TRANSFORMATIONS OF STATIC EXTERIOR SOLUTIONS OF EINSTEINS GRAVITATIONAL FIELD EQUATIONS INTO DIFFERENT SOLUTIONS BY MEANS OF CONFORMAL MAPPINGS

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RÉSUMÉ

Je me propose de montrer que l'emploi des transformations conformes permet des simplifications considérables dans le calcul par ailleurs souvent compliqué et pénible du tenseur de Ricci correspondant à une métrique donnée. Presque tous les cas connus de solutions exactes des équations du champ peuvent être traités par cette méthode qui donne bien des formules utiles.

In this report we want to show how it is possible to construct new static exterior solutions, stationary exterior solutions and stationary interior solutions of the field equations of general relativity from static exterior solutions by applying certain conformal transformations to auxiliary metrics defined on three-dimensional manifolds of distinguished paths in space-time. All theorems we shall consider are purely local. Perhaps the most interesting result is that the determination of all interior solutions in which incoherent matter moves rigidly is equivalenet to the construction of all static exterior solutions and, in fact, equivalent to the determination of all three dimensional Riemannian spaces with positive definite metrics the contracted curvature tensors of which admit a representation (6) *.

* Applications of the theorems developed in this report will be described in a series of papers by P. JORDAN and his collaborators which will appear soon in the « Abhandlungen der Akademie der Wissenschaften und der Literatur » in Mainz; there a systematic treatment of the known and of new rigorous solutions will be given.

§ 1. — Auxiliary Formulae. Notation

The following considerations make use of two well-known formulae which are given here for the convenience of the reader and in order to fix the notation.

To a first fundamental form of the type

$$G = H - V^2 (dx^o - u)^2 , \tag{1}$$

in which

$$H = h_{ik} dx^i dx^k \tag{1'}$$

is a quadratic differential form in the three variables $x^i (i, j, ... = 1, 2, 3$ here and throughout this paper),

$$u = u_i dx^i \tag{1''}$$

a linear differential form, and V a function (both u_i and V also depend on the x^i only), belongs a Ricci-tensor with components.*

$$\begin{aligned} R^o_o &= V^{-1} (V^{,i} + V^2 u_j u^{[j;i]})_{;i} , \\ R^i_o &= V^{-1} (V^3 u^{[i;j]})_{;j} , \\ R^i_j &= P^i_j + V^{-1} V^{,i}{}_{;j} + 2V^2 u^{[i;l]} u_{[l;j]} - u_j R^i_o . \end{aligned} \tag{2}$$

Here the metric operations on the right side — shifting of indices, $V^{,i} = h^{ij} V_{,j}$; covariant differentiation, $u_{i;j}$ — refer to the metric (1'), and P^i_k denotes the Riccitensor of this metric. A derivation of (2) is given in [1].

If two non-singular symmetric tensor fields on an n-dimensional manifold are related by

$$\bar{g}_{\lambda\mu} = e^{2U} g_{\lambda\mu} \tag{3}$$

where U is a scalar, the corresponding Ricci-tensors satisfy

$$e^{2U} \bar{R}^\lambda_\mu = R^\lambda_\mu + (n-2) (U_{,\mu}{}^{;\lambda} - U_{,\mu} U^{,\lambda}) + \delta^\lambda_\mu (U^{,\nu}{}_{;\nu} + (n-2) U_{,\nu} U^{,\nu}) . \tag{4}$$

Here $\bar{R}^\lambda_\mu = \bar{g}^{\lambda\nu} \bar{R}_{\nu\mu}$, and all metric operations on the right side refer to $g_{\lambda\mu}$. A proof of (4) is given in [2].

§ 2. — Reduction of static exterior fields to certain three-dimensional Riemannian spaces

Exterior fields (source-free gravitational fields) are four dimensional normal hyperbolic ** Riemannian spaces with vanishing Riccitensor. A field or « space-time » is called stationary, if it admits a one dimensional Lie-group of isometric correspondences with time-like trajectories; it is called static, if the trajectories form a normal congruence [1].

* Square brackets denote anti-symmetrization.
** We choose the signature +++—.

Let W be a static exterior field, G its first fundamental form, and $\vec{\xi}$ a Killing vector field generating the corresponding group. Then the coordinates $x^\lambda (\lambda, \mu, ... = 1, 2, 3, 0$ here and in the following) can be chosen such that

$$G = e^{-2U} H - e^{2U} (dx^o)^2 , \qquad \xi^\nu = \delta^\nu_o \tag{5}$$

with H as in (1′) and $U = U(x^i)$. H defines a positive-definite Riemannian metric on the three-dimensional manifold S formed by the trajectories of the group.

