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Propagation of high-frequency electromagnetic waves through a magnetized plasma in curved space–time. I

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This is the first of two papers on the propagation of high-frequency electromagnetic waves through a magnetized plasma in curved space–time. We first show that the nonlinear system of equations governing the plasma and the electromagnetic field in a given, external gravitational field has locally a unique solution for any initial data set obeying the appropriate constraints, and that this system is linearization stable at any of its solutions. Next we prove that the linearized perturbations of a 'background' solution are characterized by a third-order (not strictly) hyperbolic, constraint-free system of three partial differential equations for three unknown functions of the four space-time coordinates. We generalize the algorithm for obtaining oscillatory asymptotic solutions of linear systems of partial differential equations of arbitrary order, depending polynomially on a small parameter such that it applies to the previously established perturbation equation when the latter is rewritten in terms of dimensionless variables and a small scale ratio. For hyperbolic systems we then state a sufficient condition in order that asymptotic solutions of finite order, constructed as usual by means of a Hamiltonian system of ordinary differential equations for the characteristic strips and a system of transport equations determining the propagation of the amplitudes along the rays, indeed approximate solutions of the system. The procedure is a special case of a two-scale method, suitable for describing the propagation of locally approximately plane, monochromatic waves through a dispersive, inhomogeneous medium. In the second part we shall apply the general method to the perturbation equation referred to above.

1. INTRODUCTION AND ASTROPHYSICAL MOTIVATION

Since its discovery the X-ray source Cyg X-1 has been the first and so far strongest candidate to be a binary system containing a black hole. Also, Cir X-1 has been suggested to have a black hole as one of its binary components, and up to now this idea is compatible with all known data from Cir X-1. The main reasons in support of a black hole in these systems are (i) the mass estimates derived from the mass function of the system together with plausible assumptions on the inclination of the system relative to the line of sight, and (ii) the fact that the X-ray intensity shows irregular variations on a millisecond timescale. The latter property strongly suggests that the radiation is emitted within a compact region of less than 100 km in diameter, possibly during the final spiral motion of accreting matter towards the black hole which possibly has no magnetic field – as is expected from astrophysical black holes. They should neutralize quickly in case charge-separating mechanisms are active in their environments.

These features are independent of the details of the actual accretion process: whether stellar wind from the super-giant companion drives the accretion (Meszéros 1978), or whether an accretion disk forms as a result of mass transfer in a Roche lobe overflow, not to mention the difficulties one encounters with detailed disk models (Stewart 1976*a*, *b*; Drury & Stewart 1978; Thorne & Price 1975; Shakura & Sunyaev 1973; Lightmann & Eardley 1974; Shapiro *et al.* 1976; Livio & Shaviv 1977). However, the indications for the existence of a black hole in Cyg X-1 are somewhat indirect and could in principle be questioned: neutron stars with unorthodox equations of state could have higher maximal masses and could thus replace the black hole; in addition a neutron star with aligned magnetic field or an unmagnetized neutron star would show no regular pulse structure in its X-rays. What one needs is more clear-cut, direct observational evidence in favour of a black hole in Cyg X-1.

Such evidence might be provided by observation of the polarization of X-rays from sources involving neutron stars or black holes (Rees 1975). As for neutron stars, X-ray polarimetry could reveal details of the beaming and accretion mechanisms, and for black holes it would give evidence for the structure and the position angle of the accretion disk.

The polarization properties of X-rays from Cyg X-1 seem to be sensitive to strong general relativistic effects. Satellite measurements claim an upper limit of 3 % linear polarization at 2.6 keV in Cyg X-1 (Novick *et al.* 1977; Chanan *et al.* 1979). Theoretically, their polarization properties have been first calculated in the Newtonian approximation (Angel 1969; Lightman & Shapiro 1976). In a relativistic calculation Stark & Connors (1977*a*, *b*) find that for rays from the surface of an accretion disk around Cyg X-1, travelling to the observer through vacuum, the direction of linear polarization may vary as a function of energy up to 100° owing to general relativistic effects. Equally, the degree of polarization differs from the Newtonian value. In fact, for the X-ray energies of interest between 1 and 100 keV they predict about half the Newtonian value for the so-called 'one' temperature disk model. The variation of such effects with the energy of X-rays – though depending crucially on the particular disk model and assumptions about the emission mechanism – should allow to differentiate observationally between a fast-rotating black hole or, alternatively, a slowly rotating black hole or a neutron star. According to Stark & Connors, for the latter the effect should be smaller by one order of magnitude.

The influence of any magnetic fields on the accretion disk has been ignored by Stark & Connors. Although one might expect a mostly chaotic magnetic field of the order of 10^7 G (Mészáros *et al.* 1977), it is quite unclear to what extent a magnetized corona of the disk could affect the polarization pattern of the X-rays. Theoretically, a formalism is required here to describe the propagation of electromagnetic waves in a magnetized plasma embedded in a strong background gravitational field. To what extent plasma effects – dispersion, refraction and Faraday rotation – play a role for Cyg X-1 can be decided only after such a calculation is at hand and is compared with the data.

Other astrophysical situations, where plasma effects in curved space–time may influence wave propagation, can occur in galactic nuclei when a massive black hole is present, and in pulsar magnetospheres of rotating neutron stars, but also in cosmology. The cosmological microwave radiation can be affected due to intergalactic magnetic fields and when the interstellar gas is reionized at late cosmological times redshift $z \lesssim 8$) (Anile & Breuer 1977).

Generally, one cannot solve Maxwell's equations exactly for waves propagating in a relativistic medium. Instead one seeks a geometrical optics approximation, not only in the style of Hamilton's geometrical theory of rays and wave surfaces (Synge 1960), but a refined version allowing also for frequencies, polarization states and the propagation of amplitudes along rays leading to a radiative transfer equation. The standard treatments using plane wave solutions to describe wave propagation in a homogeneous plasma in flat space–time (Ginzburg 1964; Stix 1962) are no longer applicable when inhomogeneities are present in the plasma or the underlying space–time is curved. Physically, the reason for this is that the plane wave ansatz loses its meaning. There are simply no plane waves in the more complicated situations mentioned. Mathematically, plane waves are Fourier components of the original field in $x-t$ space. However, Fourier transformation is no longer possible: it presupposes that the quantities to be Fourier-analyzed are defined on a space which has an Abelian translation group.

