$N = 3$ SUPERSYMMETRY MULTIPLETS WITH VANISHING TRACE ANOMALY: BUILDING BLOCKS OF THE $N > 3$ SUPERGRAVITIES

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$N \geqslant 3$ supergravity theories with vanishing one-loop trace anomaly may be constructed from three basic $N = 3$ multiplets, one of which contains an antisymmetric tensor gauge field. As an example we construct the $N = 4$ theory and discuss its relationship to ten-dimensional supergravity.

1. $N = 3$ building blocks. In quantum supergravity the only one-loop counter-term that cannot be absorbed by a field redefinition is the Gauss–Bonnet invariant, whose integral gives the Euler number of the space–time manifold, χ

$$\chi = (32\pi^2)^{-1} \int d^4x \sqrt{g}\, {}^*R^*_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}. \tag{1.1}$$

Thus, in dimensional regularization where $n = 4 + \epsilon$ is the dimension of space–time, the one-loop counter-term may be written as

$$\Delta S = -(1/\epsilon) A \chi. \tag{1.2}$$

The contribution to the coefficient A may be calculated for each spin, and the results are tabulated in a paper of Christensen and Duff [1]. The conclusion is that A vanishes for the $N = 4$ Yang–Mills theory and the $N = 3$ supergravity, but not otherwise. However, Duff and van Nieuwenhuizen [2] have recently pointed out that these numbers depend not only on the spin, but also on the field representation. In particular, antisymmetric tensor gauge fields are inequivalent to scalars in that they contribute differently to A. Siegel [3] has used this observation to remark that the field representation of the $N = 8$ model which is given naturally by dimensional reduction of eleven-dimensional supergravity [4] is just such that the coefficient A vanishes! A is also the coefficient of the one-loop correction to the trace of the energy momentum tensor, and so the vanishing of A is also referred to as the vanishing of the trace anomaly.

As it stands, it is something of a mystery why the $N = 3$ and $N = 8$ supergravity theories should be singled out in this way. The purpose of this article is to dispel this mystery by showing that supergravity theories with $N \geqslant 3$ may all be constructed (in principle) so as to have vanishing trace anomaly [‡1]. We start from multiplets of $N = 3$ supersymmetry which can have, as highest spin, 2, 3/2, and 1. We choose the field representation to be

$$3(2) = (e_{a\mu}; \psi_\mu{}^i, A_\mu{}^i; \chi), \quad 3(3/2) = (\psi_\mu; A_\mu{}^i; \chi^i, A_{\mu\nu}, A), \quad 3(1) = (A_\mu; \lambda^i, \lambda; A^i, B^i), \tag{1.3}$$

where the notation $N(s)$ means a multiplet of N extended supersymmetry with maximum spin s. The index i runs over three values. Remarkably, the spin and field content of these multiplets is such that the corresponding lagrangian has vanishing trace anomaly, according to the results of refs. [1,2]. This is the lowest value of N for which

‡1 The general criterian for this to happen has been given in ref. [5].

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this is possible and so $N = 3$ supersymmetry is special in this respect. The supergravity theories for $N = 4$ to 8 constructed from these basic $N = 3$ multiplets will also have vanishing trace anomaly. The feature which makes this possible is the antisymmetric tensor gauge field $A_{\mu\nu}$ in 3(3/2) in place of the usual pseudoscalar. Therefore, we are led to conclude [5] that $N = 4$ supergravity should have one antisymmetric tensor, $N = 5$ two, $N = 6$ three and $N = 8$ ($\equiv N = 7$) five. (For the contribution to the A coefficient five antisymmetric tensors and 65 scalars are equivalent to seven antisymmetric tensors, 63 scalars and one $A_{\mu\nu\rho}$ field.)

