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ON A NEW CHARACTERIZATION OF SUPERSYMMETRIC THEORIES II

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A B S T R A C T

We complete the proof of a recently proposed new characterization of scalar supersymmetric theories and extend the result to "non-scalar" models.

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In a recent letter[1], we proposed to characterize scalar supersymmetric theories by the following statement: there exists a transformation of the bosonic fields which rotates the full (interacting) functional integration measure into a free measure and whose Jacobi determinant equals the Matthews-Salam-Seiler (MSS) determinant[2] of the theory. The unusual and novel feature of this approach is that it enables us to avoid the use of abstractly defined anticommuting objects, a tool which had been indispensable so far in formulating supersymmetric theories and exploring their properties. Apart from its intrinsic interest, this result deserves further attention and study for a variety of reasons; for example, one may hope for easier constructibility of supersymmetric models as compared to non-supersymmetric ones. The vanishing of the vacuum energy in supersymmetric theories is explained quite naturally if one turns around the argument to reconstruct supersymmetric models. Furthermore, it appears certainly worth while to try to do without quantities which are very convenient algebraically (e.g., in superfield perturbation theory) but, so far, have defied analytic treatment (no positivity properties and the like, only formal definability of functional integrals over superfields, etc.).

In the major part of this communication, we complete the proof of the main theorem of Ref. 1) and thereby close a gap that has been left in that proof. We then apply our result to prove global invertibility of the field transformation in some cases, verifying a conjecture made in Ref. 1). Finally, we indicate briefly how to generalize the result which, in Ref. 1), was announced for "scalar" supersymmetry only, to more sophisticated models. Our generalization will, however, not yet comprise supersymmetric gauge theories: the gauge fixing procedure[3] either explicitly violates supersymmetry[4] or, through additional ghost multiplets, renders the theory considerably more complicated[5]. In view of its importance, the discussion of this case is deferred to a future publication.

We will, in the sequel, adopt the same conventions (Euclidean metric, etc.) and notation as in Ref. 1). In particular, if ψ denotes the (Majorana) spinors of the model, we write the part of the action containing these as

$$\begin{aligned}&\tfrac{1}{2}\bar{\psi}\, M(A)\,\psi \equiv \tfrac{1}{2}\bar{\psi}\, M(\lambda;A)\,\psi \equiv \\ &\equiv \tfrac{1}{2}\int \bar{\psi}_\alpha(x)\, M_{\alpha\beta}(x,y,\lambda;A)\,\psi_\beta(y)\,dx\,dy\end{aligned} \tag{1}$$

where, as in Ref. 1), λ stands for the various coupling parameters and will be occasionally omitted; the bosonic fields are compactly denoted by A. We will make repeated use of the fact that the fermions can be "integrated out"[6], for instance

$$\int d\psi \exp\left[-\frac{1}{2}\bar{\psi}M(\lambda;A)\psi\right] = \det M(\lambda;A)^{1/2} \equiv$$
$$\equiv \left(\det M(\lambda=0;A)\right)^{1/2} \cdot \mathcal{D}(\lambda;A) \tag{2}$$

$D(\lambda;A)$ is the MSS determinant in the notation of Ref. 1). At this point, it is inessential that the action is quadratic in the fermions since, for non-quadratic actions, Berezin's integral[6] serves as well to eliminate the fermionic variables in which case, however, $D(\lambda;A)$ is no longer the square root of a determinant but some less familiar function instead. For the bosonic part of the action we will simply write $S(A) \equiv S(\lambda;A)$; $S_0(A) \equiv S(0,A)$ represents the free action.

It was proved in Ref. 1) that for supersymmetric theories there exists a transformation $A(x) \to A'(x,\lambda;A)$ which "rotates away" the interaction in the functional measure which formally defines the Schwinger functions, that is

$$e^{-S(\lambda;A)}\mathcal{D}(\lambda;A)dA = e^{-S_0(A'(\lambda;A))}dA'(\lambda;A) \tag{3}$$

This, as has been demonstrated in Ref. 1), is a consequence of the vanishing of the vacuum energy in supersymmetric theories. From (3), we infer

$$S(\lambda;A) = S_0(A') + K(\lambda;A') \tag{4}$$

$$\mathcal{D}(\lambda;A) = e^{K(\lambda;A')} \det \frac{\delta A'(x,\lambda;A)}{\delta A(y)} \tag{5}$$

It was claimed in Ref. 1) and will be proved now that, in addition to (4) and (5), supersymmetry also implies

$$K(\lambda;A) = \sum_{n=1}^{\infty} \lambda^n K_n(A) = 0 \quad ^{*)} \tag{6}$$

at least in the sense of formal power series (which is all we are concerned with here).