However, when the wavelengths are small in comparison with the scale of plasma inhomogeneities or, say, the Schwarzschild radius of the black hole, intuitively the plane wave formalism should still be valid locally. Thus the desired formalism has to satisfy two properties: locally it should resemble a Fourier transformation; globally, the long-range effects of inhomogeneities of the plasma and the gravitational fields have to show up in the propagation laws. So far, in our opinion, only several incomplete attempts are known towards a systematic derivation of such a formalism in the framework of general relativity (Madore 1974; Bičák & Hadrava 1975; Anile & Pantano 1977, 1979). It is our aim in this and the subsequent paper to provide such a rigorous treatment for a simple fluid model of the plasma which may be moving in an arbitrary manner in an inhomogeneous, non-stationary, strong gravitational field. The high-frequency approximation of electromagnetic waves in a transparent, isotropic, dispersion-free, moving medium in curved space–time has been treated by Ehlers (1967). The procedure developed in this paper is a special case of a two-scale method as given for fluid dynamics in flat space–time by Witham (1965 *a*, *b*).

In the next section we list the basic assumptions and equations for the background upon which we will perform perturbation theory later on. An existence and uniqueness theorem is established for the background. In §3 we analyse the system of differential equations governing linearized perturbations on this background resulting in a third-order evolution equation for the potential of the perturbations. In §4 we describe quite generally the formal scheme to obtain oscillatory asymptotic solutions of such evolution equations using a two-scale method.

2. Dynamics of the plasma

Consider a region of space–time (M, g_{ab}) occupied by a plasma. (For notation and conventions, see appendix A.) We idealize the plasma as a cold, i.e. pressure-free, two-component fluid and denote the number density and world velocity of the electrons by n and u^a, respectively, and write J^a for the current density of the ions. Let m be the mass, e the charge of the electron and F_{ab} the electromagnetic field. We neglect the difference between the microscopic and the macroscopic field and the velocity dispersion of the electrons and take as the equations governing the system:

$$\nabla_{[a} F_{bc]} = 0, \tag{2.1}$$

$$\nabla_b F^{ab} = enu^a + J^a, \tag{2.2}$$

$$u^b \nabla_b u^a = (e/m) F^a_b u^b, \tag{2.3}$$

$$\nabla_a (nu^a) = 0, \tag{2.4}$$

$$u_a u^a = -1. \tag{2.5}$$

(It would be preferable to absorb e in the definition of F_{ab} and use e^2/m as the only constant appearing in equations (2.1–5). We stick to the usual notation, however.) Throughout this paper we treat the metric as a given external field. For simplicity we also consider the ion-current J^a as given; obeying the conservation law

$$\nabla_a J^a = 0. \tag{2.6}$$

(In accordance with this assumption we shall in §3, etc. disregard perturbations of J^a unlike those of enu^a. This is physically justified, for high-frequency perturbations, by the large inertia of the ions compared to that of the electrons.) Under these assumptions the equations (2.1–5) determine the dynamics of the electron fluid and of the electromagnetic field.

In the remainder of this section we consider some properties of the system of equations (2.1–5) and its solutions which form the basis for the perturbation theory to be developed in the following sections.

In order to discuss the *local initial value problem* for the system (2.1–5) we assume that Σ is a smooth, spacelike hypersurface in (M, g_{ab}), and we introduce a system of Gaussian normal coordinates $(x^a) = (x^0, x^\lambda)$ with respect to Σ such that Σ is given by $x^0 = 0$. With respect to this coordinate system, equations (2.1–4) give rise to the quasi-linear, first-order system

$$\partial_{[0} F_{\lambda\mu]} = 0, \tag{2.7}$$

$$\nabla_b F^{\lambda b} = enu^\lambda + J^\lambda, \tag{2.8}$$

$$u^b \nabla_b u^a = (e/m) F^a_b u^b, \tag{2.3}$$

$$\nabla_a(nu^a) = 0 \tag{2.4}$$

of eleven *evolution equations*, and the semi-linear† system

$$\partial_{[\lambda} F_{\mu\nu]} = 0, \tag{2.9}$$

$$\nabla_b F^{0b} = enu^0 + J^0, \tag{2.10}$$

$$u_a u^a = -1 \tag{2.5}$$

of three *constraints* for the eleven unknowns (F_{ab}, u^a, n). The constraints are preserved by the evolution equations. One can freely specify $11-3=8$ initial data on Σ, corresponding to two degrees of freedom per space-point of the electromagnetic field and equally many of the electron fluid, as is well known.

Unfortunately, the nonlinear system (2.7, 8, 3, 4) of partial differential equations is not strictly hyperbolic and, as far as we are aware, it is also not equivalent to a symmetric–hyperbolic system. Therefore, the more familiar theorems do not guarantee existence, uniqueness and differentiable dependence on initial data of solutions to the Cauchy problem for this system. Nevertheless, due to results of Choquet-Bruhat (1958) the following theorem holds:

THEOREM 1. *(I) Given a set of initial data* (F_{ab}, u^a, n) *of sufficient (finite) differentiability satisfying the constraints* (2.9, 10, 5) *on a spacelike‡ hypersurface* Σ, *there exists locally a unique solution of* (2.7, 8, 3, 4), *and therefore of* (2.1–5), *having these initial values.*

(II) The system (2.1–5) *is linearization stable at any of its solutions.*

Proof. We first derive another system of differential equations from (2.1–4). Restricting attention to a simply connected domain of space–time we use (2.1) to represent F_{ab} in terms of a potential

$$F_{ab} = 2\nabla_{[a} A_{b]} \tag{2.11}$$

obeying the Lorentz gauge condition

$$\nabla_a A^a = 0. \tag{2.12}$$

In terms of the de Rham operator $\Box$ for vector fields, $\Box A^a = \nabla^b_b A^a - R^a_b A^b$, (2.2) then gives

$$\Box A^a = -enu^a - J^a. \tag{2.13}$$

Differentiating (2.13) along the electron world lines, we obtain, as a consequence of the system (2.1–4), the system

$$u^b \nabla_b(\Box A^a) = -e(nu^b\nabla_b u^a - nu^a \nabla_b u^b) - u^b \nabla_b J^a, \tag{2.14}$$

$$u^b \nabla_b u^a = 2(e/m)\, \nabla^{[a} A^{b]} u_b, \tag{2.15}$$

† A system of partial differential equations is said to be semilinear if the coefficients of the highest-order derivatives are independent of the unknown functions and their derivatives.

