ACHIEVING EFFECTIVE POSITIVE ACTION IN SUPERGRAVITY

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Received 3 April 1981

The path integral for the gravity sector of supergravity, i.e., after integration over the spin-3/2 field, is formally shown to be governed by the linearized Einstein action in terms of redefined field variables. The exponential of the action now being a product of gaussians, positivity of the action in euclidean signature is simply achieved by analytic continuation in those field components whose squares initially appear with the wrong sign.

Path integral quantization of gravity encounters the problem that in euclidean signature, the Einstein action is not positive. Because of its nonlinearity, rather complicated prescriptions have been devised to make the integration well defined [1]. Since supergravity has led to a number of other improvements over the Einstein theory, a natural question is whether the effective quantum-gravity sector defined by integrating out the spin-3/2 field there is governed by a more amenable action. We shall show that this is indeed the case as a consequence of an a priori even more surprising result: formally, this action is that of the free Einstein field in first-order form, quadratic in a redefined set of variables. Although it still contains negative terms, these occur as squares of field variables and become positive through the usual analytic continuation prescription for a gaussian integral with the wrong sign in the exponent. Our basic result is a consequence of supersymmetry, and we begin with a brief review of its antecedents. We will work in the euclidean signature throughout, which is, of course, perfectly compatible with supersymmetry [2].

Globally supersymmetric theories have the remarkable property [3] that their bosonic fields may be invertibly expanded in terms of free fields and that the Jacobi determinant of this transformation equals the MSS determinant [4] obtained upon integrating out the fermionic modes [5] in the over-all functional integral. More precisely, one may reexpress bosonic expectation values of an operator in the interacting theory by free field expectation values according to

$$\begin{aligned}\langle F(A)\rangle_\lambda &\equiv \int F(A)\,\mathrm{e}^{-S(\lambda;A)}D(\lambda;A)\,\mathrm{d}A \\ &= \int F(A(\lambda;A'))\,\mathrm{e}^{-S_0(A')}\mathrm{d}A' \\ &= \langle F(A(\lambda;A'))\rangle_0,\end{aligned} \tag{1}$$

where, as in ref. [3] whose notation we follow, (A, λ) stand for all bosonic fields and coupling constants of the theory under consideration, $S(\lambda, A)[S_0(A)]$ for the full [free] bosonic action integral and $D(\lambda, A)$ for the MSS determinant. Formula (1) is based on the fact that the transformed fields obey

$$S_0(A'(\lambda;A)) = S(\lambda;A) \tag{2}$$

and

$$\det[\delta A'(x;\lambda,A)/\delta A(y)] = D(\lambda;A) \tag{3}$$

for any value of λ. Of course, the theory is still nontrivial since the effects of the interaction now reside in the transformation and thereby in the complicated form of $F(A(\lambda, A'))$. This result is quite formal; it requires only that the vacuum functional Z be λ independent and that the original action be quadratic in the fermionic fields. As pointed out by Zumino [6], the first criterion is always met if the supersymmetry is

[1] Supported in part by NSF Grant PHY-78-09644A01.

0 031–9163/81/0000–0000/$ 02.50 © North-Holland Publishing Company

not spontaneously broken.

The procedure outlined above is also applicable in the locally supersymmetric framework of supergravity [7,8]. Formally, the vacuum functional in supergravity is constant because, at least for asymptotically flat spaces, the energy is the square of the spinor charge and the flat space (vacuum) energy vanishes [9]. (Classical gravitational energy is also positive and vanishes at flat space, but we do not expect this to persist for quantum gravity. We also mention that supergravity with a cosmological term, although supersymmetric, would not have constant Z because the cosmological constant gets multiplicatively renormalized.) The requirement that the action be quadratic in the fermion is automatically satisfied for $N = 1$ in first-order formalism [8], where the vierbein $e_{\mu a}$ and the connection $\omega_{\mu ab}$ are treated as independent fields. For higher N, it is not clear whether all quartic terms can always be removed by appropriate variants of first-order form.

There will be several ghosts coming from the various local invariances of the theory. The fermionic ghosts, due to general coordinate invariance and local SO(4) invariance, contribute to the fermionic determinant just as in the case of supersymmetric Yang–Mills theories [3]. In addition, local supersymmetry requires bosonic ghosts which would, in general, contribute to the bosonic part of the action. To avoid this we choose a flat space gauge-fixing term proportional to

$$\mathcal{L}_{\mathrm{gf}} \sim (\delta^{a\mu}\bar{\psi}_\mu\gamma_a)\gamma_b\partial_b(\delta^{c\nu}\gamma_c\psi_\nu). \tag{4}$$

The associated bosonic ghosts decouple from the physical sector of the theory and hence may be discarded. This is purely a matter of convenience as the path integral is gauge-invariant.

