ON A NEW CHARACTERIZATION OF SCALAR SUPERSYMMETRIC THEORIES

Hermann NICOLAI
CERN, Geneva, Switzerland

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It is shown that scalar and possibly other supersymmetric theories are characterized by the existence of a transformation of the Bose fields with the property that the image of the full bosonic action is the free action and the Jacobi-determinant of the transformation equals the Matthews–Salam–Seiler determinant of the model. Some possible implications of this result are discussed.

It is well known that every formulation of supersymmetry so far has employed the concepts of anticommuting numbers (Grassman algebras) and graded Lie algebras in an essential way (for a general review as well as comprehensive references, see ref. [1]). In this note, we want to point out another feature peculiar to supersymmetric models which in itself may serve as a new characterization of supersymmetry that altogether avoids these concepts. In addition, we hope that this property will not only shed more light on some of the unresolved mysteries of supersymmetry but also eventually facilitate the non-perturbative construction of supersymmetric models. Although we will only be concerned with ordinary "scalar" supersymmetric theories, i.e., spin-0 and spin-1/2 multiplets, we believe that the general principle expounded below retains its validity under more general circumstances since the arguments are sufficiently general so as to be extendable to more sophisticated examples.

Let us suppose that the multiplet contains some scalar (or pseudoscalar) fields A_i and some Majorana spinors ψ_i; auxiliary fields, if originally present, are assumed to have been eliminated. The number of dimensions is arbitrary, but the multiplet structure will, of course, depend on the dimension. Expectation values are defined via a functional integral; to define it properly, a euclidean metric is assumed (in addition to some regularization to be specified). In ref. [2], the euclideanization prescription for relativistic scalar supersymmetric models has been given.

The main result of this investigation may then be stated as follows.

Theorem: "Scalar" supersymmetric models are characterized by the existence of a generally non-linear and non-local transformation T_λ [‡1] of the Bose fields A_i

$$T_\lambda : A_i(x) \to A_i'(x, \lambda; A), \tag{1}$$

with the following properties:

(i) T_λ is invertible at least in the sense of formal power series.

(ii) $S(\lambda; A) = S_0(A'(\lambda; A))$, where S denotes the full boson action and S_0 its free part.

(iii) The Jacobi determinant of the transformation T_λ equals the Matthews–Salam–Seiler (MSS) determinant [3] obtained upon integrating out the fermions (à la Berezin [4]; if the spinors are self-conjugate, i.e., Majorana spinors, the MSS determinant is actually the square-root of a determinant).

The rest of this paper is devoted to a proof of this theorem and to a discussion of its possible consequences and applications; an explicit example will be given at the end of this paper. Before proceeding to the proof, let us clarify the significance of the theorem in the simplest case where there is only one scalar field A. If $D(\lambda; A)$ denotes

‡1 λ stands for one or more coupling parameters.

the MSS determinant, expectation values of scalar fields are formally the moments of the functional measure

$$d\mu_\lambda(A) = \exp[-S(\lambda; A)]\, D(\lambda; A)\, dA \Big/ \int \exp[-S(\lambda; A)]\, D(\lambda; A)\, dA. \tag{2}$$

The theorem implies, that for supersymmetric theories there exists a transformation T_λ such that

$$d\mu_\lambda(T_\lambda^{-1} A) = d\mu_0(A) = \exp[-S_0(A)]\, dA \Big/ \int \exp[-S_0(A)]\, dA \tag{3}$$

and the moments of the evidently non-gaussian measure $d\mu_\lambda$ have been reexpressed in terms of transforms of moments of a gaussian, i.e., free measure! A little thinking shows that this does, of course, *not* mean that supersymmetric models are trivial since the interactions now reside in the (complicated) transformation T_λ. As a further consequence, we infer from the theorem that the vacuum energy has to vanish to all orders in λ, so it is obvious that, for theories with non-vanishing vacuum energy, no transformation with properties (i), (ii), (iii) can exist. Conversely, Zumino has established that for supersymmetric theories, the vacuum energy vanishes identically in λ [5] – if supersymmetry is not exact as it is the case when the interaction is turned on in a finite volume only this is only true up to residual terms such as surface terms which (hopefully) disappear as the symmetry breaking is removed. We will now demonstrate that the vanishing of the vacuum energy is also sufficient to ensure the existence of the transformation T_λ. In accordance with the constructivists' jargon [6], we will call the measure (3) the free measure of covariance C where $C(x)$ is the inverse of the differential operator appearing in the quadratic action $S_0(A)$; usually, $C = (-\Delta + m^2)^{-1}$ but the case of higher derivative regulators as in ref. [7] is also included. The measure $d\mu_0(A)$ lives on $\mathbf{R}^n$ in the case of a lattice regularization or on a distribution space which we may take to be $\mathcal{S}'(\mathbf{R}^d)$ (Minlos' theorem, see, e.g., ref. [6]) whatever the case may be. In order to keep the notation as simple as possible, the appropriate regularization will not be specified in the lemma below; furthermore, it will be proven only for the case of one scalar field, the generalization to more complicated cases being immediate.