‡ 'Spacelike' is always to be interpreted in the sense of the geometry of (M, g_{ab}). The systems of differential equations considered in this paper are all such that geometrically spacelike hypersurfaces are also spacelike for these systems, in the sense of the theory of partial differential equations.

$$u^b \nabla_b n = -n \nabla_b u^b. \tag{2.16}$$

This system of nine quasilinear equations for the nine unknowns (A^a, u^a, n) is, as recognized by Choquet–Bruhat (1958), a strictly hyperbolic, diagonal Leray system. In fact the operators appearing on the left-hand side of (2.14, 15, 16) form a compatible set of strictly hyperbolic operators, and the derivatives of the unknowns occur such as to obey the conditions of Leray's theorem. (For a concise formulation of this theorem see, for example, Choquet–Bruhat *et al.* (1977), p. 441. In the notation used there, the 'indices' of the unknowns and equations of the system here considered are as follows:

$$m(A^a) = 3,\ m(u^a) = 2,\ m(n) = 1;\ n(14) = n(16) = 0,\ n(15) = 1.)$$

Hence, according to Leray's theorem, the system (2.14, 15, 16) has a unique solution for any system of sufficiently (finitely) differentiable initial data

$$A_a, \partial_0 A_a, \partial_{00} A_a, u^a, n. \tag{2.17}$$

(Since the metric is given, these data are equivalent to initial values for the tensors A_a, $\nabla_b A_a$, $\nabla_{bc} A_a$ on Σ, where $\nabla_{bc} = \nabla_b \nabla_c$.) Suppose ($A^a, u^a, n$) solves (2.14, 15, 16). Then, if F_{ab} is defined by (2.11), equations (2.1, 3, 4) are satisfied, and because of (2.14) and (2.16), the tensor fields ($\Box A^a + enu^a + J^a$) and $u_a u^a$ are covariantly constant on u^a-world lines. Hence, if the initial values satisfy (2.5) and (2.13), then these equations hold in the whole domain of dependence of the data. Furthermore (2.6), (2.4), (2.13), and the identity $\nabla_a \Box A^a = \Box \nabla_a A^a$ imply $\Box \nabla_a A^a = 0$. Therefore, if $\nabla_a A^a$ and $\partial_0 \nabla_a A^a$ vanish on Σ, then the Lorentz condition (2.12) holds in the domain of dependence, and in that case (2.11) and (2.13) imply (2.2). We have shown that *the system of constraints*

$$u_a u^a = -1, \tag{2.5}$$

$$\nabla_a A^a = 0, \tag{2.12}$$

$$\Box A^a = -enu^a - J^a, \tag{2.13}$$

$$\partial_0 \nabla_a A^a = 0 \tag{2.18}$$

is *preserved under the system* (2.14, 15, 16) *of evolution equations*; *any set of initial data* (2.17) *which satisfies these constraints determines a unique solution of* (2.14, 15, 16); *such a solution determines, via* (2.11) *a solution of the original system* (2.1–5). It is also clear from the preceding reasoning that *any solution of* (2.1–5) *can be obtained in this way*.

Note that the initial data set (2.17) contains 17 functions. Since there are seven constraints, ten functions can be chosen arbitrarily. However, the Lorentz-gauge is preserved under the restricted gauge transformations $A_a \to A_a + \partial_a \Lambda$ with $\Box \Lambda = 0$, and such a gauge function Λ is determined by the initial data Λ and $\partial_0 \Lambda$ on Σ. This reduces the number of physically essential initial data from ten to eight, in accordance with the previous counting.

To complete the proof of statement (I) of the theorem, we suppose that (F_{ab}, u^a, n)

are initial data on Σ which obey the constraints (2.9, 10, 5). Then, by (2.9), there exist on Σ functions A_λ such that $F_{\lambda\mu} = 2\partial_{[\lambda}A_{\mu]}$; these A_λ are determined up to changes $A_\lambda \to A_\lambda + \partial_\lambda \Lambda$. Choose A_0 arbitrarily on Σ, and define the initial data $\partial_0 A_\lambda$ and $\partial_0 A_0$ by $\partial_0 A_\lambda = F_{0\lambda} + \partial_\lambda A_0$ and $\nabla_a A^a = 0$, respectively. This provides, on Σ, A_a and $\partial_0 A_a$ or, equivalently, A_a and $\nabla_b A_a$, and by construction these data satisfy (2.11) and (2.12). Next, compute the initial values $\partial_{00} A_a$ from (2.13). This then fixes $\nabla_{ab} A_c$ on Σ. A straight-forward calculation shows that the constraint (2.10), combined with the 0-component of (2.13), implies $\partial_0(\nabla_a A^a) = 0$. Consequently *initial data of the original system* (2.7, 8, 3, 4) *satisfying the corresponding constraints* (2.9, 10, 5) *determine uniquely up to* 2 *gauge-initial data* (Λ and A_0), *an initial data set* (2.17) *obeying the constraints* (2.5, 12, 13, 18). The solution belonging to these data then furnishes a solution of (2.1–5) with the prescribed initial values (F_{ab}, u^a, n). Uniqueness follows from the uniqueness of the Cauchy-problem for (2.14–16) combined with the fact, established above, that the initial data (2.17) are determined by those of the original system exactly up to 'initial gauge transformations'.

To prove statement (II) of the theorem we first verify that the system (2.5, 12, 13, 18) of constraints is linearization stable. In fact, let (A^a, u^a, n) satisfy those constraints, and let ($\hat{A}^a$, $\hat{u}^a$, $\hat{n}$) obey the linearized constraints

$$u_a \hat{u}^a = 0, \tag{2.19}$$

$$\nabla_a \hat{A}^a = 0, \tag{2.20}$$

$$\Box \hat{A}^a = -e(\hat{n}u^a + n\hat{u}^a), \tag{2.21}$$

$$\partial_0(\nabla_a \hat{A}^a) = 0.$$

Define a 1-parameter family $(\bar{A}^a, \bar{u}^a, \bar{n})$, of fields on Σ by putting

$$\bar{A}^a := A^a + \epsilon \hat{A}^a, \bar{n}\bar{u}^a := nu^a + \epsilon(\hat{n}u^a + nu^a)$$

and requiring $\bar{u}_a \bar{u}^a = -1$ and continuity of $\bar{u}^a$ in ϵ near $\epsilon = 0$. Then this family 'passes through' the unperturbed solution for $\epsilon = 0$ in the 'direction' $(\hat{A}^a, \hat{u}^a, \hat{n})$, and a simple calculation shows that the fields $\bar{A}^a, \bar{u}^a, \bar{n}$ satisfy the constraints (2.5, 12, 13, 18) in a (finite) neighbourhood of $\epsilon = 0$. The linearization stability of the full system (2.14, 15, 16, 5, 12, 13, 18) can now be established by showing that the solution of the evolution equations is a Frechet-differentiable functional of the initial data (see, for example, Fischer & Marsden 1972). Linearization-stability of the full system is therefore inherited from that of the system of constraints. Finally, linearization stability of the original system (2.1–5) follows from that of the modified system, since both the initial data sets and the solutions are in ('physical') one-to-one, differentiable correspondence. Q.E.D.