Applying first (1) and (2) to (5) (regarding e^{-2U} H as the metric of S), then using (3) and (4) (taking H as the new metric of S) one can prove the remarkable theorem that (5) satisfies $R_{\lambda\mu} = 0$ if and only if

$$P_{ik} + 2 U_{,i} U_{,k} = 0 \tag{6}$$

is valid in (S, H); P_{ik} again denotes the contracted curvature tensor of H [2]. Therefore the determination of all static exterior fields is mathematically equivalent to the construction of those Riemannian 3-spaces the Ricci-tensors of which have the form (6).

Because of the contracted Bianchi-identity for P_{ik}

$$U^{,i}{}_{;i} = 0 . \tag{7}$$

By (5), (6) and (7) the scalar $U = \dfrac{1}{2} \log (- \xi_\lambda \xi^\lambda)$ is seen to satisfy in W with respect to G

$$U^{,\lambda}{}_{;\lambda} = 0 , \qquad \xi^\lambda U_{,\lambda} = 0 . \tag{8}$$

§ 3. — A theorem of Buchdahl

As a simple application of the theorem stated in § 2 we describe now a result found by Buchdahl in 1954 [3]. We reformulate his theorem here because our proof is much simpler than the original one and because it is similar to the statements of the following two sections.

Obviously with (H. U) also (H, — U) solves (6). This can be expressed without reference to special coordinate :

If G is the fundamental form of a static exterior space-time W whose time-like, hypersurface-normal Killing vector field is $\vec{\xi}$, then the new fundamental form

$$\bar{G} = e^{4U} G + (e^{2U} - e^{-6U}) (\vec{\xi} \cdot \vec{dx})^2, \tag{9}$$

where

$$e^{2U} \equiv | \vec{\xi} \cdot \vec{\xi} |,$$

defines again a static exterior field with the same Killing vector field $\vec{\xi}$; in fact (9) reduces to the transformation if one uses coordinates according to (5).

For this theorem and the proof given here it is not essential that $\vec{\xi}$ is timelike; they remain valid if « time-like » is replaced by « space-like ».

§ 4. — Special stationary exterior fields

If one sets up, by means of the formulae of § 1, the field equations $R_{\lambda\mu} = 0$ for a metric of the type

$$G = a \cos h(2U)\, H - (a \cos h(2U))^{-1} (dx^0 - u)^2 \tag{10}$$

with

$$a = \text{const.} > 0\,, \quad U = U(x^k)\,,$$

and H and u as in (1′), (1″) resp. (in the way described in § 2 in connection with (5)), one obtains again the relation (6) for H and U. Moreover,

$$- a\, \eta_{jkl}\, U^{,l} = u_{[j,k]} \tag{11}$$

which is to be understood with respect to H. η_{jkl} denotes the usual totally skew-symmetric tensor * with components 0, $\pm h^{1/2}$; $h = |h_{ik}|$.

The condition of integrability of (11) regarded as a system of differential equations for the unknowns u_i is (7); it is fulfilled therefore in consequence of (6). The integration of (11) is a well-known elementary procedure; it is the determination of a vector-potential for a given integrable ($F_{[ij,k]} = 0$) skew-symmetric tensor-field. If we combine this result with § 2, we get the following theorem :

If (5) describes a static exterior space time, then (10) is the metric of a stationary exterior space-time provided the u_i satisfy (11). The field given by (10) is in general non-static.

The translation of this theorem into generally covariant form reads :

If G is the fundamental form of a static exterior field W with time-like hypersurface-normal Killing vector field $\vec{\xi}$, then the equations

$$a\, \eta_{\kappa\lambda\mu\nu}\, U^{,\mu}\, \xi^{\nu} = u_{[\kappa,\lambda]}\,, \quad u_{\lambda}\, \xi^{\lambda} = -1 \tag{12}$$

(which refer to G) are integrable, (8) being the conditions of integrability. If $\vec{u}$ is a solution of equ. (12),

$$\overline{G} = a \cos h(2U)\, (e^{2U} G + (\vec{\xi} \cdot d\vec{x})^2) - (a \cos h(2U))^{-1} (\vec{u} \cdot d\vec{x})^2 \tag{13}$$

is the metric of a stationary exterior field $\overline{W}$.

