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Quantum Extended Arithmetic Veneziano Amplitude

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The Veneziano amplitude for the tree-level scattering of four tachyonic scalar of open string theory has an arithmetic analogue in terms of the p-adic gamma function. We propose a quantum extension of this amplitude using the q-extended p-adic gamma function given by Koblitz. This provides a one parameter deformation of the arithmetic Veneziano amplitude. We also comment on the dificulty in generalising this to higher point amplitudes.

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In the context of string theory, the Veneziano amplitude, which also marks the birth of this subject, describes the tree level scattering of four tachyonic scalars of the open bosonic string. The tachyon field, localised on a D-brane defined by the open string, signals an instability of this classical vacuum configuration. Understanding the dynamics of this field has been one of the subject of foremost interest in recent years and has provided us with valuable insight into non-perturbative aspect of string theory (see [1] and references therein).

This motivates one to consider the tachyon amplitudes from many different perspectives. Mathematically there are several possibilities. One that was considered soon after Veneziano's proposal[2] is a quantum extension or q-deformation[3,4] using the q-deformed gamma and beta functions. This was further investigated in the following years (see [5] for history and a list of references) and revisited more recently in [5--9].

Another direction of investigation considered the problem in the domain of local fields. In fact, two quite different theories are possible. The first starts with the integral representation of the Veneziano amplitude (and its generalisation the Koba-Nielsen amplitudes) and defines their p-adic analogues[10--13] by appropriately generalising the formulas to the field $\mathbf{Q}_p$ of p-adic numbers (see [14] for a review). The Veneziano amplitude, for example, involves complex valued Gelfand-Graev gamma function on the p-adic field[15,16]. In this theory, which we may call the *p-adic string* theory, the spacetime is the usual one and the properties of the tachyon qualitatively resembles those of the usual bosonic string[17--21]. In the second approach[22,23], one directly defines the p-adic analogues of the Veneziano formula in terms of Morita's p-adic valued p-gamma function[24,25]. This latter version will be called *arithmetic Veneziano amplitude* to distinguish it from the former one. Other arithmetic analogues of string theory can be found in Refs.[26--28].

In this note, we consider a quantum extension of the *arithmetic Veneziano amplitude* using the q-extension of the p-adic gamma function given by Koblitz[29].

Morita's p-adic gamma function $\Gamma_p : \mathbf{Z}_p \to \mathbf{Z}_p^*$ is defined on positive integers as

$$\Gamma_p(n+1) = (-)^{n+1} \prod_{\substack{m=1 \\ p \nmid m}}^{n} m, \tag{1}$$

where, $p \nmid m$ means that m is not divisible by p. This is then extended to $\mathbf{Z}_p$ by continuity[24,25] and is a generalisation of the factorial function to p-adic integers. In order to extend to all of $\mathbf{Q}_p$, one may rewrite the above as

$$\Gamma_p(n+1) = (-)^{n+1} \prod_{\substack{m=1 \\ p \not| m}}^{\infty} \frac{m}{n+m}, \tag{2}$$

and the continue by the replacement $n \to x \in \mathbf{Q}_p$. The p-adic gamma function satisfies the recursion relation

$$\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x), & \text{if } x \in \mathbf{Z}_p^*, \\ -\Gamma_p(x), & \text{if } x \in p\mathbf{Z}_p. \end{cases} \tag{3}$$

A q-extension of the above is defined by Koblitz[29]. One again starts with its form for positive integers:

$$\Gamma_{p,q}(n+1) = (-)^{n+1} \prod_{\substack{m=1 \\ p \not| m}}^{n} \frac{1-q^m}{1-q}, \tag{4}$$

where, $0 < |q-1|_p < 1$; and then extends to $\mathbf{Q}_p$ by continuity[1]:

$$\Gamma_{p,q}(x+1) = (-)^{x+1}(1-q)^{-x} \prod_{\substack{m=1 \\ p \not| m}}^{\infty} \frac{1-q^m}{1-q^{x+m}}. \tag{5}$$

Various properties of $\Gamma_{p,q}(x)$ are studied in [29]. For example,

$$\Gamma_{p,q}(x+1) = \begin{cases} -\frac{1-q^x}{1-q}\,\Gamma_{p,q}(x), & \text{if } x \in \mathbf{Z}_p^*, \\ -\Gamma_{p,q}(x), & \text{if } x \in p\mathbf{Z}_p, \end{cases} \tag{6}$$

and $\Gamma_{p,q}(1) = -1$.