Remark. Equation (2.4) or, equivalently, (2.16), restricted to a u^a-world line, shows that the sign of n cannot change on a world line. Thus if the initial value of n is positive – as it must be because of its physical interpretation – then n will be positive in the domain of dependence, as desired.

We end this section by introducing some quantities associated with a solution of (2.1–5) which will be needed later.

The local kinematics of the electron fluid are described, as usual, by the rate of rotation (vorticity) $\omega^{ab}(=-\omega^{ba})$, the rate of deformation $\theta^{ab}(=\theta^{ba})$, and the world acceleration $u^b\nabla_b u^a$, obtained from the decomposition

$$\nabla_b u^a = \omega^a_b + \theta^a_b - u_b u^c \nabla_c u^a. \tag{2.22}$$

We put

$$\theta := \theta^a_a = \nabla_a u^a \tag{2.23}$$

for the mean expansion rate.

We also need the tensor

$$h^a_b := \delta^a_b + u^a u_b \tag{2.24}$$

which projects orthogonally onto the (tangent) rest space of the electron fluid. We define the electric field

$$E^a := F^a_b u^b, \tag{2.25}$$

the magnetic field

$$B_{ab} := h^c_a h^d_b F_{cd}, \tag{2.26}$$

the Larmor angular velocities†

$$\omega_{\mathrm{L}}{}^a{}_b := (-e/m) B^a_b, \omega_{\mathrm{L}}{}^a := -\tfrac{1}{2}\eta^{abcd} u_b \omega_{\mathrm{L}\,cd}, \tag{2.27}$$

the Larmor (or cyclotron) frequency

$$\omega_{\mathrm{L}} := (\omega_{\mathrm{L}\,a}\omega_{\mathrm{L}}{}^a)^{\frac{1}{2}}, \tag{2.28}$$

and the plasma frequency

$$\omega_{\mathrm{p}} := (ne^2/m)^{\frac{1}{2}}. \tag{2.29}$$

The tensors ω^{ab}, θ^{ab}, $u^b\nabla_b u^a$, h^{ab}, E^a, B^{ab}, $\omega_{\mathrm{L}}{}^{ab}$, $\omega_{\mathrm{L}}{}^a$ are all orthogonal to u^a with respect to all indices; this will be used frequently in the remainder of the paper.

3. Dynamics of small perturbations of the electromagnetic field and the plasma

In the remainder of this paper we study the propagation of high-frequency, low-amplitude waves through a plasma. We assume the fields (F_{ab}, u^a, n) to consist of a slowly varying *background* $(\mathring{F}_{ab}, \mathring{u}^a, \mathring{n})$ and a rapidly oscillating *wave* $(\hat{F}_{ab}, \hat{u}^a, \hat{n})$. We neglect the reaction of the gravitational field g_{ab} to the wave and treat the latter as an infinitesimal perturbation of the background field, i.e. we compute the equations governing $(\hat{F}_{ab}, \hat{u}^a, \hat{n})$ by varying equations (2.1–5), leaving g_{ab} and thus ∇_a unchanged. It does not matter for our purposes whether the background fields are test fields or solutions of the coupled Einstein–Maxwell equations.

To simplify the notation we shall henceforth write $F_{ab}, \ldots$ instead of $\mathring{F}_{ab}, \ldots$ for the

† This quantity is the gyration angular velocity of the perturbed electron motion; see (3.7) below. Our convention is $\eta_{0123} = +1$ in an oriented orthonormal frame.

background fields; then equations (2.1–6) as well as the definitions (2.22–29) refer to the background.

Varying (2.1–5) we obtain as the equations governing the perturbations:

$$\nabla_{[a}\hat{F}_{bc]} = 0, \tag{3.1}$$

$$\nabla_b \hat{F}^{ab} = e(\hat{n}u^a + \hat{n}u^a), \tag{3.2}$$

$$u^b\nabla_b \hat{u}^a + \hat{u}^b\nabla_b u^a = (e/m)(F^a_b \hat{u}^b + \hat{F}^a_b u^b), \tag{3.3}$$

$$\nabla_a(\hat{n}u^a + n\hat{u}^a) = 0, \tag{3.4}$$

$$u_a \hat{u}^a = 0. \tag{3.5}$$

Theorem 1 of the preceding section implies the following

COROLLARY. (I) *The system* (3.1–5) *has unique solutions corresponding to initial data which satisfy the linearized constraints.*

(II) *If* $(F_{ab}, \ldots)$ *and* $(\hat{F}_{ab}, \ldots)$ *obey equations* (2.1–6) *and* (3.1–5), *respectively, then* $(F_{ab} + \epsilon\hat{F}_{ab}, \ldots)$ *approximates a solution of the full system* (2.1–6) *provided* ϵ *is sufficiently small.*

For the linear system (3.1–5) the decomposition into evolution equations and constraints can be carried out, in contrast to the situation for the full equations (2.1–5), in an intrinsic, physically and mathematically preferred way since $u^a\nabla_a$ is a *preferred time-derivative.*

The purpose of this section is to reduce the rather complex system (3.1–5) to a simpler one which is better suited for calculations, in particular for the treatment of high-frequency waves. We shall eliminate redundant equations and unknowns and, in particular, get rid of constraints.