Rather than construct extended supergravity theories from $N = 3$ multiplets, let us first construct a similar set of basic $N = 4$ multiplets as follows:

$$4(2) = 3(2) + 3(3/2), \quad 4(3/2) = 3(3/2) + 3(1), \quad 4(1) = 3(1). \tag{1.4}$$

The last of these follows because the $N = 3$ Maxwell multiplet has the same field content as the $N = 4$ Maxwell multiplet. Using the notation $(m, n, p)_4$ for a composite multiplet with m 4(2)'s, n 4(3/2)'s and p 4(1)'s, the multiplets for $N = 5$, 6 and 8 supergravity can be written as

$$5(2) = (1,\ 1,\ 0)_4, \quad 6(2) = (1,\ 2,\ 1)_4, \quad 7(2) = 8(2) = (1,\ 4,\ 6)_4. \tag{1.5}$$

This pattern is familiar from the construction of extended Maxwell theories from the basic fields. Here, the $N =$ 4 multiplets play the role of the basic "fields" for the construction of the remaining higher N theories.

So far, this analysis is purely speculative. One must still show that these versions of the $N \geqslant 4$ supergravities can actually be constructed. In the following we will construct the $N = 4$ theory.

2. *$N = 4$ supergravity with vanishing trace anomaly*. There are two versions of the $N = 4$ supergravity theory. One has an off-shell O(4) and an on-shell SU(4) global symmetry [6], while the other has a global off-shell SU(4) symmetry [7]. Both versions contain a scalar field ϕ and a pseudoscalar field B, but only in the latter version does B appear in the lagrangian only through $\partial_\mu B$. Our strategy [‡2] will be to construct a version of $N = 4$ supergravity with one antisymmetric tensor gauge field by making a duality transformation on B. This is only possible if B appears only through $\partial_\mu B$ and so our starting point is the lagrangian of ref. [7]. Duality transformations are standard (for example, see ref. [4]), but we will repeat the description of the steps involved in order to make it clear how it is that supersymmetry is preserved at each step of the process.

Wherever we see $\partial_\mu B$ in the lagrangian or transformation rules we will write $\frac{1}{2}L_\mu$ and then we will add to the lagrangian the term

$$\tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}M_{\mu\nu}\partial_\rho L_\sigma = \mathcal{L}_{\text{constraint}}, \tag{2.1}$$

i.e., the field $M_{\mu\nu}$ imposes the constraint that $\partial_\rho L_\sigma - \partial_\sigma L_\rho = 0$ which implies (in topologically trivial space–time) that $\frac{1}{2}L_\sigma = \partial_\sigma B$. (The factor of $\frac{1}{2}$ is for later convenience of normalization.) The new field L_μ is taken to transform exactly as $2\partial_\mu B$ did previously. It follows from this, that the original lagrangian is invariant up to terms containing the factor $\partial_\mu L_\nu - \partial_\nu L_\mu$. This is so because $\partial\mathcal{L}$ must vanish when $L_\mu = 2\partial_\mu B$. Therefore, those contributions to $\partial\mathcal{L}$ that do not cancel among themselves can be cancelled by a variation of $M_{\mu\nu}$ in $\mathcal{L}_{\text{constraint}}$. The variation of L_μ in $\mathcal{L}_{\text{constraint}}$ gives no further contribution to $\partial\mathcal{L}$ because it transforms as a total derivative. We now have an intermediate form of the theory in which L_μ and $M_{\mu\nu}$ are independent fields. We pass to the final form by eliminating L_μ by means of its algebraic equation of motion. The final transformation rules are obtained by substituting the result for L_μ into the intermediate rules. The points we wish to emphasize are that there is no need to guess the final transformation laws and that supersymmetry is *manifestly* preserved by the duality transformation.

By starting from the lagrangian of ref. [7] and following the above procedure we are led to the following new lagrangian [also in (1.5) formalism]:

[‡2] This idea is originally due to E. Cremmer. See also ref. [5].