All these parallel the case of the usual gamma function and its q-analogue $\Gamma_q(x)$

$$\Gamma_q(x+1) = (1-q)^{-x} \prod_{m=1}^{\infty} \frac{1-q^m}{1-q^{x+m}}. \tag{7}$$

The functions $\Gamma_q(x)$ appear in the q-deformed Veneziano amplitude[3--9]. (This amplitude also has an infinite number of complex poles distributed along the imaginary axis for each pole at real momentum values. This is reminiscent of the singularity structure of the other kind of p-adic string amplitudes[10-12].) It is well known that in the $q \to 1$ (classical) limit, one recovers the undeformed function: $\lim_{q\to 1} \Gamma_q(x) = \Gamma(x)$. Through this relation one recovers the usual Veneziano amplitude in the classical limit.

[1] Actually q here may be an element of $\mathbf{C}_p$, the completion of the algebraic closure of $\mathbf{Q}_p$.

Similarly in the p-adic case,

$$\lim_{q\to 1} \Gamma_{p,q}(x) \;=\; \Gamma_p(x). \tag{8}$$

Therefore, the q-extended function $\Gamma_{p,q}$ is a one parameter family of deformation of the function (1). Since, the only difference between (1) or (4) and their usual counterparts are from those factors larger than p, one finds that

$$\lim_{p\to\infty} \Gamma_p(n) = (-)^n \Gamma(n), \qquad\qquad \lim_{p\to\infty} \Gamma_{p,q}(n) = (-)^n \Gamma_q(n), \tag{9}$$

i.e., a passage to the p-adic and usual gamma functions evaluated at integers.

The arithmetic Veneziano amplitude were written in terms of the p-adic gamma function (2). Consider d-dimensional 'momenta' $k_1, \cdots, k_4$ valued in the vector space $\mathbf{Q}_p^d$ and a $\mathbf{Q}_p$-valued quadratic form $\langle . | . \rangle$ on it such that $k_i^2 \equiv \langle k_i | k_i \rangle = 2$. Let $s = (k_1 + k_2)^2$ and $t = (k_1 + k_3)^2$ are the Mandelstam variables and define linear functions $\alpha(s) = 1 + \alpha' s$ and similarly $\alpha(t)$. The proposed form of the amplitude is[22,23]

$$A_p(s,t) = \frac{\Gamma_p\left(\alpha(s)\right) \Gamma_p\left(\alpha(t)\right)}{\Gamma_p\left(\alpha(s) + \alpha(t)\right)}. \tag{10}$$

This amplitude has poles whenever the momenta in the intermediate channel is (proportional to) a negative integer not divisible by p. There are an infinite number of poles in either the s or the t channel.

A straightforward quantum extension of the above can be written in terms of the q-deformed p-adic gamma function (5):

$$A_{p,q}(s,t) = \frac{\Gamma_{p,q}\left(\alpha(s)\right) \Gamma_{p,q}\left(\alpha(t)\right)}{\Gamma_{p,q}\left(\alpha(s) + \alpha(t)\right)}. \tag{11}$$

From its definition and the property (8), it is immediately obvious that the above gives us a one-parameter family of arithmetic Veneziano amplitude. Moreover, in the $p \to \infty$ limit thanks to (9), the q-deformed arithmetic family makes contact to the usual q-deformed Veneziano amplitude. The analytic structure of the amplitude (11) is an expected combination of that of the p-adic amplitude (10) and the usual q-deformed one.

Finally, let us discuss about the difficulty in generalising the arithmetic Veneziano amplitude and its quantum extension to the higher point tree amplitudes. Recall that, in the usual case, one can use the symbol $(a;q)_n = \prod_{m=0}^{n-1}(1-q^m a)$ (also valid in the limit $n\to\infty$), to write the q-gamma function as $\Gamma_q(x) = (1-q)^{1-x}(q;q)_\infty/(q^x;q)_\infty$. There is a useful theorem, called the q-binomial theorem,

$$\frac{(zq^\alpha;q)_\infty}{(z;q)_\infty} = \sum_{m=0}^{\infty} \frac{(q^\alpha;q)_m}{(q;q)_m}\, z^m, \tag{12}$$

which may be used to re-write the q-deformed Veneziano amplitude as follows.

$$\begin{aligned} A_q(s,t) &= (1-q)(q;q)_\infty \frac{\left(q^{\alpha(s)+\alpha(t)};q\right)_\infty}{\left(q^{\alpha(s)};q\right)_\infty \left(q^{\alpha(t)};q\right)_\infty} \\ &= (1-q)(q;q)_\infty \sum_m \frac{q^{m\alpha(s)}}{(q;q)_m \left(q^{m+\alpha(t)};q\right)_\infty} \\ &= (1-q)(q;q)_\infty \sum_{m,n} \frac{q^{m\alpha(s)+n\alpha(t)+mn}}{(q;q)_m (q;q)_n}. \end{aligned} \tag{13}$$

