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The Geometry of the (Modified) GHP-Formalism*

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Let (L, N) be a *pair of future oriented null direction fields* in a temporally and spatially oriented spacetime (M, g_{ab}) with a spinor structure [1–4]. Then the collection of null-tetrads $\zeta = (l, n, m, \bar{m})$ (as defined in the preceding paper) with $l \in L$, $n \in N$ is a *principal fibre bundle over* M *with structure group* $\dot{C}$ (= multiplicative group of complex numbers), where, for $z \in \dot{C}$,

$$\zeta' = \zeta z \text{ means } (l', n', m') = \left(|z|^2 l, |z|^{-2} n, \frac{z}{\bar{z}} m\right). \tag{A.1}$$

Let B denote this bundle as well as its bundle space. B is a reduction of the bundle of oriented null tetrads over M ($\cong$ of oriented orthonormal frames).

If $\psi : M \to B$ is a cross section and (x^a) a local coordinate system of M, then (x^a, w) is a local coordinate system of B where, for $x \in M$, $\zeta_x \in B$, $w \in \dot{C} : \zeta_x = \psi_x w$. A complex valued 1-form $\bar{\omega}$ on B defines a *connection on* B if and only if it has the local representation

$$\bar{\omega} = \omega_a(x^b)\, dx^a + \frac{dw}{w}. \tag{A.2}$$

We then have $\psi^* \bar{\omega} = \omega_a \mathrm{d}x^a = \omega_\psi$, a 1-form on M depending on ψ and describing the connection relative to the tetrad field ψ. The curvature form is given by $d\bar{\omega}$ (on B) or by $\psi^* d\bar{\omega} = d\omega_\psi$ (on M).

A map η which associates with each cross section ψ of B a complex valued function η_ψ such that, for each map $z : M \to \dot{C}$,

$$\eta_{\psi z}(x) = z_x^p \bar{z}_x^q \eta_\psi(x), \tag{A.3}$$

where (p, q) is a pair of integers, is said to be a *quantity of type* (p, q). If η is of type (p, q), its complex conjugate $\bar{\eta}$ is of type (q, p). The quantities of a definite type (p, q) form a C vector space, the quantities of all types together form a graded algebra $\mathfrak{A}$.

* This note should be considered as a supplement to the preceding paper by A. Held.

If a connection is defined on B via a form $\bar{\omega}$ on B or the corresponding forms ω_ψ on M, a *covariant differential operator* D can be defined on $\mathfrak{A}$. If η is of type (p, q), then the values of the 1-form (on M)

$$(D\eta)_\psi = d(\eta_\psi) + (p\,\omega_\psi + q\,\bar{\omega}_\psi)\,\eta_\psi \tag{A.4}$$

are also in $\mathfrak{A}$, of type (p, q). If the vectors $l, n, m, \bar{m}$ of ψ_x are inserted in $(D\eta)_\psi$, one obtains four covariant *directional derivatives*

$$\begin{aligned} &\text{Þ} = D_l, \quad \tilde{\text{Þ}}' = D_n + \frac{\bar{\tau}}{\bar{\varrho}} D_m + \frac{\tau}{\varrho} D_{\bar{m}}, \\ &\tilde{\eth} = \frac{1}{\bar{\varrho}} D_m, \quad \tilde{\eth}' = \frac{1}{\varrho} D_m \end{aligned} \tag{A.5}$$

acting on $\mathfrak{A}$. (Here $\varrho \neq 0$ is assumed.)