We remark without proof, that the $\overline{W}$'s that can be constructed in this way out of W's are characterised by the existence of a time-like Killing vector field $\vec{\xi}$ which satisfies (with respect to $\overline{G}$)

$$\xi^{\varrho}\, \xi^{[\iota}{}_{;\varrho}\, \eta^{\varkappa]\lambda\mu\nu}\, \xi_{\lambda}\, \xi_{\mu,\nu} = 0\,. \tag{14}$$

If one chooses for G in (13) in particular the static axially symmetric solution of Weyl (4), one obtains stationary axially symmetric fields which, by complex transformations of coordinates, can be changed similar to, but not equivalent to the Einstein-Rosen waves **.

* We omit the difference between tensors and pseudotensors, because our considerations are purely local; we may assume that we have oriented the manifold locally.

** This application of the theorem has been described in my lecture given at the coloque about the theory of relativity held at Brussels on 19. and 20. 6. 1959.

§ 5. — Rigid motions of incoherent matter

The equation

$$\bar{R}_{\lambda\mu} - \frac{1}{2}\bar{g}_{\lambda\nu}\bar{R} + \rho\, u_\lambda u_\mu = 0 \tag{15}$$

describes a metric field $\bar{G}$ interacting with incoherent matter which has four-velocity $\vec{u}$ and proper density ρ *.

(15) implies that the streamlines are geodesics, a fact which is used tacitly throughout this section.

The scalar

$$\theta = u^\lambda{}_{;\lambda} \tag{16}$$

represents the velocity of expansion, the vector

$$\omega^\varkappa = \frac{1}{2}\bar{\eta}^{\varkappa\lambda\mu\nu}\, u_\lambda\, u_{\mu;\nu} \tag{17}$$

measures the velocity of rotation [5], and the symmetric tensor **

$$\sigma_{\mu\nu} = u_{(\mu;\nu)} - \frac{1}{3}\theta\,(\bar{g}_{\mu\nu} + u_\mu u_\nu) \tag{18}$$

with vanishing trace describes the velocity of shear [6]. We put

$$\omega = (\omega_\lambda\,\omega^\lambda)^{1/2}, \quad \sigma = \left(\frac{1}{2}\sigma_{\lambda\mu}\,\sigma^{\lambda\mu}\right)^{1/2} \tag{19}$$

(17) can be transformed into

$$u_{[\kappa,\lambda]} = \bar{\eta}_{\kappa\lambda\mu\nu}\,\omega^\mu u^\nu; \tag{20}$$

consequently we have

$$(\omega^{[\mu}u^{\nu]})_{;\nu} = 0\,, \quad \omega^\nu{}_{;\nu} = 0. \tag{21}$$

Therefore

$$\omega^\mu u^\nu{}_{;\mu} - u^\mu\,\omega^\nu{}_{;\mu} = \theta\,\omega^\nu\,. \tag{22}$$

These formulae can be used to derive the theorems of the vorticity-theory (directly in the case of geodesic streamlines, with a slight modification also in the more general case of isentropic motions of ideal fluids) developed by Synge [5], Lichnerowicz [1], M^me^ Foures [7] and Gödel [8].

The equation

$$u^\lambda\,\theta_{,\lambda} + \frac{1}{3}\theta^2 + 2(\sigma^2 - \omega^2) + \frac{1}{2}\rho = 0 \tag{23}$$

is a consequence of (15) as has been shown by Raychandhuri [9].

* The metric tensor is denoted $\bar{g}_{\lambda\mu}$ because we shall use a second metric $g_{\lambda\mu}$ below. Nevertheless covariant differentiation; and index-shifting refer to $\bar{G}$ unless otherwise stated.

** Round brackets denote symmetrization.

We propose now to study those solutions of (15) in which matter moves without changing its form (locally); that means we postulate $\theta = 0$ and $\sigma = 0$ or, equivalenty by (16) and (18),

$$u_{(\mu;\nu)} = 0 . \qquad (24)$$

Rigid motions have been treated by several authors [10], [11], [12], [13]; but rigorous solutions of (15), (24) have been given only by van Stockum [10] in 1937. We shall specify below how van Stockum's results are contained in our general theorem.