Lemma: Let $d\mu_0(A)$ be the free measure of covariance C and $F(\lambda; A)$ a functional of A with an asymptotic expansion in λ and vanishing identically for $\lambda = 0$. If moreover

$$\int \exp[-F(\lambda; A)]\, d\mu_0(A) = 1, \tag{4}$$

for all $\lambda \geqslant 0$, there exists a generally non-linear and non-local transformation of the boson fields $A(x) \to A'(x, \lambda; A)$, invertible at least in the sense of formal power series such that

$$d\mu_0(A(\lambda; A'))\, \exp[-F(\lambda; A(\lambda; A'))] = d\mu_0(A'). \tag{5}$$

Proof: We expand

$$F(\lambda; A) = \sum_{\nu=1}^{n} \lambda^\nu F_\nu(A) + O(\lambda^{n+1}) \tag{6}$$

and show by induction on n that the terms F_n can be transformed away order by order. For $n = 0$, there is nothing to prove since the transformation is the identity. Now, assume that all terms up to some order $n - 1 \geqslant 0$ have been eliminated by successive transformations. Thus, we have

$$\exp[-F(\lambda; A)]\, d\mu_0(A) = \exp\left[-\lambda^n \sum_{\nu=0}^{N(n)} \int dx_1 \ldots dx_\nu R_n^{(\nu)}(x_1, \ldots, x_\nu) A'(x_1) \ldots A'(x_\nu) - O(\lambda^{n+1})\right] d\mu_0(A') \quad (n \geqslant 1), \tag{7}$$

for some $A'(x) = A'(x, \lambda; A)$. Note that, without loss of generality, the coefficient functions $R_n^{(\nu)}$ are assumed to be totally symmetric in their arguments; some of them may be identically zero. From eq. (4), the symmetry of

$R_n^{(\nu)}$, and from the fact that the expansion in λ is asymptotic, it follows that

$$0 = \sum_{\nu=0}^{N(n)} \int d\mu_0(A') \int dx_1 \dots dx_\nu R_n^{(\nu)}(x_1, \dots, x_\nu) A'(x_1) \dots A'(x_\nu)$$
$$= \sum_{\nu=0}^{[N(n)/2]} \frac{(2\nu)!}{2^\nu \nu!} \int dx_1 \dots dx_{2\nu} R_n^{(2\nu)}(x_1, \dots, x_{2\nu}) C(x_1 - x_2) \dots C(x_{2\nu-1} - x_{2\nu}). \quad (8)^{‡2}$$

There is no restriction on $R_n^{(\nu)}$ with ν odd. We now define

$$A''(x, \lambda; A') = A'(x) + \lambda^n \int dx_1 C(x - x_1) R_n^{(1)}(x_1)$$
$$+ \lambda^n \sum_{\nu=2}^{N(n)} \int dx_1 \dots dx_\nu C(x - x_1) R_n^{(\nu)}(x_1, \dots, x_\nu) A'(x_2) \dots A'(x_\nu). \quad (9)$$

Hence,

$$\frac{1}{2} \int dx_1 dx_2 A'(x_1) C^{-1}(x_1 - x_2) A'(x_2) = \frac{1}{2} \int dx_1 dx_2 A''(x_1) C^{-1}(x_1 - x_2) A''(x_2)$$
$$- \lambda^n \sum_{\nu=1}^{N(n)} \int dx_1 \dots dx_\nu R_n^{(\nu)}(x_1, \dots, x_\nu) A''(x_1) \dots A''(x_\nu) + O(\lambda^{2n}) \quad (10)$$

and the unwanted terms are cancelled with the exception of $R_n^{(0)}$. However, we get a new $O(\lambda^n)$ contribution from the Jacobi determinant of the transformation

$$\det \delta A'(x, \lambda; A'')/\delta A''(y) = \exp \operatorname{Tr} \log \delta A'(x, \lambda; A'')/\delta A''(y)$$
$$= \exp\left[-\lambda^n \sum_{\nu=2}^{N(n)} (\nu - 1) \int dx_1 \dots dx_\nu C(x_1 - x_2) R_n^{(\nu)}(x_1, \dots, x_\nu) A''(x_3) \dots A''(x_\nu) - O(\lambda^{2n})\right]. \quad (11)$$