For the proof of Eq. (6), we split the full supersymmetric action in two parts as follows

$$S(A,F,\psi) = S(A,F) + \frac{1}{2}\bar{\psi}M(A)\psi \tag{7}$$

where, in two dimensions,

$$S(A,F) = \int \left[\frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}F^2 + iFp(A) \right] dx \tag{8}$$

F is an auxiliary field and $p(A)$ an arbitrary polynomial in A. Even though (8) is special to two space time dimensions, there is no difficulty in generalizing (8) to other cases, and all steps in the arguments below extend naturally to other supersymmetric models. All we need is that the action is quadratic in the fermions and the auxiliary fields. Our main input is the infinite set of identities $(n \in \mathbb{N})$

$$\int S(A,F,\psi)^n \, e^{-S(A,F,\psi)} \, dA\,dF\,d\psi = 0 \tag{9}$$

valid for all values of the coupling parameters. These identities follow from supersymmetry, and one may convince oneself that they only hold for supersymmetric models[**)]. Introducing a shifted field $\tilde{F}(x) = F(x) + ip(A(x))$ and replacing F by $\tilde{F}$ in (8), we obtain

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*) For simplicity, we assume that there is only one coupling constant λ.

**) The identities (9) are generated from the supersymmetry relation

$$\int \exp\left[-(1+\alpha)S(A,F,\psi)\right] dA\,dF\,d\psi = 1$$

by differentiation with respect to α. Although, for a non-supersymmetric theory, the vacuum energy may always be made to vanish by adding a suitable function $f(\alpha)$ to the action, the identities (9) are only obtained if $f'(\alpha) = 0$ and therefore $f(\alpha) = 0$ since $f(0) = 0$ which is only possible in supersymmetric theories. I am indebted to E. Seiler for having raised this point.

$$S(A,F) = S(A) + \frac{1}{2}\int \tilde{F}(x)^2 dx \tag{10}$$

with

$$S(A) = \frac{1}{2}\int [(\partial_\mu A)^2 + p(A)^2]\, dx \tag{11}$$

Thus, Eq. (9) becomes

$$\begin{aligned} 0 &= \int dA\, d\tilde{F}\, d\psi \left(S(A) + \frac{1}{2}\int \tilde{F}^2 dx + \frac{1}{2}\bar{\psi} M(A)\psi \right)^n \cdot \\ &\quad \cdot \exp\left[S(A) + \frac{1}{2}\int \tilde{F}^2 dx + \frac{1}{2}\bar{\psi} M(A)\psi \right] = \\ &= \sum_{k=0}^{n} \binom{n}{k} c_k \int S(A)^{n-k} e^{-S(A)} \det M(A)^{1/2}\, dA \end{aligned} \tag{12}$$

The crucial observation is now that the numerical coefficients c_k do <u>not</u> depend on the various couplings*) of the model as may be seen by making the substitution

$$\psi' = M(\lambda;A)^{1/2}\psi \quad ; \quad d\psi' = \det M(\lambda;A)^{-1/2} d\psi \tag{13}$$

in the Berezin integral. Therefore, (12) is also fulfilled for the free theory

$$\sum_{k=0}^{n} \binom{n}{k} c_k \int S_0(A)^{n-k} e^{-S_0(A)} dA = 0 \tag{14}$$

In the functional integral (12), we now substitute $A \to A'$, so, by (3) the functional measure becomes a free measure. Equation (4) tells us that (12) is equivalent to

$$\sum_{k=0}^{n} \binom{n}{k} c_k \int \left(S_0(A) + K(\lambda;A) \right)^{n-k} e^{-S_0(A)} dA = 0 \tag{15}$$

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*) It is here that we need the action to be quadratic in ψ and $\tilde{F}$.

