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SUPERSYMMETRY WITHOUT ANTICOMMUTING VARIABLES

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A B S T R A C T

We present a new characterization of supersymmetric theories which does not make use of anticommuting variables and discuss several examples involving "scales" supersymmetric theories as well as supersymmetric gauge theories.

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1. INTRODUCTION

We all know about the importance of anticommuting variables in the context of supersymmetry[1]. They play an essential role in the formulation of supersymmetric theories and they greatly facilitate complex calculations, for example in perturbation theory. Nonetheless, as mathematical objects, they are not always as convenient and wieldy as ordinary numbers. For instance, they do not have any positivity properties, and we do not know how to attribute an intrinsic meaning to a functional integral over superfields beyond perturbation theory (of course, Gaussian integrals can always be defined) without going back to the component fields. Thus, their main advantage lies in algebraic applications such as proving divergence cancellations, whereas they appear to be unsuitable for analytical applications such as proving correlation inequalities for theories containing fermions in interaction with bosons.

It is therefore of interest that there does exist a characterization of supersymmetric theories directly in terms of the functional measure by means of which expectation values of a supersymmetric theory are defined[2]. This result, which enables us to reconstruct supersymmetric models without recourse to anticommuting variables, is essentially a consequence of the fact that in supersymmetric theories the vacuum energy vanishes identically[3]. In the absence of interactions, this just means that the bosonic and fermionic degrees of freedom add up to zero (fermions being counted as negative), a basic principle known to all those working in supersymmetry. This counting rule is sufficient to determine the supermultiplets, but not to deduce the possible non-trivial supersymmetric interactions.

Our theorem may be viewed as a generalization of this rule to encompass the interacting case as well.

Let us suppose that the multiplet contains some bosonic fields A_i -- where the index i may be either vectorial, internal, or both -- and some Majorana spinors ψ_i; auxiliary fields are assumed to have been eliminated. The number of space-time dimensions is arbitrary, but the multiplets will, of course, depend on the dimension. A Euclidean metric is assumed throughout the rest of this paper. We will furthermore make repeated use of the fact that the fermions may be "integrated out"[4]. If

$$\frac{1}{2}\bar{\psi}M(A)\psi \equiv \frac{1}{2}\int \bar{\psi}_{i\alpha}(x)M_{i\alpha,j\beta}(x,y,\lambda;A_k)\psi_{j\beta}(y)\ dx\ dy \tag{1.1}$$

denotes the fermionic part of the action, which we henceforth assume to be quadratic in the fermions, we have, for instance*,

$$\int d\psi\ \exp\left[-\frac{1}{2}\bar{\psi}M(A)\psi\right] = \det\ M(\lambda;A)^{1/2} \equiv \left[\det\ M(\lambda=0;A)\right]^{1/2}D(\lambda;A)\ , \tag{1.2}$$

since the fermions are Majorana ($\bar{\psi} = \psi^T\mathbb{C}$). $D(\lambda;A)$ is the Matthews-Salam-Seiler (MSS) determinant[5] of the model. We can now state our main theorem.

Theorem: Supersymmetric theories are characterized by the existence of a generally non-linear and non-local transformation T_λ of the bosonic fields

$$T_\lambda\ :\ A_i(x) \to A'_i(x,\lambda;A) \tag{1.3}$$

with the following properties

i) T_λ is invertible in the sense of formal power series.

ii) $S(\lambda;A) = S_0(A'(\lambda;A))$, where S denotes the full bosonic part of the action and S_0 its free part.

iii) The Jacobi determinant of the transformation T_λ equals the MSS determinant in the case of "scalar" supersymmetry and the product of the MSS determinant and the Faddeev-Popov determinant[6] in the presence of an additional gauge symmetry.

*λ (or g) stands for the various coupling parameters.

We will not give the details of the proof in this talk, but rather concentrate on some explicit examples which illustrate and, we hope, clarify the content of the theorem.