Equation (3.4) is implied by (3.2) and can therefore be omitted from the system. Next, we insert (2.22) into (3.3) and use (3.5) to get

$$u^b\nabla_b \hat{u}^a + (\omega^a_b + \theta^a_b)\,\hat{u}^b = (e/m)\,(F^a_b \hat{u}^b + \hat{F}^a_b u^b). \tag{3.6}$$

The equation obtained by transvecting this equation with u_a is implied by (2.3) and (3.5) whence (3.6), or rather (3.2), can be replaced by its 'spatial part',

$$h^a_b u^c\nabla_c \hat{u}^b + (\omega^a_b + \theta^a_b + \omega^a_b)\,\hat{u}^b = (e/m)\,\hat{F}^a_b u^b. \tag{3.7}$$

Moreover, by (3.5), (3.2) is equivalent to the pair

$$e\hat{n} = -u_a \nabla_b \hat{F}^{ab}, \tag{3.8}$$

$$en\hat{u}^a = h^a_b \nabla_c \hat{F}^{bc} \tag{3.9}$$

which can be used to eliminate the electron variables $(\hat{u}^a, \hat{n})$ from the system of perturbation equations altogether. (This presupposes $n \neq 0$. In vacuum this elimination procedure is not possible.) Note that if $\hat{u}^a$ is computed from (3.9) it automatically obeys the constraint (3.5) whence the latter can also be omitted from the set of essential perturbation equations. Inserting (3.9) into (3.7) gives the second-order equation

$$h^a_b u^d \nabla_{dc} \hat{F}^{bc} + (\omega^a_b + \omega_{\mathrm{L}}{}^a{}_b + \theta^a_b + \theta h^a_b + (e/m) E^a u_b) \nabla_c \hat{F}^{bc} - \omega_p^2 \hat{F}^{ab} u_b = 0. \quad (3.10)$$

The pair (3.1, 10) of equations for $\hat{F}_{ab}$ is equivalent to the system (3.1–5) modulo the background equations (2.1–5); i.e. any solution of (3.1–5) satisfies (3.1, 10); and if $\hat{F}_{ab}$ obeys (3.1, 10) and $\hat{u}^a, \hat{n}$ are defined by (3.8, 9), then (3.1–5) holds.

The reduced system (3.1, 10) consists of seven partial differential equations for the six components of the Faraday tensor $\hat{F}^{ab}$. Only six of these equations are evolution equations; the space-part of (3.1) is a constraint. To get rid of this constraint and to reduce further the number of equations and unknowns we introduce a potential $\hat{A}_a$ for $\hat{F}_{ab}$,

$$\hat{F}_{ab} = 2\nabla_{[a} \hat{A}_{b]}, \quad (3.11)$$

and impose on it the *Landau gauge condition*

$$u^a \hat{A}_a = 0, \quad (3.12)$$

which can be done without loss of generality. The remaining gauge freedom, $\hat{A}_a \rightarrow \hat{A}_a + \nabla_a \Lambda$, is restricted by the condition $u^a \nabla_a \Lambda = 0$ which says that Λ is constant on the unperturbed electron world lines. The gauge-freedom associated with (3.12) is, therefore, *less* than that associated with the Lorentz gauge. Landau gauge initial data are determined, for a given field $\hat{F}_{ab}$, up to only *one* arbitrary function on the initial hypersurface.

If (3.11) is inserted into (2.10) and the gauge condition (3.12) is used to simplify some terms, the equation

$$\mathrm{D}^b_a \hat{A}_b := \{h^{ac} u^d \nabla_d (\nabla^b_c - \delta^b_c \nabla^e_e) + (\omega^{ac} + \omega_{\mathrm{L}}{}^{ac} + \theta^{ac} + \theta h^{ac} + (e/m) E^a u^c)(\nabla^b_c - \delta^b_c \nabla^e_e) + \omega_p^2 h^{ab} u^d \nabla_d + \omega_p^2 (\theta^{ab} - \omega^{ab})\} \hat{A}_b = 0 \quad (3.13)$$

emerges.

In order to satisfy the gauge condition (3.12) identically and to specify (3.13) as a system of differential equations completely we choose, once and for all, an orthonormal tetrad field $(E_{(a)}^{\ b})$ where $E_{(o)}^{\ a} = u^a$ is the background electron velocity. Denote the dual basis field as $(E^{(a)}_{\ b})$, and write $\hat{A}_a = \hat{A}_{(\lambda)} E^{(\lambda)}_{\ a}$. This expression is to be inserted into (3.13) and the three real functions $\hat{A}_{(\lambda)}$ of the space–time coordinates are to be considered as the (unconstrained) unknown functions. Although we shall not, in the following, use such a tetrad explicitly, we always assume such a representation and take the operator D^b_a in (3.13) to be a map from a space of fields obeying (3.12) into another such space.

The linear, third-order, 3×3-system of partial differential equations (3.13) is hyperbolic, but not strictly hyperbolic; it has multiple characteristics. In spite of this we can prove on the basis of §1

THEOREM 2. *Given a set*

$$\hat{A}_{(\alpha)}, \partial_0 \hat{A}_{(\alpha)}, \partial_{00} \hat{A}_{(\alpha)} \quad (3.14)$$

(*or, equivalently*,

$$\hat{A}_a, \nabla_b \hat{A}_a, \nabla_{cb} \hat{A}_a) \quad (3.15)$$

of initial data on a spacelike hypersurface Σ, there exists locally a unique solution of (3.13) *having these initial data.*

Proof. From initial data (3.15) one can find uniquely, on Σ, values for $\hat{F}_{ab}$ and $\nabla_b \hat{F}^{ab}$, by restricting (3.11) and its derivative to Σ. These data provide via (3.8) and (3.9), values for $\hat{u}^a$ and $\hat{n}$ on Σ. By construction, the functions $\hat{F}_{ab}$, $\hat{u}^a$, $\hat{n}$ thus obtained form an initial data set for the system (3.1–5) which obeys the appropriate constraints. The corresponding solution of (3.1–5) (which exists according to the corollary at the beginning of this section) can be represented in the form (3.11) with (3.12), the potential $\hat{A}_a$ then satisfies (3.13). If one exploits the initial-gauge transformations permitted by (3.12) by taking covariant derivatives of $u^a \nabla_a \Lambda = 0$ and restricting the results to Σ, one verifies that one can always gauge-transform $\hat{A}_a$ such that, on Σ, it takes those initial values (3.15) with which we began the proof. This finishes the existence-proof.