Observe that the first $O(\lambda^n)$ term in eq. (9) drops out upon differentiation and that we have again used the symmetry of $R_n^{(\nu)}$. Setting

$$A'''(x, \lambda; A'') = A''(x) + 2\lambda^n \int dx_1 dx_2 dx_3 C(x - x_1) C(x_2 - x_3) R_n^{(3)}(x_1, x_2, x_3)$$
$$+ \lambda^n \sum_{\nu=4}^{N(n)} (\nu - 1) \int dx_1 \dots dx_\nu C(x - x_1) C(x_2 - x_3) R_n^{(\nu)}(x_1, \dots, x_\nu) A''(x_4) \dots A''(x_\nu) \quad (12)$$

and repeating the procedure described before, one easily verifies that all unwanted terms are cancelled up to

$$\lambda^n \left[R_n^{(0)} + \int dx_1 dx_2 C(x_1 - x_2) R_n^{(2)}(x_1, x_2)\right], \quad (13)$$

but that new ones arise from the Jacobi-determinant. Continuing in this manner and collecting the uncancelled terms, we arrive after a finite number of steps at the following expression

$$\exp\left[-\lambda^n \sum_{\nu=0}^{[N(n)/2]} \frac{(2\nu)!}{2^\nu \nu!} \int dx_1 \dots dx_{2\nu} R_n^{(2\nu)}(x_1, \dots, x_{2\nu}) C(x_1 - x_2) \dots C(x_{2\nu-1} - x_{2\nu}) - O(\lambda^{2n})\right], \quad (14)$$

‡2 $[N(n)/2]$ denotes the smallest integer $\leqslant N(n)/2$.

which is exp $O(\lambda^{2n})$ on account of eq. (8). Together with eq. (7) this implies that the remaining terms are exp $O(\lambda^{n+1})$; since the transformation is always the identity in zeroth order in λ, it can be inverted in the sense of formal series. This concludes the proof of the lemma.

The theorem now follows from the fact already mentioned, namely that in supersymmetric theories, the vacuum energy is identically zero in λ and upon using the formula $(\det)^{1/2} = \exp \frac{1}{2}\mathrm{Tr}\log$ for the MSS determinant to define the functional $F(\lambda; A)$.

What we find most remarkable about the result is that it allows us to reformulate the problem of defining the renormalized measure (2) in terms of the transformation T_λ. For instance, if some UV regularized version of eq. (2) with UV cut-off κ has been realized on, say, $\mathcal{S}'(\mathbf{R}^d)$, then $T_\lambda = T_\lambda(\kappa)$ is a non-linear and non-local map of the distribution space into itself. Is it possible, then, to "renormalize" the map $T_\lambda(\kappa)$ as κ tends to infinity and is the limit still well-defined on $\mathcal{S}'(\mathbf{R}^d)$ such that the products $\Pi(T_\lambda^{-1}A)(f_i), f_i \in \mathcal{S}(\mathbf{R}^d)$ are $\mathrm{d}\mu_0$ measurable? Another interesting question is whether the maps $T_\lambda(\kappa)$ are injective; we conjecture that they are in case of a one-well scalar field potential and that special precautions have to be taken in presence of more than one absolute minimum – for an illustrative example, see below. As the inductive construction in the proof of the foregoing lemma is only order by order there also remains the important question of whether the series representation for T_λ is actually convergent, Borel summable [8] or only asymptotic.

