination of the generators of the three mutually commuting  $\text{Sp}_2$  subalgebras:

$$\hat{T}_a = \{ \alpha I_a + \beta J_a + \gamma L_a \} , \quad a = 1, 2, 3 . \tag{3.5}$$

Now we see that  $[\text{Sp}_2^I, \mathbf{T}] \subset \text{Sp}_2^I$ , and since  $\text{Sp}_2^I$  lies in  $\mathbf{H}$ , reductivity requires  $\alpha = 0$ . It is easy to verify that  $\alpha = 0$  is in fact a sufficient condition for the decomposition (3.3) to be reductive. In order to implement the orthogonality condition for  $\mathbf{H}$  and  $\mathbf{T}$

we need more explicit information on the structure constants of  $\mathfrak{G}$ . As these are also necessary for the forthcoming computations, we make a digression to introduce the nomenclature of the generators and the indices we have adopted.

We have chosen the following basis for  $\mathfrak{G}$ :

$$\begin{aligned}
 \mathfrak{G} &= \{X_M\} = \mathfrak{H} + \mathfrak{K}, & (M, N, \dots), \\
 \mathfrak{H} &= \{H_i\} = \begin{cases} I_i & (i, j, k, \dots), \\ (J + L)_i & (\hat{i}, \hat{j}, \hat{k}, \dots), \end{cases} \\
 \mathfrak{K} &= \{K_a\} = \mathfrak{Q} + \mathfrak{T}, \\
 \mathfrak{Q} &= \begin{cases} S & (0), \\ T_a & (a, b, c, \dots), \end{cases} \\
 \mathfrak{T} &= \hat{T}_a, & (\hat{a}, \hat{b}, \hat{c}, \dots). \tag{3.6}
 \end{aligned}$$

In (3.6) we have grouped the generators together according to their transformation properties under the subgroup  $\text{Sp}_2^{I+J+L}$ . We see that  $\mathfrak{H} = \{\mathbf{3} + \mathbf{3}\}$ ,  $\mathfrak{T} = \{\mathbf{3}\}$ , and  $\mathfrak{Q} = \{\mathbf{3} + \mathbf{1}\}$ . The  $\mathfrak{Q}$  decomposition follows from the familiar fact that the  $(\mathbf{2}, \mathbf{2})$  representation of  $\text{SO}_4$  decomposes in a  $\mathbf{3} + \mathbf{1}$  under the diagonal subgroup or alternatively that in quantum mechanics the product of two spinors yields a triplet and a singlet.

We proceed by compiling the non-vanishing structure constants defined by

$$[X_M, X_N] = c_{MN}^K X_K \tag{3.7}$$

in a by now familiar notation:

$$\begin{aligned}
 c_{ij}^k &= \epsilon_{ij}^k, & c_{ij}^{\hat{k}} &= \epsilon_{ij}^k, \\
 c_{ia}^0 &= c_{0i}^a = \frac{1}{2} \delta_{ai}, & c_{a0}^i &= \frac{10}{81} \delta_a^i, \\
 c_{ia}^b &= \frac{1}{2} \epsilon_{iab}, & c_{ab}^i &= \frac{10}{81} \epsilon_{abi}, \\
 c_{ia}^0 &= c_{0i}^{\hat{a}} = -\frac{1}{2} \delta_{ai}, & c_{0\hat{a}}^i &= \frac{2}{27} \delta_a^i, \\
 c_{ia}^b &= \frac{1}{2} \epsilon_{iab}, & c_{ab}^{\hat{i}} &= \frac{2}{27} \epsilon_{abi}, \\
 c_{i\hat{a}}^{\hat{b}} &= \epsilon_{iab}, & c_{\hat{a}\hat{b}}^{\hat{i}} &= \frac{4}{27} \epsilon_{abi}, \\
 c_{\hat{a}\hat{b}}^{\hat{c}} &= -\frac{1}{9} \sqrt{2} \epsilon_{abc}, & c_{\hat{a}\hat{b}}^c &= c_{bc}^{\hat{a}} = \frac{1}{9} \sqrt{2} \epsilon_{abc}, \\
 c_{\hat{a}\hat{b}}^0 &= c_{0\hat{a}}^{\hat{b}} = c_{b\hat{a}}^0 = -\frac{1}{9} \sqrt{2} \delta_{ab}. \tag{3.8}
 \end{aligned}$$