2. SOME "SCALAR" SUPERSYMMETRIC EXAMPLES

As a prelude, let us begin with an extremely simplified example in zero space-time dimensions, where no non-local (kinetic) couplings exist and where the functional integral becomes an ordinary integral. In a nutshell, this example neatly displays all the relevant features. For a "multiplet" A, F real, ψ_1, ψ_2 anticommuting, consider the "Lagrangian"

$$\mathcal{L} = \frac{1}{2}F^2 + iFp(A) - \frac{1}{2}p'(A)\psi_\alpha \varepsilon^{\alpha\beta}\psi_\beta \ , \tag{2.1}$$

where p(A) is a globally invertible but otherwise arbitrary C^1 function. This "Lagrangian" is invariant under the supersymmetry transformations

$$\begin{gathered} \delta A = \zeta_\alpha \varepsilon^{\alpha\beta}\psi_\beta \ , \quad \delta\psi_\alpha = i\zeta_\alpha F \ , \quad \delta F = 0 \\ \zeta_\alpha \text{ anticommuting} \ , \quad \varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{gathered} \tag{2.2}$$

"Vacuum expectation values" are given by

$$\frac{1}{2\pi}\int R(A)e^{-\mathcal{L}(A,F,\psi_\alpha)} \ dA \ dF \ d\psi_1 \ d\psi_2 \ . \tag{2.3}$$

Integrating out F (the "auxiliary field") and ψ_1, ψ_2, we find

$$\langle R(A)\rangle = \frac{1}{\sqrt{2\pi}} \ R(A)e^{-1/2[p(A)]^2} p'(A) \ dA \ , \tag{2.4}$$

where $p'(A)$ is the MSS determinant. Indeed, the transformation $A \to A' = p(A)$ reduces (2.4) to a Gaussian integral and has Jacobi determinant $p'(A)$! Moreover, we can derive "super Ward identities" without using ζ's, so, for instance, the identity $\langle\psi_\alpha\psi_\beta\rangle = -i\varepsilon_{\alpha\beta}\langle FA\rangle$ is nothing but

$$\int_{-\infty}^{+\infty} \frac{\partial}{\partial A}\left[Ae^{-1/2 \ p(A)^2}\right] dA = 0 \ , \tag{2.5}$$

and all other "Ward identities" can be obtained in a similar fashion. Conversely, if we had postulated the existence of a transformation which reduces the measure to a Gaussian, we would have been immediately led to (2.1).

In two space-time dimensions, the minimal supermultiplet contains a scalar A and a two-component Majorana spinor ψ_α [7]. The most general invariant action is[7]

$$S(A,\psi) = \frac{1}{2}\int dx\left[(\partial_\mu A)^2 + p(A)^2\right] + \\ + \frac{1}{2}\int dx\ \overline{\psi}_\alpha\left[\gamma^\mu_{\alpha\beta}\partial^\mu + \delta_{\alpha\beta}p'(A)\right]\psi_\beta \tag{2.6}$$

and, for definiteness, we will take $p(A) = mA + \lambda A^3$ which $m, \lambda > 0$. From (2.6), we may now compute the MSS determinant*

$$\left[\det\left\{\gamma^\mu_{\alpha\beta}\partial^\mu + \delta_{\alpha\beta}(m+3\lambda A^2(x))\right\}\delta(x-y)\right]^{1/2} = \\ = \det\ (-\Delta+m^2)^{1/2}\ \exp\left[3m\lambda C(0)\int A^2(x)\ dx + O(\lambda^2)\right], \tag{2.7}$$

where $C(x) = (-\Delta+m^2)^{-1}(x)$ is the usual propagator. The transformation of A that we are looking for reads up to first order

$$A'(x,\lambda;A) = A(x) + m\lambda\int C(x-y)A^3(y)\ dy + O(\lambda^2) \tag{2.8}$$

(it has been written out up to third order in Ref. 2). It is not difficult to verify that indeed

$$\det\frac{\delta A'(x,\lambda;A)}{\delta A(y)} = \text{MSS determinant} + O(\lambda^2) \tag{2.9}$$

<u>and</u>

$$\frac{1}{2}\int dx\left[(\partial_\mu A')^2 + m^2A'^2\right] = \frac{1}{2}\int dx\left[(\partial_\mu A)^2 + m^2A^2 + 2m\lambda A^4\right] + O(\lambda^2)\ . \tag{2.10}$$

Had we chosen, say, a Yukawa coupling constant other than that prescribed by supersymmetry, either (2.9) or (2.10) would cease to hold. For $\lambda \to 0$, (2.8) becomes the identity transformation, whereas in the ultralocal limit we obtain $mA' = mA + \lambda A^3$, and the map is invertible only if $m, \lambda > 0$ (or both < 0). The question of <u>global</u> invertibility

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*Remember, it is the determinant of a matrix with space-time indices (= x,y) <u>and</u> spinor indices (= α,β).

of the transformation is thus related to the question of vacuum degeneracy, since the condition $m, \lambda > 0$ just ensures that the scalar field potential has only one absolute minimum.