$$
\begin{aligned}
\mathcal{L} = & -(4\kappa^2)^{-1} VR - \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu{}^i\gamma_5\gamma_\nu D_\rho\psi_\sigma{}^i + \tfrac{1}{2}\mathrm{i}V\bar{\chi}^i \not{D}\chi^i + \tfrac{1}{2}Vg^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi \\
& - \tfrac{1}{8}V\exp(-4\kappa\phi)\{[\partial_\nu M_{\rho\sigma} - \kappa(\boldsymbol{A}_\nu\cdot\boldsymbol{A}_{\rho\sigma} + \boldsymbol{B}_\nu\cdot\boldsymbol{B}_{\rho\sigma})]\,\epsilon^{\mu\nu\rho\sigma}\}^2 - \tfrac{1}{4}V\exp(-2\kappa\phi)(\boldsymbol{A}_{\mu\nu}\cdot\boldsymbol{A}^{\mu\nu} + \boldsymbol{B}_{\mu\nu}\cdot\boldsymbol{B}^{\mu\nu}) \\
& - [\partial_\nu M_{\rho\sigma} - \kappa(\boldsymbol{A}_\nu\cdot\boldsymbol{A}_{\rho\sigma} + \boldsymbol{B}_\nu\cdot\boldsymbol{B}_{\rho\sigma})]\,V\exp(-2\kappa\phi) \\
& \times \epsilon^{\mu\nu\rho\sigma}[-(\mathrm{i}/\sqrt{2})\,\bar{\chi}^i\gamma_5\gamma^\nu\gamma_\mu\psi_\nu{}^i - \tfrac{3}{8}\,\bar{\chi}^i\gamma_5\gamma_\mu\chi^i - \tfrac{1}{8}\mathrm{i}\epsilon^{\mu'\alpha\beta\gamma}\bar{\psi}_\alpha{}^i\gamma_\beta\psi_\gamma{}^i g_{\mu\mu'} \\
& - \tfrac{1}{2}\mathrm{i}\kappa V\exp(-\kappa\phi)\bar{\psi}_\rho{}^i C^{\mu\nu}{}_{ij}\sigma_{\mu\nu}\gamma^\rho\chi^j + (\kappa/2\sqrt{2})V\exp(-\kappa\phi)[\bar{\psi}_\mu{}^i(C^{\mu\nu} + \hat{C}^{\mu\nu})_{ij}\psi_\nu{}^j - \mathrm{i}\,\bar{\psi}_\mu{}^i\gamma_5(\tilde{C}^{\mu\nu} + \tilde{\hat{C}}^{\mu\nu})_{ij}\psi_\nu{}^j] \\
& - (\kappa/\sqrt{2})V\bar{\psi}_\mu{}^i\gamma^\nu\gamma^\mu\chi^i\partial_\nu\phi - \tfrac{1}{4}\kappa^2 V[(\bar{\chi}^i\gamma_5\gamma^\nu\gamma^\mu\psi_\nu{}^i)(\bar{\psi}_\mu{}^j\gamma_5\chi^j) - (\bar{\chi}^i\gamma^\nu\gamma^\mu\psi_\nu{}^i)(\bar{\psi}_\mu{}^j\chi^j)] \\
& - 2\kappa^2 V[-(\mathrm{i}/2\sqrt{2})\,\bar{\chi}^i\gamma_5\gamma^\nu\gamma^\mu\psi_\nu{}^i - \tfrac{3}{8}\,\bar{\chi}^i\gamma_5\gamma^\mu\chi^i - \tfrac{1}{8}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\nu{}^i\gamma_\rho\psi_\sigma{}^i]^2 . \qquad (2.2)
\end{aligned}
$$

The conventions are those of Cremmer et al. [7]. The index i runs over 1 to 4; $\boldsymbol{A}_{\mu\nu} = (A^n_{\mu\nu};\, n = 1,2,3)$ is the field strength of $\boldsymbol{A}_\mu$, and $C^{\mu\nu}{}_{ij} = \alpha^n{}_{ij}A^{n\mu\nu} + \beta^n{}_{ij}B^{n\mu\nu}\mathrm{i}\gamma_5$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the matrices defined in ref. [7]. The transformation rules are