Rayner studies instead of (15), (24) the equation

$$\bar{R}^{\lambda}_{\mu}\, u^{\mu} = \frac{1}{2}\, \rho\, u^{\lambda} \qquad (25)$$

together with $\theta = 0$, $\sigma = 0$ and does not put any restrictions on the tensions inside matter. As (25) follows from (15) our considerations are, as far as only (25) is concerned, similar to those of Rayner; but as we have the additional condition $u^{\lambda} u^{\nu}{}_{;\lambda} = 0$ even in this part of the treatment we get some more specific results, namely (26), (27), (29).

With (24) Raychandhuri's equation (23) simplifies to

$$\rho = 4\, \omega^2 .\text{*} \qquad (26)$$

(We use — see (15) — natural relativistic units, namely $c = 1$, $f = \frac{1}{8\pi}$. In cgs-units (26) reads $\omega^2 = 2\pi f \rho$ with f as Newton's constant of gravitation.)

By (20), (24), and (22) we have

$$\omega^{\mu} u^{\nu}{}_{;\mu} = u^{\mu} \omega^{\nu}{}_{;\mu} = 0 , \qquad (27)$$

thus in a rigid, geodesic motion the four-velocity is parallel propagated along the vorticity-lines, and the vorticity-vector is parallel propagated along the streamlines.

By (24) and the Ricci-identity we have $u^{[\lambda;\mu]}{}_{;\mu} = \bar{R}^{\lambda}_{\mu}\, u^{\mu}$, therefore, by (25), $(\delta^{\nu}_{\lambda} + u^{\nu} u_{\lambda})\, u^{[\lambda;\mu]}{}_{;\mu} = 0$ which, by (20), can be reformulated as

$$u_{[\lambda}\, \omega_{\mu,\nu]} = 0 . \qquad (28)$$

Contraction with u^{λ} gives $\omega_{[\mu,\nu]} = (u_{\mu}\, \omega_{[\nu,\lambda]} + u_{\nu}\, \omega_{[\lambda,\mu]})\, u^{\lambda}$ which equals, because of (27), $u^{\lambda}\, \omega_{\lambda\,[;\mu}\, u_{\nu]}$ and, by $u_{\lambda}\, \omega^{\lambda} = 0$, equals $\omega^{\lambda}\, u_{\lambda\,[;\nu}\, u_{\mu]}$ which vanishes because of (20) and (24) :

$$\omega_{[\mu,\nu]} = 0 . \qquad (29)$$

(21) and (29) show that in a rigidly moving fluid without pressure $\vec{\omega}$ is a harmonic vector field; therefore a scalar U exists with

$$\omega_{\nu} = U_{,\nu} , \quad U^{,\nu}{}_{;\nu} = 0 . \qquad (30)$$

Because of (20) and (24)

$$\bar{\eta}_{\kappa\lambda\mu\nu}\, U^{,\mu}\, u^{\nu} = u_{\kappa;\lambda} \ : \qquad (31)$$

* (26) shows that $\rho \geqslant 0$ need not be postulated; it follows from (15), (24).

because of (26)

$$\rho = 4 U_{,\nu} U^{,\nu}. \tag{32}$$

It is easy to show that, conversely, (31) and (32) imply (24) and (25); so we may state : The system of equations (25), (24) is equivalent to the system (31), (32).

We wish now to consider the complète system (15), (24). Suppose we are given a solution $(g_{\lambda\mu}, u_\lambda, \rho)$; then we can determine U by (30) so that (31), (32) hold. We take an arbitrary scalar t satisfying

$$u^\lambda t_{,\lambda} = 1 \tag{33}$$

and construct the quadratic differential form

$$G = e^{-2U}(\bar{G} + (\vec{u} \cdot d\vec{x})^2) - e^{2U} dt^2. \tag{34}$$

We assert that G is normal-hyperbolic and that the corresponding Ricci-tensor $R_{\lambda\mu}$ vanishes; this statement is the main result of this section.

To prove this statement we introduce « comoving coordinates » with

$$u^\lambda = \delta^\lambda_0, \qquad t = x^0, \tag{35}$$

which is possible because of (33). Then

$$\bar{G} = H - (dx^0 - u)^2, \tag{36}$$

and by (34)

$$G = e^{-2U} H - e^{2U}(dx^0)^2, \tag{37}$$

with H and u as in (1′), (1″). U is independent of x^0 because of (30), (35) and $u^\lambda \omega_\lambda = 0$. (31) and (32) reduce to the equations

$$-\eta_{ijk} U^{,k} = u_{[i,j]}, \ \rho = 4 h_{ik} U^{,i} U^{,k} \tag{38}$$

with respect to the metric H.