Having satisfied its assumptions, we can now take over the results of ref. [3] : there exists a non-linear and non-local transformation of the bosonic fields [‡1]

$$e_{\mu a} \to e'_{\mu a}(e, \omega), \quad \omega_{\mu ab} \to \omega'_{\mu ab}(e, \omega), \tag{5}$$

such that the functional measure of the full theory is mapped into a free measure (we set $\kappa = 1$)

$$\exp\left(\int \hat{R}(e, \omega)\, \mathrm{d}^4x\right) D(e, \omega)\, \mathrm{d}e\, \mathrm{d}\omega$$
$$= \exp\left(\int \hat{R}_0(e'(e, \omega), \omega'(e, \omega))\, \mathrm{d}^4x\right) \tag{6}$$
$$\times\, \mathrm{d}e'(e, \omega)\, \mathrm{d}\omega'(e, \omega).$$

Here, $\hat{R}$ denotes the sum of the usual scalar curvature density R and its associated gauge-fixing term f^2; $\hat{R}_0$, R_0 and f_0 represent the corresponding quantities of the free massless spin-2 theory; we have neglected the gravitational boundary term. Note that in pure quantum gravity the fermionic spin-3/2 determinant, which is essential for the cancellation of the Jacobi determinant of the field transformation, is absent. The action then acquires an extra part proportional to the logarithm of this determinant and is no longer gaussian. Thus, although the transformation still exists, the considerations below are not applicable there.

This linearization is perhaps less surprising in view of the fundamental result of Morse theory [10] which states that any function can be written as a quadratic form in some open neighbourhood of each extremum if $\det f'' \neq 0$. The transformation leading to the quadratic form is not unique; the remarkable property of supersymmetric theories is that, for them, there exists one such reparametrization whose Jacobi determinant cancels the MSS determinant.

At this point, it would be convenient to integrate out the connection ω' in order to reach the usual second-order form of the free action in terms of e'. However, this cannot be done in general since even the expectation value of functions of $e_{\mu a}$ alone will involve both e' and ω'. This is the price of having to use the first-order form, and is traceable to the presence of torsion in supergravity. Nevertheless, the action $\int \hat{R}_0(e', \omega')$ is still quadratic which is sufficient for our purpose. From its definition it can be written as

$$\int R_0(e', \omega')\, \mathrm{d}^4x = \int (\bar{\omega}_{\mu\nu a}\bar{\omega}_{\nu\mu a} - \bar{\omega}_a\bar{\omega}_a)\, \mathrm{d}^4x$$
$$+ \int [-h_{\alpha\beta,\mu}h_{\alpha\beta,\mu} + \tfrac{1}{2}h_{\alpha\alpha,\mu}h_{\alpha\alpha,\mu} + f^2(h)]\, \mathrm{d}^4x. \tag{7}$$

Here, $\bar{\omega} \equiv \omega' - \omega'(e')$, $\bar{\omega}_\mu \equiv \bar{\omega}_{a a\mu}$ and $f^2(h)$ is equal to, and therefore cancelled by, the usual harmonic gauge choice

[‡1] Throughout this paper we expand about a flat-space background. However, it is conceivable that the results also extend to curved backgrounds.

$$f^2(h) \equiv 2(h_{\mu\alpha,\alpha} - \tfrac{1}{2}h_{\alpha\alpha,\mu})^2, \tag{8}$$

and we have used the fact that, in second-order form, only the symmetric field $h_{\mu\nu} \equiv e_{\mu\nu} + e_{\nu\mu}$ propagates. There are negative terms of two types. The first are those in the non-propagating $\bar{\omega}$ part and would in any case also be present if we could freely carry out the $\bar{\omega}$ integration. Defining $2\omega^{\pm}_{\mu\nu} \equiv \bar{\omega}_{\mu\nu c} \pm \bar{\omega}_{\nu\mu c}$, it is clear that $-\omega^2_{\mp}$ is the negative term there. The second negative term comes from the $(\partial_\mu h_{\alpha\alpha})^2$ trace mode, corresponding to the conformal one in Einstein gravity, which cannot be removed by any gauge choice.