Consider the bundle Sp of spin frames [2, 3] (o, ι). Its structure group is $SL_2(C)$, and its connection can be described [1], relative to a cross section χ, by a matrix $\omega_\chi = (\omega^A_{\chi B})$ of complex valued 1-forms on M, with $tr\,\omega_\chi = 0$. Under the change $(o, \iota) \to (o, \iota)\Lambda$ of cross section, where $\Lambda_x \in SL_2(C)$, one has $\omega_{\chi\Lambda} = \Lambda^{-1}(\omega\Lambda + d\Lambda)$. If $\Lambda = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$, then

$$(\omega_{\chi\Lambda})^1_2 = z^2(\omega_\chi)^1_2, \quad (\omega_{\chi\Lambda})^2_1 = z^{-2}(\omega_\chi)^2_1, \tag{A.6}$$

and

$$(\omega_{\chi\Lambda})^1_1 = (\omega_\chi)^1_1 + \frac{dz}{z}. \tag{A.7}$$

We may consider B as well as a reduction of Sp with respect to the subgroup $C^{\cdot} = \left\{\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right\} \subset SL_2(C)$. (A.6) shows that ω^1_2 and ω^2_1, restricted to cross sections of B, are 1-forms whose values are quantities of type (2, 0) and (−2, 0), respectively. The components of these 1-forms are precisely the NP spin coefficients [1, 4] ϱ, σ, $\varkappa$, τ, ϱ', σ', $\varkappa'$, τ' (in the notation of the preceding paper). These are, therefore, quantities of various types as defined in (A.3). On the other hand, it is easily seen from (A.2) that ω_ψ [defined below (A.2)] transforms precisely like $\omega^1_{\chi 1}$; in fact, that behaviour is equivalent to the fact that $\bar{\omega}$ in (A.2) is a coordinate-independent 1-form on B.

The preceding argument shows: If we take, from Sp, the form $(\omega_\psi)^1_1$, where ψ is a cross section of $B \subset$ Sp, and use this form as the first term in (A.2), then we get a connection on B.

Now, $(\omega_\psi)^1_1$ contains precisely the spin coefficients α, β, γ, ε; they therefore give rise to a connection on B. This connection, however, may be changed into another one by adding to the first term in (A.2) any form ω_ψ of type (0, 0). This freedom may be exploited to construct a new connection $\bar{\omega}$ on B the corresponding directional derivatives (A.5) of which, called $\text{Þ}, \tilde{\text{Þ}}', \tilde{\eth}, \tilde{\eth}'$ in the preceding paper, have the property (3.6); that is, the subalgebra $\mathfrak{A}^0 \subset \mathfrak{A}$ which is annihilated by $\text{Þ}(\text{Þ}\mathfrak{A}^0 = 0)$ is invariant with respect to the derivations $\text{Þ}', \tilde{\eth}, \tilde{\eth}'$. For this purpose, one puts

$$\omega_\psi = \frac{1}{2}\left(\bar{m}^b \nabla_a m_b - n^b \nabla_a l_b + \left\{2\frac{\tau\bar{\tau}}{\bar{\varrho}} - \frac{\bar{\Psi}_2}{\bar{\varrho}}\right\} l_a - 2\frac{\varrho\bar{\tau}}{\bar{\varrho}} m_a\right) dx^a \quad \text{(A.8)}$$

where ψ is the cross section given by the tetrad field $(l, n, m, \bar{m})$. The operator D defined by (A.4) and (A.8) is the same as Held's operator $\tilde{\theta}_a$, Eq. (2.7).

To obtain the field equations and Bianchi-identities of Section 4, one can use Bichteler's [1] spinorial structure equations of Sp and reduce them to B according to the idea sketched in this note.

It appears that similar "formalisms" could be set up for different subgroups of $O(3, 1)$ or $SL_2(C)$. Whenever one succeeds in reducing the orthonormal frame bundle (or the spin bundle) to another principal bundle with a linear connection $\tilde{\nabla}$, this connection will induce a metric connection on (M, g_{ab}). The information which, in the standard description of spacetimes, is contained in the Riemannian curvature, is in any such "formalism" contained *partly in the curvature*, *partly in the torsion* of the modified, metric connection, and partly in some additional structure. (This illuminating remark is due to Dr. B. Schmidt.) For the investigation of *null fields*, e.g., it appears best to take, instead of $\dot{C}$, the subgroup of *null rotations* about 0, which is the additive group C^+.

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