If we now work out the space-components $\bar{R}^i_k + \frac{1}{2} \rho \, \delta^i_k = 0$ of (15) for the metric (36), using the last line of (2) and taking into account (38) as well as $\bar{R}^i_0 = 0$ (which follows from (15) and (35)) we find that H and U satisfy (6). But this means that (37) is an exterior metric according to § 2. Moreover (35) and (37) show that G is static and that $\vec{u}$ is a Killing vector also with respect to G.

With help of (33) and (34) we get :

$$\xi_\lambda \equiv g_{\lambda\mu} u^\mu = -e^{2U} t_{,\lambda}, \ e^{2U} = -g_{\lambda\mu} u^\lambda u^\mu, \tag{39}$$

(notice that by definition $u_\lambda = \bar{g}_{\lambda\mu} u^\mu$) therefore (34) can be solved for $\bar{G}$:

$$\bar{G} = e^{2U} G + (\xi_\lambda dx^\lambda)^2 - (u_\lambda dx^\lambda)^2. \tag{40}$$

It is also easy, by using the coordinates with (35), to reformulate (31) in four-dimensional form with respect to G :

$$\eta_{\kappa\lambda\mu\nu} U^{,\mu} u^\nu = u_{[\kappa,\lambda]}. \tag{41}$$

Again we can obtain a conserve. We can start with a static exterior field with metric G and Killing-field $\vec{u} = (u^\lambda)$, solve (41) simultaneously with $u^\lambda u_\lambda = -1$ to obtain u_λ, construct $\overline{G}$ by (40), and calculate ρ by (32). Then (25) is satisfied because of (41) and (32), and the space-components of (15) with respect to a coordinate system with (35) are fulfilled in consequence of $R_{\lambda\mu} = 0$.

Now our proof of the equivalence of (15), (24) to the equations $\{R_{\lambda\mu} = 0, \quad \xi_{(\lambda;\mu)} = 0, \quad \xi_{[\lambda} \xi_{\mu,\nu]} = 0, \quad \xi_\lambda \xi^\lambda < 0\}$ is accomplished; the transformation $(\bar{g}_{\lambda\mu}, u_\lambda, \rho) \rightarrow (g_{\lambda\mu}, \xi_\lambda)$ is given by (34) where U and t are determined by (30) resp. (33), and $(g_{\lambda\mu}, \xi_\lambda) \rightarrow (\bar{g}_{\lambda\lambda}, \rho)$ is given by $e^{2U} = -\xi_\lambda \xi^\lambda$, (41), (40), and (32).

Van Stockum's general solution is obtained from our theorem by specialising G to Weyl's axially symmetric static vacuum metric (4); van Stockum's special solution with a rotating fluid cylinder corresponds to that Weyl-solution the « potential » $\frac{1}{2} \log |g_{oo}|$ of which is independent of Weyl's « canonical » radial coordinate.

Finally I mention that solutions of (15), (24) with constant density do not exist [15] and that, because of Bochner's lemma [16] and (7), solutions in which the space-time manifold W is (globally) a topological product of the real line R and a three-dimensional compact orientable manifold S such that the points of S correspond to the streamlines also do not exist.

The most interesting questions in the further investigation of these solutions are : Are there everywhere regular, complete solutions of (15), (24) with a finite total mass ? Is it possible to connect smoothly such matter fields with exterior solutions in other cases than the one that has been treated by van Stockum ?

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Remarque faite après la conférence du Docteur Ehlers (Royaumont, 1959)

C. B. Rayner

Je voudrais indiquer brièvement quelques-uns des résultats que j'ai obtenus dans la théorie du mouvement rigide en relativité générale, puisque je les crois être pertinents au sujet traité par le Docteur Ehlers.