In the second line above, one sees the poles in the t-channel — alternatively one could have displayed the singularities in the s-channel. More interesting is the symmetric form in the last line. It suggests a possible generalisation to the higher point amptitudes[4,5]. For the n-point amplitude, consider the set of $n(n-3)/2$ independent planar channels labelled by $\{i,j\}$, $1 \le i < j < n$ (excluding $\{1, n-1\}$) corresponding to the set of tachyons numbers $i, i+1, \cdots, j$. Let, $s_{ij} = (p_i + p_{i+1} + \cdots + p_j)^2$ and $\alpha_{ij} \equiv \alpha(s_{ij})$. The n-point tree amplitude is:

$$\mathcal{A}_q^{(n)} = [(1-q)(q;q)_\infty]^{n-3} \sum_{\{\ell\}} \prod_{ij} \frac{q^{\ell_{ij}\alpha_{ij}}}{(q;q)_{\ell_{ij}}} \prod_{mn;m'n'} q^{\ell_{mn}\ell_{m'n'}}, \tag{14}$$

where, ℓ_{ij} is the summation index in the ij channel and the first product above is over all single channels and the second over all distinct pairs of overlapping ones.

Let us try to follow the above steps as closely as possible in the p-adic case. First in writing Koblitz' q-extended p-gamma function (5), we are naturally led to the symbol

$$(a;q)_{p,n} = \prod_{\substack{m=0 \\ p\nmid m+1}}^{n-1} (1-q^m a), \tag{15}$$

in terms of which

$$\Gamma_{p,q}(x) = (-)^x (1-q)^{1-x} \frac{(q;q)_{p,\infty}}{(q^x;q)_{p,\infty}}. \tag{16}$$

Let us look for a generalisation of the q-binomial theorem starting with the ratio of p-adic symbols:

$$\frac{(q^{\alpha}z;q)_{p,\infty}}{(z;q)_{p,\infty}} = \prod_{\substack{m=0\\ p\not| m+1}}^{\infty} \frac{1-q^{m+\alpha}z}{1-q^{m}z}. \tag{17}$$

For an integer α, i.e., $\alpha = n$ in the $q \to 1$ limit the above reduces to $(1-z)^{-n+\nu_p(n)}$, where the function $\nu_p(n) = \sum_{m\le n, p|m} 1$, counts the numbers upto and including n that are divisible by the given prime p. Incidentally, $\nu_p(n)$ seems to be inherently more arithmetic that what we have encountered so far. Using standard tricks of manipulation of infinite products, we remove the restriction by putting in extra terms and compensate for it by another product running over integers which are multiples of $m+1$. This leads to

$$\frac{(q^{\alpha}z;q)_{p,\infty}}{(z;q)_{p,\infty}} = \frac{(q^{\alpha}z;q)_{\infty} \,/\, (z;q)_{\infty}}{\left(q'^{\alpha/p}z';q'\right)_{\infty} \Big/ \, (z';q')_{\infty}}, \tag{18}$$

where, $q' = q^p$ and $z = q^{p-1}z$. Now one can use the q-binomial theorem (12) to simplify this. In particular, the q-extended p-adic Veneziano amplitude is

$$\begin{aligned} A_{p,q}(s,t) &= (1-q)\,(q;q)_{p,\infty}\; \frac{\left(q^{\alpha(s)+\alpha(t)};q\right)_{p,\infty}}{\left(q^{\alpha(s)};q\right)_{p,\infty}\left(q^{\alpha(t)};q\right)_{p,\infty}} \\ &= \frac{(1-q)\,(q;q)_{p,\infty}\;\left(q^{\alpha(s)+\alpha(t)};q\right)_{\infty} \Big/ \left(q^{\alpha(s)};q\right)_{\infty}\left(q^{\alpha(t)};q\right)_{\infty}}{\left(q'^{(\alpha(s)+\alpha(t))/p}q^{p-1};q'\right)_{\infty} \Big/ \left(q'^{\alpha(s)/p}q^{p-1};q'\right)_{\infty}\left(q'^{\alpha(t)/p}q^{p-1};q'\right)_{\infty}}, \end{aligned} \tag{19}$$

which is expressed as a ratio of the ordinary q-symbols and thus in turn can be simplified with (12). Despite the initial impression, the form above does not lead to something like the last line of (13). The extra factors of q^{p-1} spoil our effort. The case $q^{p-1} = 1$ may appear to be promising, however, it gives the trivial result $A_{p,q}(s,t) = 1$.

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References

[1] A. Sen, *Tachyon dynamics in open string theory*, Int. J. Mod. Phys. **A20** (2005) 5513, [hep-th/0410103].

[2] G. Veneziano, *Construction of a crossing-symmetric, Regge behaved amplitude for linearly rising trajectories*, Nuovo Cim. **A57** (1968) 190.

[3] D. Coon, *Uniqueness of the Veneziano representation*, Phys. Lett. **B29** (1969) 669.