$$
\begin{aligned}
& \delta\phi = (1/\sqrt{2})\bar{\epsilon}^i\chi^i, \quad \delta V_\mu{}^a = -\mathrm{i}\kappa\bar{\epsilon}^i\gamma^a\psi_\mu{}^i, \\
& \delta\boldsymbol{A}_\mu = (1/\sqrt{2})\exp(\kappa\phi)[\bar{\epsilon}^i\boldsymbol{\alpha}_{ij}\psi_\mu{}^j + (\mathrm{i}/\sqrt{2})\bar{\epsilon}^i\boldsymbol{\alpha}_{ij}\gamma_\mu\chi^j], \\
& \delta\boldsymbol{B}_\mu = (\mathrm{i}/\sqrt{2})\exp(\kappa\phi)[\bar{\epsilon}^i\boldsymbol{\beta}_{ij}\gamma_5\psi_\mu{}^j + (\mathrm{i}/\sqrt{2})\bar{\epsilon}^i\boldsymbol{\beta}_{ij}\gamma_5\gamma_\mu\chi^j], \\
& \delta M_{\mu\nu} = [\sqrt{2}\bar{\epsilon}^i\sigma_{\mu\nu}\chi^i - \tfrac{1}{2}\mathrm{i}\,\bar{\epsilon}^i(\gamma_\mu\psi_\nu{}^i - \gamma_\nu\psi_\mu{}^i)]\exp(2\kappa\phi) \\
& \quad + \{(\kappa/\sqrt{2})\exp(\kappa\phi)[(\bar{\epsilon}^i\boldsymbol{\alpha}_{ij}\psi_\mu{}^j + (\mathrm{i}/\sqrt{2})\bar{\epsilon}^i\boldsymbol{\alpha}_{ij}\gamma_\mu\chi^j)\cdot\boldsymbol{A}_\nu + (\mathrm{i}\bar{\epsilon}^i\boldsymbol{\beta}_{ij}\gamma_5\psi_\mu{}^j - (1/\sqrt{2})\bar{\epsilon}^i\boldsymbol{\beta}_{ij}\gamma_5\gamma_\mu\chi^j)\cdot\boldsymbol{B}_\nu] - (\mu \leftrightarrow \nu)\}, \\
& \delta\bar{\chi}^i = (\mathrm{i}/\sqrt{2})\bar{\epsilon}^i[\hat{D}_\mu\phi + \tfrac{1}{2}\mathrm{i}\gamma_5\exp(2\kappa\phi)\hat{L}_\mu]\gamma^\mu - \tfrac{1}{2}\exp(-\kappa\phi)(\bar{\epsilon}C^{\alpha\beta}\sigma_{\alpha\beta})^i - (3/2\sqrt{2})\kappa(\bar{\epsilon}^j\gamma_5\chi^j)\bar{\chi}^i\gamma_5, \\
& \delta\bar{\psi}_\mu{}^i = \kappa^{-1}\bar{\epsilon}^i\overleftarrow{D}_\mu - \tfrac{1}{4}\mathrm{i}\exp(2\kappa\phi)\bar{\epsilon}^i\gamma_5\hat{L}_\mu - (\mathrm{i}/2\sqrt{2})\exp(-\kappa\phi)(\bar{\epsilon}C^{\alpha\beta})^i\gamma_\mu\sigma_{\alpha\beta} \\
& \quad - (\kappa/2\sqrt{2})[(\bar{\epsilon}^j\gamma_5\chi^j)\bar{\psi}_\mu{}^i - (\bar{\psi}_\mu{}^j\gamma_5\chi^j)\bar{\epsilon}^i]\gamma_5 + (\kappa/\sqrt{2})\epsilon^{ijkl}[(\bar{\epsilon}^k\psi_\mu{}^j)\bar{\chi}^l - (\bar{\epsilon}^k\gamma_5\psi_\mu{}^j)\bar{\chi}^l\gamma_5] \\
& \quad + \tfrac{1}{4}\mathrm{i}\kappa[(\bar{\chi}^j\gamma_5\gamma^\rho\chi^j)\bar{\epsilon}^i\gamma_5 + (\bar{\chi}^i\gamma^\rho\chi^j)\bar{\epsilon}^j - (\bar{\chi}^i\gamma_5\gamma^\rho\chi^j)\bar{\epsilon}^j\gamma_5]\gamma_\mu\gamma_\rho . \qquad (2.3)
\end{aligned}
$$