We observe that the various terms in the action can be written as squares of independent irreducible parts (e.g., the traceless $\tilde{h}_{\mu\nu}$ and $h_{\alpha\alpha}$) of the variables. Thus, the present theory, although inequivalent to pure gravity, also has a negative action problem. However, it is easy to provide a mathematically acceptable solution in this case. The analyticity properties of the gaussian suggest that one simply rotates the integration contour and makes the redefinitions $h_{\alpha\alpha} \to \mathrm{i}h_{\alpha\alpha}$, etc., as advocated by the Cambridge school [1]. In this way, the exponent of the action becomes negative while all expectation values remain real, any odd powers vanishing upon integration. Also, since the rotations are carried out on irreducible (with respect to the flat background metric) parts of the field, they will not spoil covariance. We mention in this context that even in euclideanized globally supersymmetric models, the auxiliary fields enter with the wrong sign and must be dealt with in a similar fashion [2], being akin in this respect to the "auxiliary" $\bar{\omega}$ fields.

In view of the formal nature of the above considerations, we now exhibit a transformation which linearizes the Einstein action to cubic order. For simplicity, we proceed in second-order metric form and use the linearized gauge-fixing term (8) with no cubic contribution. Then, it is to be shown that starting from the free gauge-fixed action

$$\int \hat{R}_0(h')\,\mathrm{d}^4x = \int h'_{\mu\nu}(-\Delta)(h'_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}h')\,\mathrm{d}^4x, \tag{9}$$

there exists a transformation $h'_{\mu\nu} = h_{\mu\nu} + H_{\mu\nu}(h)$ such that its insertion in (9) yields the gauge-fixed Einstein action through cubic order. It suffices to know that the cubic terms can all be schematically written in the form $2\int(h_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}h)[\partial h\cdot\partial h]_{\mu\nu}\mathrm{d}x$. Since

$$\int \hat{R}_0(h'(h))\,\mathrm{d}^4x = \int \hat{R}_0(h)\,\mathrm{d}^4x + 2\int H_{\mu\nu}(h)(-\Delta)(h_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}h)\,\mathrm{d}^4x + \mathrm{O}(h^4), \tag{10}$$

it follows that $H_{\mu\nu}$ equals $\int C(x-y)[\partial h\cdot\partial h]_{\mu\nu}(y)\,\mathrm{d}y$ where C is the inverse of $-\Delta$, i.e., the usual propagator. This defines the desired transformation. Its Jacobi determinant to $\mathrm{O}(h)$ has the generic form (dropping space–time indices)

$$\det(\delta h'/\delta h) = \exp[\mathrm{tr}\log(\delta h'/\delta h)] = 1 + \int C(x-y)\partial_\mu h(y)(\partial/\partial x^\mu)\delta(y-x)\,\mathrm{d}^4x\,\mathrm{d}^4y + \mathrm{O}(h^2). \tag{11}$$

The integral vanishes because $\partial_\mu C(0) = 0$ in any reasonable regularization scheme. The MSS determinant for our special gauge choice (4) becomes ($S \equiv \not{\partial} C$)

$$\left[\det\left(\delta^{\mu\nu}\delta(x-y) + \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}\int \mathrm{d}^4z\,S(x-z) \times [\gamma_5\gamma_a h_{\alpha a}(z)\partial_\beta + \gamma_5\gamma_\alpha\sigma^{ab}\partial_a h_{\beta b}(z)]\,\delta(z-y)\right)\right]^{1/2}$$

and it, too, equals unity to first order in h. We are fully aware of the fact that this agreement to lowest order does not provide a very strong check on our theorem. However, the algorithm given in (3) in principle allows the construction of the transformation to all orders.

While our procedure is clearly valid to arbitrary κ and loop order [since these enter only through $F(e(e', \omega'))$ and not in the free action], it is highly formal and as such open to mathematical objections. The most important is perhaps that we really know nothing about the behaviour of the transformation which turns the interacting measure into a free measure. However, at least in the global case, there is some reason to believe that this transformation is well behaved in the sense that the transformed fields do not increase exponentially as functions of the non-transformed fields. For, if in eq. (2), we scale the field A' by a factor t, the left-hand side grows as t^2 which, by eq. (2), dictates the scaling behaviour of $A(tA')$. If the highest power of A in $S(A)$ is p, then we must have

$$A(tA') = \mathrm{O}(t^{2/p}) \quad \text{as } t\to\infty \text{ for fixed } A', \tag{13}$$

which suggests that products of $A(\lambda, A')$ are integrable with respect to the gaussian measure $d\mu_0(A')$. A similar argument may be given in the first-order formulation of gravity. In spite of the frailty of such arguments, we view our result as a further indication that the difficulties of quantizing gravity may be overcome in the wider framework of supergravity.

We thank B. Zumino for insisting on further clarification of an earlier version.

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