To prove uniqueness it is sufficient, because of the linearity of (3.13), to show that a solution with vanishing initial data (3.15) vanishes everywhere. Let, therefore, $\hat{A}_a$ be a solution with initial zero data. It determines a solution of (3.1–5) with zero initial data, hence, because of the uniqueness of that system, $\hat{F}_{ab} = 2\nabla_{[a}\hat{A}_{b]} = 0$ everywhere. Hence $\hat{A}_a = \nabla_a \Lambda$, with $u^a \nabla_a \Lambda = 0$. The vanishing of the functions (3.15) on Σ then implies $\Lambda|_\Sigma =$ const., which in turn implies $\Lambda =$ const. and therefore $\hat{A}_a = 0$. Q.E.D.

The point of the preceding argument was to establish existence and uniqueness of a non-strictly hyperbolic system with multiple characteristics by relating it to a Leray-system.

Note that the set (3.14) consists of nine functions. Since the initial gauge transformations allowed by (3.12) are determined by a single function, we have again eight physically essential functions, as it should be.

We have now established (3.13) as the *basic equation governing the perturbations* of the original system (2.1–5). This equation contains the influence of the gravitational field as well as that of the (in general inhomogeneous, moving) plasma and the background electromagnetic fields on the 'wave' $\hat{F}_{ab}$, via the covariant derivatives, the various matter terms, and the ${\omega_{\mathrm{L}}}^{ab}$- and E^a-contributions, respectively. Our plasma-model does not exclude charge-separations, so there may be 'electrostatic' fields.

Since the operator D^{ab} in (3.13) is linear and its coefficients are real-valued, we may determine its physical (real) solutions as real parts of complex-valued solutions, as usual.

4. Oscillatory asymptotic solutions as approximations to high-frequency solutions of linear, hyperbolic systems of partial differential equations†

Exact solutions of complicated equations such as (3.13) are rarely available. However, approximate high-frequency solutions may be constructed by means of W.K.B.-methods. Before doing this for (3.13) we review one such method for a general class of equations, for two reasons. First, we need an *algorithm* for obtaining oscillatory asymptotic solutions of linear systems of partial differential equations of higher than first order, the operator of which depends polynomially on a small parameter. Such an equation will be obtained in part II from (3.13) by introducing dimensionless variables, in order to be able to treat efficiently dispersive properties. The published presentations† of such formalisms of which we are aware are not sufficiently general to cover equations such as (3.13). It appears therefore to be of some interest to outline the generalized version in spite of its straightforward nature.

Secondly, we wish to consider briefly the question of *justifying* this algorithm as an approximation method. This is necessary since (3.13) is neither strictly hyperbolic nor, as far as we know, equivalent to a symmetric-hyperbolic system, so that standard results about error estimates do not apply. We are not able to provide a general justification, but we state a sufficient condition under which this gap could be closed.

Let

$$\mathrm{D}(\epsilon) := \sum_{j=0}^{p} \epsilon^j \sum_{k=0}^{j} A_k^{j-k} \tag{4.1}$$

be a linear differential operator depending on a real parameter ϵ, in which A_k^r denotes an $(m \times m)$ matrix the elements of which are forms of degree r in the partial differential operators $\partial_a = \partial/\partial x^a$ $(a = 1, \ldots, n)$ with smooth, real coefficients depending on $x \in \mathbb{R}^n$. $\mathrm{D}(\epsilon)$ maps $\mathbb{R}^m$-valued functions on $\mathbb{R}^n$ into functions of the same kind. Thus

$$\mathrm{D}(\epsilon) \cdot U(\epsilon) = 0 \tag{4.2}$$

is a parameter-dependent linear, homogeneous partial differential equation of order p for an m-vector U on $\mathbb{R}^n$. (In the subsequent paper (3.13) will be transformed locally into such an equation with $p = 3$, $m = 3$, $n = 4$.) For convenience we study complex-valued solutions of (4.2), taking real parts at the end.

We want to obtain asymptotic solutions of (4.2) of the form

$$U(x, \epsilon) = \mathrm{e}^{(\mathrm{i}/\epsilon) S(x)} V(x, \epsilon) \underset{\epsilon \to 0}{\simeq} \mathrm{e}^{(\mathrm{i}/\epsilon) S(x)} \sum_{n=0}^{\infty} (\epsilon/\mathrm{i})^n V_n(x) \tag{4.3}$$

where the *eiconal* S is a $\mathbb{C}$-valued function and the *amplitudes* V, V_n are $\mathbb{C}^m$-valued on $\mathbb{R}^n$.

† For this section compare, for example, Courant & Hilbert (1962); Duistermaat (1974) Lax (1957); the treatment given here is a straightforward generalization of those given in these references.

We first describe an *algorithm* for determining S and the V_n. We write d: $= (\partial_1, \ldots, \partial_n)$ for the gradient operator and put

$$l := \mathrm{d}S. \tag{4.4}$$

By means of the operator identity

$$\mathrm{e}^{(-\mathrm{i}/\epsilon)S}\,\mathrm{d}\mathrm{e}^{(\mathrm{i}/\epsilon)S} = \mathrm{d} + (\mathrm{i}/\epsilon)\,l \tag{4.5}$$

the equation $\mathrm{D}(\mathrm{e}^{(\mathrm{i}/\epsilon)S}V) = 0$ is transformed into $L \cdot V = 0$ where the coefficients of $L(\epsilon) = \mathrm{e}^{(-\mathrm{i}/\epsilon)S}\,\mathrm{D}(\epsilon)\,\mathrm{e}^{(\mathrm{i}/\epsilon)S}$ contain the wave covector l. Application of (4.5) to (4.1) gives

$$L(\epsilon) = \sum_{j=0}^{p} \epsilon^j L_j, \tag{4.6}$$

when L_j is a differential operator of order j. L_0, the *principal matrix* of $\mathrm{D}(\epsilon)$, is obtained by substituting ∂_a in $\Sigma_{l=0}^{p} A_0^l$ by $\mathrm{i}l_a = \mathrm{i}\partial_a S$. Thus L_0 is an $(m \times m)$ numerical matrix, with complex elements even if $\mathrm{D}(\epsilon)$ has real coefficients, depending on x and, polynomially, on the l_a. The substitution rule (4.5) also shows that

$$L_1 = \left(\frac{\partial}{\partial l_a} L_0\right) \frac{1}{\mathrm{i}}\,\partial_a + L_1', \tag{4.7}$$

where L_1' is obtained by substituting ∂_a in $\Sigma_{l=0}^{p} A_1^l$ by $\mathrm{i}l_a$. This L_1 is a linear first-order differential operator, the principal part of which is determined by L_0.