$\hat{L}_\mu$ is the following supercovariant quantity:

$$
\begin{aligned}
\hat{L}^\mu = & \ \epsilon^{\mu\nu\rho\sigma}[\partial_\nu M_{\rho\sigma} - \kappa(\boldsymbol{A}_\nu\cdot\boldsymbol{A}_{\rho\sigma} + \boldsymbol{B}_\nu\cdot\boldsymbol{B}_{\rho\sigma})]\exp(-4\kappa\phi) \\
& + [\tfrac{3}{2}\kappa\,\bar{\chi}^i\gamma_5\gamma^\mu\chi^i + 2\sqrt{2}\,\mathrm{i}\kappa\,\bar{\chi}^i\gamma_5\sigma^{\nu\mu}\psi_\nu{}^i + \tfrac{1}{2}\mathrm{i}\kappa\,\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\nu{}^i\gamma_\rho\psi_\sigma{}^i]\exp(-2\kappa\phi). \qquad (2.4)
\end{aligned}
$$

The antisymmetric tensor field $M_{\mu\nu}$ appears only through the following quantity

$$G^\mu = \epsilon^{\mu\nu\rho\sigma}[\partial_\nu M_{\rho\sigma} - \kappa(\boldsymbol{A}_\nu\cdot\boldsymbol{A}_{\rho\sigma} + \boldsymbol{B}_\nu\cdot\boldsymbol{B}_{\rho\sigma})], \qquad (2.5)$$

which is not only invariant with respect to $\partial M_{\mu\nu} = \partial_\mu\xi_\nu - \partial_\nu\xi_\mu$, but also with respect to the transformations

$$\delta\boldsymbol{A}_\mu = \partial_\mu\boldsymbol{\lambda}, \quad \delta\boldsymbol{B}_\mu = \partial_\mu\boldsymbol{\rho}, \quad \delta M_{\mu\nu} = \kappa(\boldsymbol{A}_\mu\cdot\partial_\nu\boldsymbol{\lambda} + \boldsymbol{B}_\mu\cdot\partial_\nu\boldsymbol{\rho}) - (\mu \leftrightarrow \nu), \qquad (2.6)$$

which generalize the usual gauge transformations of $\boldsymbol{A}_\mu$ and $\boldsymbol{B}_\mu$. The transformations (2.6) constitute an invariance of the action that ensures that the non-physical degrees of freedom in $\boldsymbol{A}_\mu$ and $\boldsymbol{B}_\mu$ decouple. The unusual form of this symmetry will mean that the ghosts required for the quantization of $M_{\mu\nu}$ will mix with those required for $\boldsymbol{A}_\mu$ and $\boldsymbol{B}_\mu$. One can check that the symmetry of (2.6) commutes with supersymmetry and that it also appears in the commutator of two supersymmetry transformations.

3. Comments. It is interesting to note that just as the $N = 8$ supergravity can be obtained by dimensional reduction from eleven-dimensional supergravity, so can the $N = 4$ supergravity theory, coupled to $N = 4$ matter, be obtained by dimensional reduction of ten-dimensional supergravity [8,9]. The field content of the $N = 4$ theory obtained in this way would have one antisymmetric tensor replacing the pseudoscalar and is therefore expected to be equivalent to the theory constructed in this paper. It is remarkable that in both cases dimensional reduction automatically gives the field content with vanishing trace anomaly, this is surely more than a coincidence! No analogous statement holds for the $N = 5$, and 6 supergravities, but we do expect that the $N = 5$, 6 and 8 theories with 2, 3 and 5 antisymmetric tensors, respectively, can be constructed by duality transformations and truncations of the usual $N = 8$ theory. This is because, while not all scalars can be replaced by antisymmetric tensors, all antisymmetric tensors can be converted into scalars. In the $N = 8$ theory as obtained from eleven dimensions we start with seven antisymmetric tensors and so we need only trade in some of them for scalars. To avoid a misunderstanding, perhaps we should make it clear here that duality transformations provide a means of obtaining one supersymmetric theory from another *inequivalent* one. If the two theories related by this transformation were completely equivalent, then there would be no point in performing the transformation.

The $N = 4$ supergravity with one antisymmetric tensor also has a natural interpretation in superspace. Instead of the usual on-shell $N = 4$ field strength chiral superfield V in which both scalar and pseudoscalar fields appear as the first component [10], one can consider a real on-shell $N = 4$ superfield in which only the scalar appears in the first component. The antisymmetric tensor appears through its field strength later in the θ expansion [11].

Finally, the message of this paper is that $N = 3$ supersymmetry is special. This is the lowest value of N for which all supersymmetric theories can be constructed so as to have vanishing one-loop trace anomaly.

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