Le mouvement rigide, au sens de Born, a été étudié par lui, par Herglotz, Rosen, Saltzmann et d'autres. Rosen a été le premier à donner les équations :

$$\sigma_{\alpha\beta} + \sigma_{\beta\alpha} = 0\,; \qquad \sigma_{\alpha\beta} = \nabla_\beta \lambda_\alpha + \varkappa_\alpha \lambda_\beta\,; \qquad \varkappa_\alpha = \lambda^\gamma \nabla_\gamma \lambda_\alpha \tag{1}$$

auxquelles doivent satisfaire le vecteur unitaire λ^α, $(\lambda^\gamma \lambda_\gamma = -1)$, tangent aux lignes d'univers du mouvement rigide. On peut considérer le mouvement rigide en relativité générale en identifiant λ^α avec le vecteur propre orienté dans le temps du tenseur d'Einstein $G_{\alpha\beta}$. On a :

$$G_{\alpha\beta} = -T_{\alpha\beta} = -\rho \lambda_\alpha \lambda_\beta - S_{\alpha\beta}, \tag{2}$$

où ρ est la densité propre, et $S_{\alpha\beta}$ le tenseur de pression. On peut éliminer $S_{\alpha\beta}$ de (2) en multipliant par λ^β :

$$f^\alpha \equiv G^\alpha_\beta \lambda^\beta - \rho\lambda^\alpha = 0; \qquad \rho = -G_{\gamma\delta} \lambda^\gamma \lambda^\delta. \tag{3}$$

Si on peut satisfaire à (1), (3) avec un système $(g_{\alpha\beta}, \lambda_\gamma)$, les tensions internes qui résultent du mouvement sont données par (2).

J'ai étudié [1] le système (1), (3) et montré que lorsque (1) est satisfaite, le vecteur f^α défini par (3) peut se présenter sous la forme :

$$f^\alpha \equiv \nabla_\beta \sigma^{\alpha\beta} + \sigma^{\alpha\beta} \varkappa_\beta - 2\sigma\lambda^\alpha; \; 2\sigma = \sigma^{\beta\gamma} \sigma_{\beta\gamma}. \tag{4}$$

Une conséquence de (4) est que :

$$\nabla_\alpha f^\alpha + \varkappa_\alpha f^\alpha \equiv -\lambda^\alpha \partial_\alpha \rho = -3\lambda^\alpha \partial_\alpha \sigma. \tag{5}$$

Si $\theta^\alpha \equiv \eta^{\alpha\beta\gamma\delta} \lambda_\beta \partial_\delta \lambda_\gamma$ ($\eta^{\alpha\beta\gamma\delta}$ étant le tenseur élément de volume) est le vecteur moment-angulaire, on montre aisément que :

$$\sigma^{\alpha\beta} = -(1/2)\, \eta^{\alpha\beta\gamma\delta} \lambda_\gamma \theta_\delta, \quad \sigma = (1/4)\, g_{\alpha\beta}\, \theta^\alpha \theta^\beta. \tag{6}$$

Ainsi, par (5), la densité propre et la grandeur du vecteur moment-angulaire sont constantes le long des lignes d'univers d'un mouvement rigide.

(1) *C. R. Acad. Sc.*, t. *248* (1959) 929.

On voit que (1) se réduit aux équations de Killing si $\varkappa_\alpha$ est un gradient. (Si $\varkappa_\alpha = v^{-1}\,\partial_\alpha v$, $v\lambda^\alpha$ est le vecteur de Killing). J'ai obtenu (2) une solution générale du système $f^\alpha = 0$, $\nabla_\gamma \xi_\beta + \nabla_\beta \xi_\gamma = 0$. Me servant de celle-ci, j'ai montré (3) que le problème de la résolution des équations extérieures d'Einstein $R_{\alpha\beta} = 0$ dans le cas stationnaire est réductible à celui de trouver un tenseur défini-positif $\tilde{g}_{ij}\,(i, j = 1, 2, 3)$ et deux scalaires α, β pour satisfaire au problème en trois dimensions :

$$\tilde{R}_{ij} + (1/2)\,\omega_{ij} = 0\;; \qquad \omega_{ij} = \alpha^{-2}(\partial_i \alpha \partial_j \alpha + \partial_i \beta \partial_j \beta). \tag{7}$$

Ici $\tilde{R}_{ij}$ est le tenseur de Ricci déterminé par $\tilde{g}_{ij}$. Il est intéressant de remarquer que la forme différentielle $\omega_{ij}\,dx^i\,dx^j \equiv \alpha^{-2}(d\alpha^2 + d\beta^2)$ est la métrique d'un espace V_2 à courbure constante négative.

(2) *C. R. Acad. Sc.*, t. *248* (1959) 1725.
(3) Dans trois notes qui paraîtront bientôt dans les *Comptes Rendus*.