Inserting (4.3) into (4.2) and using (4.5) one obtains formally the asymptotic equation

$$\sum_{j=0}^{p} \epsilon^j L_j \cdot \sum_{n=0}^{\infty} (\epsilon/\mathrm{i})^n V_n \sim 0 \quad \text{for} \quad \epsilon \to 0. \tag{4.8}$$

In zeroth order this equation requires

$$L_0 \cdot V_0 = 0, \tag{4.9}$$

which in turn demands

$$\det L_0 = 0 \tag{4.10}$$

to be solvable nontrivially.

Det L_0, the *principal polynomial* of $\mathrm{D}(\epsilon)$, is a polynomial of degree mp in the l_a, with coefficients depending on x. Henceforth *we shall assume this polynomial to have real coefficients* although L_0 may be complex. (This will indeed turn out to be the case for the transformed version of (3.13), and it holds also for other equations of mathematical physics.) The *dispersion relation* (4.10) then defines an (in general many-branched) *principal variety* in $\mathbb{R}^{2n}$, to be considered here as the cotangent bundle of $\mathbb{R}^n$. Its branches may intersect or coalesce.

Equation (4.10) imposes on S a first-order partial differential equation of degree mp, the *eiconal equation*.

Suppose $H(x, l)$ is a real factor of the principal polynomial corresponding to one or several branches of the principal variety. Then one can construct, according to Monge–Hamilton–Jacobi, the real solutions of the (reduced) eiconal equation

$$H(x, \mathrm{d}S) = 0 \tag{4.11}$$

from the characteristic strips of (4.11) in $\mathbb{R}^{2n}$ which obey the *canonical equations*

$$\dot{x} = \frac{\partial H}{\partial l}, \quad \dot{l} = -\frac{\partial H}{\partial x} \tag{4.12}$$

and the constraint

$$H(x, l) = 0. \tag{4.13}$$

This constraint restricts the initial values and is preserved by (4.12). The projections of these strips into $\mathbb{R}^n$ are the *rays*; the values of l on a ray determine the infinitesimal pieces of the phase-hypersurfaces $S = \text{const}$ at the points of the ray.

The dispersion relation (3.13) and the canonical equations (4.12) imply, at least if $\partial H/\partial l \neq 0$ on a ray, that the ray-velocity with respect to any observer equals the phase velocity.

The restriction of $\partial H/\partial l$ to a solution S of the eiconal equation (4.11) is a vector field T on $\mathbb{R}^n$ which is called the *transport vector* associated with S, for reasons to be explained below.

On a branch of the principal variety where the principal matrix has constant rank $r < m$ there exist smooth basis fields of $(m-r)$ linearly independent left null vectors and right null vectors of L_0. Here we restrict attention to a nondegenerate branch, $r = m-1$, and write

$$N \cdot L_0 = 0, \quad L_0 \cdot R = 0 \tag{4.14}$$

for a left- and a right-null vector, respectively. R represents the *polarization state* of the branch or 'mode' considered, compare (4.16) below.

By varying (4.14) at a fixed point x with respect to l one obtains

$$L_0 \cdot \mathrm{d}R + (\partial L_0/\partial l_a) \cdot R \, \mathrm{d}l_a = 0,$$

$$N \cdot (\partial L_0/\partial l_a) \cdot R \, \mathrm{d}l_a = 0.$$

If the branch belongs to the factor H of $\det L_0$, then the last equation holds whenever $(\partial H/\partial l_a = 0$; hence if on that branch $\partial H/\partial l \neq 0$ then

$$N \cdot \left(\frac{\partial}{\partial l} L_0\right) \cdot R = \lambda \frac{\partial H}{\partial l} \tag{4.15}$$

where λ is some scalar. If S is a corresponding solution of the eiconal equation and (4.15) is restricted to the section of $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ determined by S, the right-hand side becomes λT, T being the transport vector defined above.

We now return to the evaluation of (4.8). Suppose we have found a system of rays and a real eiconal S by integrating Hamilton's equations (4.12), belonging to a nondegenerate branch, and assume we have computed N and R so that (4.14, 15) hold. Then the amplitude condition (4.9) requires

$$V_0 = v_0 R \tag{4.16}$$

with an as yet unrestricted complex scalar amplitude v_0. This is all the information obtained in the first step. It suffices to determine all coefficients of the operator (4.6).

In first order in ϵ, (4.8) demands

$$L_0 \cdot V_1 + \mathrm{i} L_1 \cdot V_0 = 0. \tag{4.17}$$

Using (4.14, 16, 7, 15), recalling the definition of T and assuming $\lambda \neq 0$, we get from the last equation the *transport equation*

$$T^a \partial_a v_0 + f v_0 = 0, \tag{4.18}$$

where f is known from the first step. (Both T^a and f depend on the choice of N and R in a neighbourhood of the ray considered. In applications one chooses R such that it represents unit amplitude or unit intensity so that v_0 measures the magnitude of the wave.) This linear, homogeneous equation determines the amplitude v_0 along each ray via initial conditions and, by (4.16), the leading contribution V_0 to the amplitude V. The initial conditions for v_0 determine the *profile* of the wave. If the initial value of v_0 has compact support, V_0 will be non-zero only in a spatially compact ray bundle; this case corresponds to a pulse-wave.

Once V_0 has thus been determined, (4.17) is solvable for V_1 since $N \cdot (iL_1 \cdot V_0) = 0$. If R_1 denotes one particular solution of

$$L_0 \cdot R_1 + \mathrm{i} L_1 \cdot V_0 = 0, \tag{4.19}$$

the general solution of (4.17) is

$$V_1 = R_1 + v_1 R \quad (v_1 \in \mathbb{C}). \tag{4.20}$$

After the second step one therefore knows $\mathrm{e}^{(\mathrm{i}/\epsilon)S}\, V_0$ everywhere, and one knows the form $\mathrm{e}^{(\mathrm{i}/\epsilon)S}([v_0 - \mathrm{i}\epsilon v_1] R - \mathrm{i}\epsilon R_1)$ of U in (4.3) to first order, but not yet the value of v_1.

If the coefficients of $\mathrm{D}(\epsilon)$ and the initial values of $S, v_0, v_1, \dots$ are C^∞ or if one takes derivatives in a distributional sense, this procedure can be continued indefinitely. The higher-order amplitudes $v_1, v_2, \dots$ are determined by initial values and *inhomogeneous* transport equations of the form

$$T^a \partial_a v_j + f_j(v_0, \dots v_{j-1})\, v_j + g_j(v_0, \dots v_{j-1}) = 0, \quad j \geqslant 1. \tag{4.21}$$

They all propagate along the rays. This justifies the name *transport vector* of T.