It is important to note that in this list we have set  $\gamma$  in (3.5) equal to  $\gamma = -\frac{3}{2}\beta$ , the argument will be given shortly. The normalization of the generators  $H_i$  is chosen

such that the two  $Sp_2$  subalgebras obey the standard angular momentum commutation relations. For the coset generators, we have chosen an, at this stage, somewhat arbitrary normalization where

$$[K_a, K_b] = -\frac{1}{3}\sqrt{2} a_{ab}{}^c K_c, \tag{3.9}$$

in which the constants  $a_{ab}{}^c$  are the structure constants of the octonions defined by the relation

$$e_a e_b = a_{ab}{}^c e_c - \delta_{ab}. \tag{3.10}$$

This choice will prove to be convenient in sect. 4.

Let us now return briefly to the Killing metric  $\gamma_{MN}$ , and in particular to the orthogonality of  $\mathbb{H}$  and  $\mathbb{T}$ . Assuming the generators  $\hat{T}_a$  again to be of the form  $\hat{T}_a = \beta J_a + \gamma L_a$ , we may compute  $\gamma_{i\hat{a}}$  and  $\gamma_{\hat{a}\hat{a}}$  using the structure constants given in (3.8) except that now the commutation relations involving  $\hat{T}_a$  will still depend on  $\beta$  and  $\gamma$ . Using that in that case

$$\begin{aligned} c_{\hat{a}\hat{b}}{}^{\hat{c}} &= (\beta + \gamma) \epsilon_{abc}, \\ c_{\hat{a}\hat{b}}{}^c &= \frac{1}{2} \beta \epsilon_{abc}, \\ c_{\hat{a}\hat{b}}{}^0 &= -\frac{1}{2} \beta \delta_{ab}, \end{aligned} \tag{3.11}$$

one finds that

$$\gamma_{i\hat{a}} = 0, \quad \gamma_{\hat{a}\hat{a}} = -(3\beta + 2\gamma); \tag{3.12}$$

so that demanding  $\gamma_{i\hat{a}} = 0$  indeed yields  $\gamma = -\frac{3}{2}\beta$ , the value used in (3.8) (the normalization (3.9) fixed  $\beta = \frac{2}{3}\sqrt{2}$ ).

In the basis we have described in detail the Killing form is actually completely diagonal, though not proportional to the unit matrix, in fact we have that

$$\gamma_{ij} = -3\delta_{ij}, \quad \gamma_{i\hat{j}} = -5\delta_{ij}, \quad \gamma_{00} = -\frac{20}{27}, \quad \gamma_{ab} = \gamma_{\hat{a}\hat{b}} = -\frac{20}{27}\delta_{ab}. \tag{3.13}$$

With the help of (3.6) it is straightforward to calculate the curvature forms  $R_b^a$  in terms of the vielbeins  $e^a$  using eq. (2.12). We find the following expressions

$$\begin{aligned} R^o_a &= c \left[ -\frac{17}{20} e^o e^a - \frac{1}{3} \epsilon_{abc} e^{\hat{b}} e^{\hat{c}} \right], \\ R^o_{\hat{a}} &= c \left[ -\frac{1}{20} e^o e^{\hat{a}} + \frac{1}{3} \epsilon_{abc} e^b e^{\hat{c}} \right], \\ R^a_b &= c \left[ -\frac{17}{20} e^a e^b - \frac{2}{3} e^{\hat{a}} e^{\hat{b}} \right], \\ R^{a\hat{c}}_b &= c \left[ -\frac{1}{20} e^a e^{\hat{b}} + \frac{1}{3} \epsilon_{abc} e^o e^{\hat{c}} - \frac{1}{3} e^{\hat{a}} e^b - \frac{1}{3} \delta_{ab} e^c e^{\hat{c}} \right], \\ R^{\hat{a}\hat{c}}_b &= c \left[ -\frac{2}{3} \epsilon_{abc} e^o e^c - \frac{2}{3} e^a e^b - \frac{5}{4} e^{\hat{a}} e^{\hat{b}} \right]. \end{aligned}$$

Apart from the over-all constant  $c = -\frac{10}{81}$  these are precisely the expressions given by Awada, Duff and Pope [6].