[4] M. Baker and D. D. Coon, *Dual Resonance Theory With Nonlinear Trajectories*, Phys. Rev. D **2**, 2349 (1970).

[5] L. Romans, *Deforming the Veneziano model*, in Proceedings, High energy physics and cosmology, 278-287, Trieste (1989); also *A new family of dual models ('Q Strings')*, Preprint USC88/HEP-014 (as quoted in proc. above).

[6] M. Chaichian, J. Gomes and P. Kulish, *Operator formulation of q-deformed dual string model*, Phys. Lett. **B311** (1993) 93;
M. Chaichian, J. Gomes and R. Gonzalez-Felipe, *New phenomenon of nonlinear trajectory and quantum dual string theory*, Phys. Lett. **B341** (1994) 147.

[7] L. Jenkovszky, M. Kibler and A. Mishchenko, *Two parametric quantum deformed dual amplitude*, Mod. Phys. Lett. **A10** (1995) 51, [hep-th/9407071].

[8] D. Fairlie and J. Nuyts, *A fresh look at generalized Veneziano amplitudes*, Nucl. Phys. **B433** (1995) 26 [hep-th/9406043].

[9] N. Mebarki, A. Boudine, H. Aissaoui, A. Benslama, A. Bouchareb and A. Maasmi, *No-ghost theorem for a q-deformed open bosonic string*, J. Geom. Phys. **30** (1999) 103.

[10] P.G.O. Freund and M. Olson, *Nonarchimedean strings*, Phys. Lett. **B199** (1987) 186.

[11] P.G.O. Freund and E. Witten, *Adelic string amplitudes*, Phys. Lett. **B199** (1987) 191.

[12] L. Brekke, P.G.O. Freund, M. Olson and E. Witten, *Non-archimedean string dynamics*, Nucl. Phys. **B302** (1988) 365.

[13] P.H. Frampton and Y. Okada, *The p-adic string N point function*, Phys. Rev. Lett. **60** (1988) 484; *Effective scalar field theory of p-adic string*, Phys. Rev. **D37** (1988) 3077;
P.H. Frampton, Y. Okada and M.R. Ubriaco, *On adelic formulas for the p-adic string*, Phys. Lett. **B213** (1988) 260.

[14] L. Brekke and P.G.O. Freund, *p-Adic numbers in physics*, Phys. Rep. **233** (1993) 1, and references therein.

[15] I.M. Gelfand, M.I. Graev and I.I. Pitaetskii-Shapiro, *Representation theory and automorphic functions*, Saunders (1969).

[16] V.S. Vladimirov, I.V. Volovich and E.I. Zelenov, *p-Adic Analysis and Mathematical Physics*, (Series on Soviet and East European Mathematics, Vol.1) World Scientific, 1994.

[17] D. Ghoshal and A. Sen, *Tachyon condensation and brane descent relations in p-adic string theory*, Nucl. Phys. **B584** (2000) 300, [hep-th/0003278].

[18] D. Ghoshal, *Exact noncommutative solitons in p-adic strings and BSFT*, JHEP **0409** (2004) 41, [hep-th/0406259].

[19] D. Ghoshal and T. Kawano, *Towards p-adic string in constant B-field*, Nucl. Phys. **B710** (2005) 577, [hep-th/0409311].

[20] For a review including recent developments, see P. Freund, *p-adic strings and their applications*, [hep-th/0510192].

[21] D. Ghoshal, *p-Strings vs. strings*, Preprint HRI-P-0605001, to appear in the proceedings of the '12th Regional Conference in Mathematical Physics', Islamabad, Pakistan, World Scientific.

[22] I.V. Volovich, *p-Adic string*, Class. Quant. Grav. **4** (1987) L83.

[23] B. Grossman, *p-Adic strings, the Weyl conjectures and anomalies*, Phys. Lett. **B197** (1987) 101.

[24] N. Koblitz, *p-Adic numbers, p-adic analysis and zeta functions*, GTM 58, Springer-Verlag (1977).

[25] A.M. Robert, *A course in p-adic analysis*, GTM 198, Springer-Verlag (2000).

[26] P. Candelas, X. de la Ossa and F. Rodriguez-Villegas, *Calabi-Yau manifolds over finite fields, I*, [hep-th/0012233]; *Calabi-Yau manifolds over finite fields, II*, [hep-th/0402133].

[27] R. Schimmrigk, *Arithmetic of Calabi-Yau varieties and rational conformal field theory*, J. Geom. Phys. **44** (2003) 555, [hep-th/0111226]; *Arithmetic spacetime geometry from string theory*, [hep-th/0510091].

[28] M. Kontsevich, A. Schwarz and V. Vologodsky, *Integrality of instanton numbers and p-adic B-model*, [hep-th/0603106].

[29] N. Koblitz, *q-Extension of the p-adic gamma function*, Trans. Am. Math. Soc. **260** (1980) 449.