Let us now turn to the case of four dimensions. Here, the minimal multiplet consists of one scalar A, one pseudoscalar B, and one four-component Majorana spinor[8] ψ_α and the simplest non-trivial action is given by[8]

$$S(A,B,\psi) = \frac{1}{2}\int dx\Big[(\partial_\mu A)^2 + (\partial_\mu B)^2 + m^2(A^2+B^2) + 2mgA(A^2+B^2) + g^2(A^2+B^2)^2\Big] + \frac{1}{2}\int dx\, \bar{\psi}_\alpha\Big[\gamma^\mu_{\alpha\beta}\partial^\mu + \delta_{\alpha\beta}(m+2gA) - 2g\gamma^5_{\alpha\beta}B\Big]\psi_\beta \,. \qquad (2.11)$$

As before, we calculate the MSS determinant from this action

$$\Big[\det\Big\{\gamma^\mu_{\alpha\beta}\partial^\mu + \delta_{\alpha\beta}\big[m+2gA(x)\big] - 2g\gamma^5_{\alpha\beta}B(x)\Big\}\delta(x-y)\Big]^{1/2} = \det(-\Delta+m^2)\exp\Big[4mgC(0)\int A(x)\,dx + O(g^2)\Big] \qquad (2.12)$$

and, as before, we verify that the transformation

$$\begin{aligned} A'(x,g;A,B) &= A(x) + mg\int C(x-y)(A^2(y)-B^2(y))\,dy + O(g^2) \\ B'(x,g;A,B) &= B(x) + 2mg\int C(x-y)A(y)B(y)\,dy + O(g^2) \end{aligned} \qquad (2.13)$$

has all the desired properties up to first order in g. The novel feature is that now, in the ultralocal limit, we get

$$mA' = mA + g(A^2-B^2)\,, \qquad mB' = mB + 2gAB\,, \qquad (2.14)$$

so, although being locally invertible almost everywhere, the transformation is no longer globally invertible. In general, if there are at least two scalar fields, the winding number of the transformation equals the number of absolute minima of the potential. As a technical remark, we mention that, in quantum field theory, the bosonic fields are distributions in general and therefore the transformations (2.8) and (2.13) are not really well defined as they stand. However, it is not difficult to find a supersymmetry-preserving UV cut-off which remedies this defect.

3. SUPERSYMMETRIC GAUGE THEORIES

Supersymmetric gauge theories[9] are physically more interesting. Because of the additional gauge invariance, a gauge-fixing procedure is needed[6] which either explicitly violates supersymmetry[10] or, through additional ghost multiplets, renders the theory considerably more complicated[11]. We will disregard the second possibility, since the statement loses much of its transparency in that case. In a Euclidean space-time, the model is given by the Lagrangian[9]

$$\mathcal{L} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2}\bar{\psi}^a \not{D}\psi^a + \frac{1}{2} D^{a2} \ , \tag{3.1}$$

where, obviously,

$$F^a_{\mu\nu}(A) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{3.2}$$

and

$$\not{D}\psi^a = \gamma^\mu \partial^\mu \psi^a + g f^{abc} A^b_\mu \gamma_\mu \psi^c \ . \tag{3.3}$$

It is by now standard folklore that (3.1) is invariant (up to a total divergence) with respect to the transformations

$$\delta A^a_\mu = -\bar{\varepsilon}\gamma_\mu \psi^a \ , \quad \delta D^a = \bar{\varepsilon}\gamma_5 \not{D}\psi^a \ , \quad \delta\psi^a = (\sigma_{\alpha\beta} F^a_{\alpha\beta} - \gamma^5 D^a)\varepsilon \ . \tag{3.4}$$