As an application of our result, we will now deduce the most general supersymmetric lagrangian with one scalar field A and one Majorana spinor ψ in two dimensions [9] by requiring the MSS determinant to be a Jacobi determinant in two limiting cases [‡3]; the derivation will be formal but it can be easily made rigorous (on a lattice, for instance). We start with a free lagrangian for A

$$\mathcal{L} = \tfrac{1}{2}Z(\partial_\mu A)^2 + (m^2/2Z)A^2 \quad (Z > 0). \tag{15}$$

From this lagrangian, we get formally (normalization factors have been inserted accordingly for convenience)

$$\int \prod_x \frac{\mathrm{d}A(x)}{\sqrt{2\pi Z}} \exp\left[-\tfrac{1}{2}Z \int \mathrm{d}x\, A(x)(-\Delta + m^2/Z^2)A(x)\right] = \det(-Z^2\Delta + m^2)^{-1/2}. \tag{16}$$

This is cancelled if we add to the lagrangian the free lagrangian of a two-component Majorana spinor ψ

$$\mathcal{L} \to \mathcal{L} + \tfrac{1}{2}\bar{\psi}(Z\partial\!\!\!/ + m)\psi \quad (\psi = \varphi\bar{\psi}^{\mathrm{T}}), \tag{17}$$

because [4]

$$\int \prod_x \prod_{\alpha=1,2} \mathrm{d}\psi_\alpha(x) \exp\left[-\frac{1}{2}\int \mathrm{d}x\, \bar{\psi}(x)(Z\partial\!\!\!/ + m)\,\psi(x)\right] = \det(Z\partial\!\!\!/ + m)^{1/2} = \det(-Z^2\Delta + m^2)^{1/2}. \tag{18}$$

By the same argument we would have found the multiplet in four dimensions: a four-component Majorana-spinor ψ and *two* scalar fields A and B refs [5,11]. To reconstruct the interaction, we replace (15) by the ansatz

$$\mathcal{L} = \tfrac{1}{2}Z(\partial_\mu A)^2 + \tfrac{1}{2}(1/Z)V(A), \quad V(A) = m^2A^2 + \ldots, \quad V(A) \geqslant 0 \tag{19}$$

and the fermionic part of the lagrangian by

$$\mathcal{L}_{\mathrm{F}} = \tfrac{1}{2}Z\bar{\psi}\partial\!\!\!/\psi + \tfrac{1}{2}g(A)\bar{\psi}\psi, \quad g(A) = m + \ldots. \tag{20}$$

In the limit $Z \to 0$, the kinetic terms may be dropped, and we have

$$\lim_{Z\to 0} \frac{1}{\sqrt{2\pi Z}} \exp[-(1/2Z)V(A)] = \delta(\sqrt{V(A)}) = \sum_i \frac{1}{|(\sqrt{V(A)})'|}\,\delta(A - A^{(i)}), \quad V(A^{(i)}) = 0. \tag{21}$$

Moreover [4]

‡3 See also ref. [10], sections 3 and 4.

$$\int \prod_x \prod_{\alpha=1,2} \mathrm{d}\psi_\alpha(x) \exp\left(-\frac{1}{2}\int \mathrm{d}x\, g(A)\bar{\psi}\psi\right) = \prod_x g(A(x)). \tag{22}$$

Defining $p(A) = V(A)^{1/2}$, our requirement yields

$$|p'(A)| = g(A) \geqslant 0, \quad p(A) = c + mA + \lambda A^2 + \dots . \tag{23}$$

It should be clear from this derivation that in order to stay out of trouble we must demand $p'(A) > 0$ (or < 0) for all A; the local but non-linear map $A \to p(A)$ can only be inverted if the potential has just one absolute minimum at $V = 0$ [‡4]. The full lagrangian is now given by

$$\mathcal{L} = \tfrac{1}{2}Z(\partial_\mu A)^2 + \tfrac{1}{2}(1/Z)(p(A))^2 + \tfrac{1}{2}Z\bar{\psi}\partial\!\!\!/\psi + \tfrac{1}{2}p'(A)\bar{\psi}\psi. \tag{24}$$

It is equivalent to

$$\mathcal{L} = \tfrac{1}{2}Z[(\partial_\mu A)^2 + \bar{\psi}\partial\!\!\!/\psi + F^2] + \mathrm{i}Fp(A) + \tfrac{1}{2}p'(A)\bar{\psi}\psi, \tag{25}$$

on account of the formula

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}F}{\sqrt{2\pi}} \exp[-(Z/2)F^2 - \mathrm{i}Fp(A)] = \frac{1}{\sqrt{Z}} \exp[-\tfrac{1}{2}(1/Z)(p(A))^2], \tag{26}$$

where we have introduced an auxiliary field F. Note the reappearance of the factor $Z^{-1/2}$ which had already been anticipated in eq. (16). The simplest example is now obtained with $p(A) = mA + \lambda A^3$, where both m and λ are positive, so $p'(A) > 0$. In this case, the transformation T_λ up to third order reads as follows