(we have dropped the primes). From this identity, from (14) and from the fact that $c_0 \neq 0$, we get

$$\int (S_0(A) + K(\lambda;A))^n e^{-S_0(A)} dA = \int S_0(A)^n e^{-S_0(A)} dA \tag{16}$$

for all n by induction. Upon inserting the asymptotic expansion (6) into (16), the left-hand side becomes

$$\int \Big(S_0(A) + \sum_{\nu=1}^{\infty} \lambda^\nu K_\nu(A)\Big)^n e^{-S_0(A)} dA \tag{17}$$

Differentiating this expression with respect to λ and setting $\lambda = 0$, we obtain

$$\sum_{k=1}^{n} \binom{n}{k} \sum_{\nu_1+\ldots+\nu_k=m} \Big\langle S_0(A)^{n-k} K_{\nu_1}(A)\ldots K_{\nu_k}(A)\Big\rangle_0 = 0 \tag{18}$$

For every $m \in \mathbb{N}$, these identities impose infinitely many constraints on the functionals $K_1(A), \ldots, K_m(A)$. Taking into account that the polynomiality of $p(A)$ entails that each $K_n(A)$ is also polynomial in the bosonic fields, we conclude that all $K_n(A)$ vanish identically which completes the proof of (6) and thus of the main theorem of Ref. 1).

An interesting consequence of the equality of the MSS determinant and the Jacobi determinant of the transformation is that whenever we are able to show that

$$\det \frac{\delta A'(x,\lambda;A)}{\delta A(y)} = \mathcal{D}(\lambda;A) > 0 \tag{19}$$

the transformation $A \to A'$ is locally invertible everywhere. But then we can also prove that it has a global inverse because, by supersymmetry,

$$\int e^{-S(\lambda;A)} \mathcal{D}(\lambda;A) dA = \text{const.} \tag{20}$$

Taking the limit $\lambda \to 0$ and assuming continuity at $\lambda = 0$, we find that the "winding number" of the transformation equals one which, together with local invertibility, implies global invertibility. Since for Majorana spinors, $D(\lambda;A)$ is the square root of a determinant, (19) is usually more difficult to verify than for Dirac spinors where $D(\lambda;A)^2$ would be the relevant quantity; so, for example, the inequality $D(\lambda;A)^2 \geq 0$, first established in the third paper of Ref. 2), is insufficient for (19). In the two-dimensional case, we find, employing the methods of Ref. 7), that (19) is satisfied if $p'(A) > 0$ in agreement with the conjecture made in Ref. 1); but $p'(A) > 0$ (for all λ) is also the condition that ensures continuity at $\lambda = 0$ in (20). In four dimensions, the simplest model[8)] already contains both Yukawa and pseudo-Yukawa interactions and it is not known whether (19) is true or not.

Chiefly for reasons of conceptual clarity and notational simplicity the scope of the present investigation has been confined to scalar supersymmetric theories until now. No great effort is required, however, to drop that restriction if the model under consideration has zero vacuum energy, satisfies identities which correspond to (9) and is quadratic in the fermion fields. Formally, of course, the first two requirements are met in every supersymmetric theory, but, as was already emphasized in Ref. 1), some care must be exercised when using purely formal arguments. Attaching vectorial and internal indices to all the fields, we discover that we can carry over our proof virtually unchanged. As a technical remark, we mention that it is sometimes advantageous and more convenient not to assume the coefficient functions $R_n^{(\nu)}(x)$ introduced in Ref. 1) to be totally symmetric in their arguments. The only change is that, in Eqs (7) to (14) of Ref. 1), we have to write out all contractions.

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REFERENCES

1) H. Nicolai, preprint CERN TH.2758 (1979), to appear in Phys. Lett. B.

2) T. Matthews and A. Salam, Nuovo Cim. 12 (1954) 563, and Nuovo Cim. 2 (1955) 120;
E. Seiler, Comm. Math. Phys. 42 (1975) 163.

3) L.D. Faddeev and V.N. Popov, Phys. Lett. 25B (1967) 29.

4) B. de Wit and D.Z. Freedman, Phys. Rev. D12 (1975) 2286.

5) B. de Wit, Phys. Rev. D12 (1975) 1628;
S. Ferrara and O. Piguet, Nucl. Phys. B93 (1975) 261;
F. Honerkamp, M. Schlindwein, F. Krause and M. Scheunert, Nucl. Phys. B95 (1975) 397.

6) F.A. Berezin, "The Method of Second Quantization", (Academic Press, New York 1966).

7) H. Nicolai, Comm. Math. Phys. 59 (1978) 71.

8) J. Wess and B. Zumino, Phys. Lett. 49B (1974) 52.