We now fix a gauge by adding a term $1/2(\partial_\mu A^a_\mu)^2$ to the Lagrangian[12], so the purely bosonic part of the action becomes (dropping auxiliary fields)

$$\frac{1}{4}\int F^a_{\mu\nu}(A)^2 \, dx + \frac{1}{2}\int (\partial_\mu A^a_\mu)^2 \, dx = \\ = \frac{1}{2}\int A^a_\mu (-\delta_{\mu\nu}\delta^{ab}\Delta) A^b_\nu \, dx + O(g) \ . \tag{3.5}$$

It is only for convenience that we have set the conventional gauge parameter $\alpha = 1$, for we could have even chosen a non-linear gauge-fixing term without altering our final result. To compensate for the explicit breaking of gauge invariance, the functional measure has to be weighted with the Faddeev-Popov determinant[6] which, in our case, reads

$$(\det \delta^{ac}\Delta) \cdot \det \left\{ \delta^{ac}\delta(x-y) - g f^{abc} \partial_\mu C(x-y) A^b_\mu(y) \right\} \ , \tag{3.6}$$

where, now, $C(x) = -\Delta^{-1}(x)$. The MSS determinant is obtained from

$$\left[\det\left\{\delta^{ac}\gamma^{\mu}_{\alpha\beta}\partial^{\mu} + gf^{abc}\gamma^{\mu}_{\alpha\beta}A^{b\mu}(x)\right\}\delta(x-y)\right]^{1/2} . \tag{3.7}$$

After a slightly more tedious calculation, we get for the product of the two determinants

$$\exp\Bigg[ng^2\int dx\, dy\,\Big\{\frac{3}{2}\,\partial_\mu C(x-y)A^a_\mu(y)\partial_\nu C(y-x)A^a_\nu(x) - \\ - \partial_\mu C(x-y)A^a_\nu(y)\partial_\mu C(y-x)A^a_\nu(x) + \\ + \partial_\mu C(x-y)A^a_\nu(y)\partial_\nu C(y-x)A^a_\mu(x)\Big\} + O(g^3)\Bigg] , \tag{3.8}$$

where we made use of $f^{abc}f^{a'bc} = n\delta^{aa'}$ and omitted a trivial factor $(\det \delta^{ac}\Delta)^2$ (it is not entirely trivial, because it is exactly the factor needed for the elementary counting rule cited in the Introduction, as the reader may easily check!). The required field transformation is found to be

$$A'^{a}_{\mu}(x,g;A) = A^a_\mu(x) + gf^{abc}\int dy\, \partial_\lambda C(x-y)A^b_\mu(y)A^c_\lambda(y) + \frac{g^2}{2}f^{abc}f^{bde} \\ \int dy\, dz\,\Big\{\partial_\nu C(x-y)A^c_\lambda(y)\partial_\nu C(y-z)A^d_\mu(z)A^e_\lambda(z) - \\ - \partial_\nu C(x-y)A^c_\lambda(y)\partial_\lambda C(y-z)A^d_\mu(z)A^e_\nu(z) + \\ + \partial_\nu C(x-y)A^c_\lambda(y)\partial_\mu C(y-z)A^d_\lambda(z)A^e_\nu(z)\Big\} \tag{3.9}$$

up to and including second order in g. It satisfies both

$$\det \frac{\delta A'^{a}_{\mu}(x,g;A)}{\delta A^b_\nu(y)} = (3.8) + O(g^3) \tag{3.10}$$

and

$$\frac{1}{2}\int A'^{a}_{\mu}(-\delta_{\mu\nu}\delta^{ab}\Delta)A'^{b}_{\nu}\, dx = \frac{1}{4}\int F^a_{\mu\nu}(A)^2\, dx + \frac{1}{2}\int(\partial_\mu A^a_\mu)^2\, dx + O(g^3). \tag{3.11}$$

The general proof to all orders in g in the case of supersymmetric gauge theories will be given in a forthcoming publication[13]. We note that, in contradistinction to the "scalar" supersymmetric case, the problem of making (3.9) mathematically acceptable for distribution-valued $A_\mu^a(x)$ has not been solved: there exists up to date no non-perturbative regularization prescription which respects both supersymmetry and gauge invariance.

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