$$\begin{aligned} A'(x,\lambda;A) = A(x) &+ m\lambda \int C(x-x_1)A^3(x_1)\,\mathrm{d}x_1 - \tfrac{3}{2}\lambda^2 Z^2 \int \partial_\mu C(x-x_1)A^2(x_1)\partial_\mu C(x_1-x_2)A^3(x_2)\,\mathrm{d}x_1\mathrm{d}x_2 \\ &+ \tfrac{9}{2}\lambda^3 m Z^2 \int \partial_\mu C(x-x_1)A^2(x_1)C(x_1-x_2)A^2(x_2)\partial_{\mu}C(x_2-x_3)A^3(x_3)\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 + \mathrm{O}(\lambda^4), \end{aligned} \tag{27}$$

where

$$C(x) = \int \frac{\mathrm{d}^2k}{(2\pi)^2} \mathrm{e}^{\mathrm{i}kx}/(Z^2k^2 + m^2), \tag{28}$$

is the usual propagator. By use of the Leibniz product rule and partial integration, the reader may check that formally

$$\frac{1}{2Z}\int \mathrm{d}x\, A'(x,\lambda;A)(-Z^2\Delta + m^2)A'(x,\lambda;A) = \int \mathrm{d}x\,[\tfrac{1}{2}Z(\partial_\mu A(x))^2 + \tfrac{1}{2}(1/Z)(mA(x) + \lambda A^3(x))^2] + \mathrm{O}(\lambda^4) \tag{29}$$

and

$$\{\det[\delta(x-y) + 3\lambda(Z\partial\!\!\!/ + m)^{-1}(x-y)A^2(y)]\}^{1/2} = \det \delta A'(x,\lambda;A)/\delta A(y) + \mathrm{O}(\lambda^4). \tag{30}$$

A lattice regularization of supersymmetry appropriate for the derivation of eqs. (29) and (30) from eq. (27) has been outlined in ref. [12].

Finally, we would like to comment on possible generalizations of our result to more complicated supersymmetric models. From the superalgebra relation (in relativistic space–time)

$$\{Q_{\alpha M}, \bar{Q}_{\dot{\beta}}^{\ N}\} = 2\delta_M^{\ N}\sigma^\mu_{\ \alpha\dot{\beta}}P_\mu, \tag{31}$$

‡4 Actually, in two dimensions, it may occur that $V(A)$ has no zeros at all (spontaneous breaking of supersymmetry) but, as our argument is formal anyway, we will not worry about such additional complications.

it follows by a well-known argument [7] that the hamiltonian H (or a suitably regularized version thereof) always obeys the inequality $H \geqslant 0$. If supersymmetry is not spontaneously broken, the ground state Ω_0 satisfies $H\Omega_0 = 0$ and the vacuum energy is zero indeed, so we expect our arguments to apply. If, however, supersymmetry is spontaneously broken, the hamiltonian is bounded below by a strictly positive constant, and, in fact, naively using the formula (21), we would find that the vacuum energy turns out to be infinite. Evidently, this case requires a more careful examination.

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References

[1] P. Fayet and S. Ferrara, Phys. Rep. C32 (1977) 249.
[2] H. Nicolai, Nucl. Phys. B140 (1978) 294.
[3] T. Matthews and A. Salam, Nuovo Cimento 12 (1954) 563; 2 (1955) 120;
E. Seiler, Commun. Math. Phys. 42 (1975) 163.
[4] F.A. Berezin, The method of second quantization (Academic Press, New York, 1966).
[5] B. Zumino, Nucl. Phys. B89 (1975) 535.
[6] B. Simon, The $P(\phi)_2$ (euclidean) quantum field theory (Princeton U.P., 1974).
[7] J. Iliopoulos and B. Zumino, Nucl. Phys. B76 (1974) 310.
[8] G. Hardy, Divergent series (Oxford, 1939).
[9] B. Zumino, in: Renormalization and invariance in quantum field theory (Plenum Press, New York, 1973);
S. Ferrara, Lett. Nuovo Cimento 13 (1975) 629;
S. Browne, Phys. Lett. 59B (1979) 253.
[10] H. Nicolai, Nucl. Phys. B156 (1979) 177.
[11] J. Wess and B. Zumino, Nucl. Phys. B70 (1974) 39.
[12] H. Nicolai, in: Group theoretical methods in physics (Springer, Heidelberg, 1978).