#### 4. The massless excitations on squashed $S^7$

The analysis of the low-energy sector of the theory obtained by spontaneous compactification of  $N = 1$  supergravity in eleven dimensions on the squashed  $S^7$  proceeds essentially along the same lines as in the case of the round  $S^7$  [2, 5], however, with some important technical differences. One expands all fields of the eleven-dimensional theory around the given (squashed) background solution in terms of eigenmodes of certain operators on the squashed  $S^7$  and then identifies the massless excitations as the zero modes of these operators. The fact that the theory under consideration has a residual  $N = 1$  supersymmetry suggests that the massless states can be grouped into  $N = 1$  supermultiplets, and this is indeed the case. Let us here already anticipate the final result, assuming that there exists a consistent truncation in which the eigenmodes which correspond to massive excitations can be discarded. The effective low-energy theory describes the interactions of an  $N = 1$  ( $2, \frac{3}{2}$ ) supergravity multiplet with an  $Sp_4 \times Sp_2$  ( $1, \frac{1}{2}$ ) vector multiplet, at least at the linearized level, and we expect the non-linear interactions to coincide with the ones that one would obtain from  $N = 1$  tensor calculus in the usual way [10]. In particular, there are *no* massless  $N = 1$  ( $\frac{1}{2}, 0^\pm$ ) chiral multiplets. This means that there is no way to break  $N = 1$  supersymmetry within the massless sector, and the theory which has both squashing and torsion cannot be interpreted as a spontaneously broken version of the theory with squashing and without torsion of [6].

It was pointed out in ref. [5] that, before analyzing the various differential operators on the seven-dimensional manifold (i.e. squashed  $S^7$  in our case), it is convenient to diagonalize the kinetic terms, i.e. the terms containing space-time derivatives. For the fermionic fields, this is accomplished by the redefinitions [11]\*

$$\psi_\mu(x, y) = \psi'_\mu(x, y) + \frac{1}{2} \gamma_5 \gamma_\mu \Gamma^\alpha \psi'_\alpha(x, y), \quad (4.1)$$

$$\psi_\alpha(x, y) = \psi'_\alpha(x, y), \quad (\Gamma_\alpha \text{ is } 8 \times 8). \quad (4.2)$$

For the bosonic fields, similar redefinitions are needed but since we will not need the explicit expressions here we refer the reader to refs. [2, 5] for more details.

One next inserts (4.1), (4.2) and the analogous expressions for the bosonic fields into the lagrangian of eleven-dimensional supergravity and expands it about the given background to second order in the fluctuations. In our case, the background is,

\* In accordance with our conventions in sect. 3, we will denote the internal seven-dimensional indices by  $\alpha, \beta, \gamma, \dots$  if they are curved and by  $a, b, c, \dots$  if they are flat.

of course, characterized by eq. (1.4) for the four-index field strength and by

$$g_{MN}(x, y) = \begin{pmatrix} g_{\mu\nu}^{(0)}(x) & 0 \\ 0 & g_{\alpha\beta}^{(0)}(y) \end{pmatrix} \tag{4.3}$$

for the metric in eleven dimensions. In (4.3),  $g_{\mu\nu}^{(0)}$  denotes the usual AdS background metric, while  $g_{\alpha\beta}^{(0)}(y)$  is the metric on the squashed  $S^7$ . In this way one obtains second-order differential operators for the bosonic fields and first-order differential operators for the fermionic fields, such that the mass spectrum of the theory is in one-to-one correspondence with the eigenvalues of these operators. For the spin-2, spin- $\frac{3}{2}$  and spin-1 excitations the massless modes are comparatively easy to identify; for spin- $\frac{1}{2}$ , the analysis is considerably more complicated and will be presented in detail below.

In accordance with the general theory, there will be only one massless graviton. We can therefore write

$$h'_{\mu\nu}(x, y) = h_{\mu\nu}(x) + \dots, \tag{4.4}$$

where  $h'_{\mu\nu}(x, y)$  denotes the fluctuation about the AdS background  $g_{\mu\nu}^{(0)}(x)$  (up to a Weyl-rescaling) and the dots in (4.4) stand for massive excitations. By  $N = 1$  supersymmetry, there must be a massless gravitino, i.e. the fermionic partner of  $h_{\mu\nu}(x)$ . Since, for the squashed  $S^7$  there is precisely one covariantly constant spinor, the gravitino of the four-dimensional theory is easily identified in the expansion of  $\psi'_\mu(x, y)$ , viz.

$$\psi'_\mu(x, y) = \psi_\mu(x)\eta(y) + \dots. \tag{4.5}$$

In conjunction with (1.9), this guarantees that

$$\delta\psi_\mu(x) = D_\mu \varepsilon(x) + \dots, \tag{4.6}$$

which is the expected transformation law. Explicit calculation confirms that  $\psi_\mu(x)$  is indeed massless\*.