After $(n+1)$ steps one knows

$$U^{(n-1)} := \mathrm{e}^{(\mathrm{i}/\epsilon)S} \sum_{j=0}^{n=1} \left(\frac{\epsilon}{\mathrm{i}}\right)^j V_j$$

as well as the form of $U^{(n)}$, but not v_n.

Note that it is the set $(A_0^p, A_0^{p-1}, \dots, A_0^1)$ of the principal parts of the terms of the original operator D in (4.1) which determines the dispersion relation, the eiconal equation and the ray-congruences along which the amplitude are transported. The lower-order parts of D determine the coefficients f, f_j, g_j of the transport equations.

According to the construction of the series (4.3) one has

$$\mathrm{D}(\epsilon)\, U^{(n)}(\epsilon) = O(\epsilon^{n+1}), \qquad n = 0, 1, \dots, \tag{4.22}$$

where the order-symbol is to be understood uniformly in any compact part of $\mathbb{R}^n$ in which $U^{(n)}$ has been determined, i.e. *$U^{(n)}$ is an asymptotic solution of order n of* (4.2).

The last statement does not imply that $U^{(n)}$ actually approximates a solution of (4.2). So far, we did not assume hyperbolicity of (4.2); all we needed was that the principal variety $\det L_0 = 0$ contains some n-dimensional submanifolds. Let us now *assume*, however, that (4.2) is hyperbolic in the (most general) sense that it has a unique and sufficiently differentiable solution for any set of initial data depending smoothly on the data, a property which we did establish for (3.13) in theorem 2. Then, given a spacelike hypersurface Σ, the restrictions of

$$U^{(n)}(\epsilon),\ \partial U^{(n)}(\epsilon), \ldots, \partial^{p-1} U^{(n)}(\epsilon)$$

to Σ define a one-parameter family of initial data for (4.2). Let $U(\epsilon)$ be the family of (exact) solutions of (4.2) belonging to these initial data, and let

$$E^{(n)}(\epsilon) := U(\epsilon) - U^{(n)}(\epsilon) \tag{4.23}$$

be the deviation of U from $U^{(n)}$, the 'error'. Then, by (4.2) and (4.22)

$$\mathrm{D}(\epsilon)\, E^{(n)}(\epsilon) = O(\epsilon^{n+1}). \tag{4.24}$$

Also, by construction, $E^{(n)}(\epsilon)$ has zero initial data on Σ. Therefore, an estimate of an absolute magnitude (i.e. a suitable norm or seminorm on the space of W's) of the solution $W(\epsilon)$ of the inhomogeneous equation

$$\mathrm{D}(\epsilon)\, W(\epsilon) = Q(\epsilon) \tag{4.25}$$

with zero initial data on Σ in terms of the magnitude of the source $Q(\epsilon)$ gives an estimate of the error $E^{(n)}(\epsilon)$. If, as the form of (4.1) and simple examples suggest, (4.25) implies

$$|W(\epsilon)| \leqslant \epsilon^{-p} \cdot |Q(\epsilon)| \cdot x^0 \cdot \text{const}, \tag{4.26}$$

where x^0 is the 'time distance' from the initial hypersurface $x^0 = 0$, then (4.24) implies for a compact domain

$$E^{(n)}(\epsilon) = O(\epsilon^{n+1-p}). \tag{4.27}$$

Moreover, then

$$|U - U^{(n-p)}| = |U - U^{(n)} + O(\epsilon^{n+1-p})| = O(\epsilon^{n+1-p}). \tag{4.28}$$

That is, *if the operator* $\mathrm{D}(\epsilon)$ *is such that* (4.25) *implies* (4.26), *then*

$$|U - U^{(n)}| = O(\epsilon^{n+1}),\ n = 0, 1, 2, \ldots. \tag{4.29}$$

In particular, under this assumption the lowest order approximation, $U^{(0)} = \mathrm{e}^{(i/\epsilon)S} v_0 R = \mathrm{e}^{(i/\epsilon)S} V_0$, with v_0 obeying (4.18), is an *approximate solution* to (4.2) with an error of order ϵ.

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Appendix A. Notation

We assume a simply connected background space–time $(M, g_{ab}(x^c))$ with a Lorentzian metric g_{ab}; signature $(-+++)$; Latin indices range from 0 to 3. Units are chosen such that $c = 1$. Then we use the following notation:

∇_a	covariant derivative with respect to x^a;
∇_{ab}	second covariant derivative $\nabla_{ab} := \nabla_a \nabla_b$
ϵ	small parameter ($\ll 1$) of the order of the ratio between the wavelength of the perturbation and a macroscopic scale of the gravitational field;

$x \cdot y = g_{ab} x^a y^b$	inner product of two vectors x^a and y^a;
η_{abcd}	volume element, with $\eta_{0123} = +1$ in an oriented orthonormal frame.

On the gravitational background we assume to have a plasma, not influencing the metric, described by the following quantities:

$n, u^a,$	number density and four-velocity of electrons;
J^a	ion current density;
e, m	(negative) electron charge, electron mass;
$\omega_p^2 = e^2 n/m$	square of plasma frequency;
$D_u = u^a D_a$	derivation along u^a;
$\nabla_b u^a = \theta^a_b + \omega^a_b - \nabla_u(u^a) u_b$	decomposition of electron velocity gradient;
θ^a_b	deformation velocity;
$\theta := \theta^a_a = \nabla_a u^a$	expansion rate;
ω^a_b	angular velocity (vorticity);
$h^a_b = \delta^a_b + u^a u_b$	projector on to three-space orthogonal to u^a;
F^{ab}	unperturbed electromagnetic field;
$E^a = F^{ab} u_b$	unperturbed electric field;
$B_{ab} = h^c_a h^d_b F_{cd}$	unperturbedmagneticfield; $= \frac{1}{2}\eta_{abcd} B^c u^d, B = (B_a B^a)^{\frac{1}{2}}$;
${\omega_L}^a{}_b = -(e/m) B^a_b$	Larmor angular velocity;
$\omega_L = -(e/m) B$	Larmor frequency;
F_{ab}, u^a, n	perturbations of F_{ab}, u^a, n.

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Non-Linear Dispersive Waves

G. B. Whitham

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