As for spin-1, we again rely on the general theory which predicts as many massless spin-1 fields as there are Killing vectors on the internal manifold. The latter will therefore carry the same labels as the generators of the isometry group which, for the squashed  $S^7$ , is  $Sp_4 \times Sp_2$  [12]. The massless spin-1 fields are therefore given by the ansatz

$$h'_{\mu a}(x, y) = \sum_M A_\mu^{(M)}(x) K_{a(M)}(y) + \dots \tag{4.7}$$

\* Of course, this means that there is an *apparent* mass term for  $\psi_\mu$  whose coefficient is fixed by the requirement of masslessness in AdS space.

where  $M$  labels the generators of the group  $\text{Sp}_4 \times \text{Sp}_2$ . Again, masslessness of the fields  $A_\mu^{(M)}$  can be established by an explicit calculation.

Up to this point, the analysis has been rather straightforward. For the spin- $\frac{1}{2}$  sector, a more detailed investigation is required. We have already used  $N = 1$  supersymmetry to infer the existence of one massless gravitino from the existence of one massless graviton. In a similar fashion, one can argue that, by  $N = 1$  supersymmetry, there must be 13 massless spin- $\frac{1}{2}$  fields which pair up with the 13 spin-1 fields of (4.7) into  $N = 1$  vector multiplets. Denoting these spin- $\frac{1}{2}$  fields by  $\lambda^{(M)}(x)$ , we thus have

$$\psi'_\alpha(x, y) = \sum_M \phi_\alpha^{(M)}(y) \lambda_{(M)}(x) + \dots, \tag{4.8}$$

and it now remains to construct the corresponding zero modes  $\phi_\alpha^{(M)}(y)$ . Besides the masslessness condition to be discussed below, there is another constraint on the form of  $\phi_\alpha^{(M)}(y)$  which follows from the transformation law (1.9). Inserting (4.7) into the left-hand side of the elfbein transformation law  $\delta e_M^A = \frac{1}{2} \bar{\epsilon} \Gamma^A \psi_M$  and (4.1), (4.5) and (4.8) into its right-hand side, we find upon equating the resulting two expressions that

$$\delta A_\mu^{(M)}(x) K_\alpha^{(M)}(y) + \dots = -\frac{1}{4} (\bar{\eta} \Gamma_\alpha \Gamma^\beta \phi_\beta^{(M)}(y)) (\bar{\epsilon}(x) \gamma_5 \gamma_\mu \lambda^{(M)}(x)) + \dots. \tag{4.9}$$

We next observe that, owing to the constancy of  $\eta$ , any spinor (with arbitrary  $y$  dependence) can be expressed as a linear combination of  $\eta$  and  $\Gamma_\alpha \eta$  with  $y$  dependent coefficients. Specializing to  $\phi_\alpha^{(M)}$ , we recognize that

$$\phi_\alpha^{(M)}(y) = A_\alpha^{(M)}(y) \eta + B_{\alpha\beta}^{(M)}(y) \Gamma^\beta \eta, \tag{4.10}$$

which, upon insertion into (4.9), yields the condition

$$A_\alpha^{(M)}(y) + a_\alpha^{\beta\gamma} B_{\beta\gamma}^{(M)}(y) = \text{const. } K_\alpha^{(M)}(y). \tag{4.11}$$

Here, we used the fact that the octonionic structure constants  $a_{mnp}$  are given by [7]

$$a_{abc} = -i \bar{\eta} \Gamma_{abc} \eta. \tag{4.12}$$

Consequently, (4.11) imposes a restriction on  $\phi_\alpha^{(M)}(y)$  but clearly does not entirely determine  $\phi_\alpha^{(M)}(y)$ . To determine the zero modes, we have to analyze the spin- $\frac{1}{2}$ , spin- $\frac{1}{2}$  part of the fermionic mass operator which, after some calculation, is found to be

$$(M\phi)^\alpha = \delta^{(\alpha\beta} \Gamma^\gamma) D_\beta \phi_\gamma - \frac{1}{3} i \sqrt{2} (\Gamma^\alpha \Gamma^\beta - \frac{1}{4} \delta^{\alpha\beta}) \phi_\beta. \tag{4.13}$$

What are the possible ansätze that could satisfy  $M\phi = 0$ ? From (4.10), we know that there are two undetermined functions  $A_\alpha^{(M)}(y)$  and  $B_{\alpha\beta}^{(M)}(y)$ . Possible ansätze can be constructed out of the Killing vectors  $K_\alpha^{(M)}(y)$ , the octonion structure constants  $a_{abc}$  and their covariant derivatives as well as the Weyl tensor (which here does not vanish unlike for the round  $S^7$ ). For example, for  $A_\alpha^{(M)}$ , one may use  $K_\alpha^{(M)}$  itself or  $a_\alpha^{\beta\gamma}D_\beta K_\gamma^{(M)}$ . However, from sects. 2 and 3, we know that there is a relation

$$a_\alpha^{\beta\gamma}D_\beta K_\gamma^{(M)} = \frac{1}{3}\sqrt{2} K_\alpha^{(M)}, \tag{4.14}$$

which immediately eliminates the second possibility. Eq. (4.14) is really the crucial relation of this section: it turns out that if it did not hold, (4.13) could not have zero eigenmodes! Using the relation [13]

$$a_{abe}a^{cde} = 2\delta_{ab}^{cd} - \frac{1}{6}\epsilon_{ab}{}^{cdefg}a_{efg}, \tag{4.15}$$

we can invert (4.14)

$$D_\alpha K_\beta^{(M)} = a_{\alpha\beta\gamma}K^\gamma{}^{(M)} - \frac{1}{12}\epsilon_{\alpha\beta\gamma\delta\epsilon}{}^{\zeta\eta}a^{\gamma\delta\epsilon}D_\zeta K_\eta^{(M)}, \tag{4.16}$$

which shows that out of three possible ansätze for  $B_{\alpha\beta}^{(M)}$  present in (4.16), only two are independent. From (4.13), we recognize that if the ansatz contains the first derivative of  $K_\alpha^{(M)}$ , we must evaluate the second derivative on  $K_\alpha^{(M)}$ . This can be done by means of the well-known identity [14]

$$D^\beta D_\gamma K_\delta^{(M)} = R^{\alpha\beta}{}_{\gamma\delta} K_\alpha^{(M)} = (C^{\alpha\beta}{}_{\gamma\delta} - 2\delta_{\gamma\delta}^{\alpha\beta}) K_\alpha, \tag{4.17}$$

where we have made use of the fact that the curvature tensor on squashed  $S^7$  differs from that of the (maximally symmetric) round  $S^7$  only by the Weyl tensor  $C^{\alpha\beta}{}_{\gamma\delta}$  [7]. Furthermore, the calculation is facilitated by the relation

$$\alpha^{\alpha\beta\gamma}C_{\beta\gamma\delta\epsilon} = 0, \tag{4.18}$$

which follows from the relation [6]

$$C_{\alpha\beta}{}^{\gamma\delta}\Gamma_{\gamma\delta}\eta = 0, \tag{4.19}$$

by multiplication with  $\bar{\eta}\Gamma_r$  from the left. Eqs. (4.17) and (4.18) suffice to completely eliminate the Weyl tensor whenever it occurs in the actual calculation of  $\phi_m^{(M)}$ . Finally, we need the following properties of the quantity  $a_{\alpha\beta\gamma}$ , namely

$$D_\alpha a_{\beta\gamma\delta} = \frac{1}{6}\epsilon_{\alpha\beta\gamma\delta\epsilon\zeta\eta}a^{\epsilon\zeta\eta}(-ic), \tag{4.20}$$

which follows directly from the fact that  $\eta$  is a covariantly constant spinor and the formula (4.12), and the identity

$$a_{[\alpha\beta\gamma}a_{\delta]\epsilon\xi} = -\frac{1}{4}\epsilon_{\alpha\beta\gamma\delta ab[\epsilon}a_{\xi]}^{ab}, \quad (4.21)$$

which follows by use of Fierz identities for spinors in seven dimensions [13].

The final result for the zero modes  $\phi_m^{(M)}$  of the mass operator (4.16) is found to be

$$\phi_\alpha^{(M)}(y) = \eta K_\alpha^{(M)}(y) - \frac{1}{2}i\Gamma^\beta\eta\left(D_\alpha K_\beta^{(M)}(y) - \frac{1}{24}\epsilon_{\alpha\beta\gamma\delta\epsilon}{}^\xi\eta a^{\gamma\delta\epsilon}D_\xi K_\eta^{(M)}(y)\right), \quad (4.22)$$

after some calculation where all the relations given above are needed. Clearly, there is the required number of massless spin- $\frac{1}{2}$  fields in the adjoint representation of  $\text{Sp}_4 \times \text{Sp}_2$ , and the relation (4.14) ensures that the constraint (4.11) is also satisfied. Hence putting everything together, we get

$$\delta A_\mu^{(M)}(x) \sim \bar{\epsilon}(x)\gamma^5\gamma_\mu\lambda^{(M)}(x), \quad (4.23)$$

which is the correct transformation law.

To see that there are no further massless excitations we note that these would have to be present in the form of chiral  $(\frac{1}{2}, 0^\pm)$  multiplets. The pseudoscalars would have to be found in the expansion of the three-index field of the eleven-dimensional theory in the form

$$A_{\alpha\beta\gamma}(x, y) = i\bar{\eta}\Gamma_{\alpha\beta\gamma}\eta B(x) + \dots, \quad (4.24)$$

in complete analogy with the case of the round  $S^7$  [2, 5]. However, the  $\eta$  which appears in (4.24) has the wrong parity and the spinor of opposite parity is not available in the case of squashed  $S^7$ , in contrast to the case of the round  $S^7$ , as we pointed out already in the introduction\*. Hence, the field  $B(x)$  in (4.24) is a *massive* excitation. From the absence of massless pseudoscalar excitations we infer the absence of massless scalar as well as further massless spin- $\frac{1}{2}$  excitations by  $N = 1$  supersymmetry (the latter statement is confirmed by an explicit calculation). This spares us the trouble of going through a detailed analysis of the scalar spin-0 sector (which, in the case of round  $S^7$ , is the most tedious).

Furthermore, one can easily see that supergravity obtained by compactification on the squashed  $S^7$  *cannot* be interpreted as a spontaneously broken version of  $N = 8$  supergravity. From the constancy of the covariantly constant spinor  $\eta$ , it follows that  $\eta$  cannot be a linear combination with constant coefficients of the eight  $\eta^I(y)$  of the

\* Remember that the parity of  $\eta$  is unambiguously fixed by the requirement of supersymmetry, i.e. eq. (1.10).



round  $S_7$ , because the latter are not constant, i.e.

$$\eta \neq \sum a^I \eta^I(y), \quad \text{with constant } a^I, \quad (4.25)$$

and therefore the eigenmode expansion of  $\eta$  will involve higher modes on the round  $S^7$ . If the zero mass sector of the squashed  $S^7$  were to lead to a spontaneously broken version of  $N = 8$  supergravity, the remaining supersymmetry would have to be a linear combination of the eight supersymmetries of  $N = 8$  supergravity which, from (4.25), it manifestly is not. We have seen that this conclusion is confirmed by the detailed analysis of the low-energy spectrum of the squashed solution. In terms of the original expansion into  $SO(8)$  covariant eigenmodes, this means the compactification on the squashed  $S^7$  shifts the zero modes from the sector which is spanned by the eight  $\eta^I$ 's into another sector spanned by the higher modes\*. Whether the *original* sector is interpretable as a spontaneously broken version of  $N = 8$  supergravity where now *all* eight supersymmetries are broken and whether it corresponds also to a consistent truncation is, at present, an open question (see ref. [13] for a more detailed discussion of this problem).

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#### Note added

It has recently been shown that the  $G_2$  symmetry of the parallelized solution [3] can be enlarged to  $SO(7)$  [15, 13, 16, 17].

#### Note added in proof:

After this paper was submitted for publication we learnt of ref. [18] which contains work along similar lines. The authors of [18] have identified yet another solution with squashing and without torsion but *without* supersymmetry. They have also noted the occurrence of "level crossing" for the squashed solution, but they have not derived the zero modes.

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\* In this case, the squashed solution would be the first example where "level-crossing" in the sense of ref. [5] occurs.

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