Arch. Rational Mech. Anal. 200 (2011) 313–360
Digital Object Identifier (DOI) 10.1007/s00205-011-0405-3

Optimal Gradient Estimates and Asymptotic Behaviour for the Vlasov–Poisson System with Small Initial Data

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Communicated by Y. Brenier

Abstract

The Vlasov–Poisson system describes interacting systems of collisionless particles. For solutions with small initial data in three dimensions it is known that the spatial density of particles decays as t^{-3} at late times. In this paper this statement is refined to show that each derivative of the density which is taken leads to an extra power of decay, so that in N dimensions for $N \geqq 3$ the derivative of the density of order k decays as t^{-N-k}. An asymptotic formula for the solution at late times is also obtained.

1. Introduction

The Vlasov–Poisson system provides a statistical description of the dynamics of a large number of particles which are acted upon by a force field which they generate collectively. One class of applications of this system is in plasma physics, where the force is electrostatic and the particles are electrons or ions [10]. Another is in stellar dynamics, where stars play the role of particles. The particle treatment is justified in models of galaxies where the distance between stars is much larger than their diameters. In this case the force is gravitational [2]. The equations in these two cases differ only by a sign and much of the mathematical theory works in exactly the same way for both. This applies in particular to the results of this paper. For surveys of results on the Vlasov–Poisson and related systems, see [8] and [1].

The distribution function f of the particles satisfies the Vlasov equation, while the potential ϕ for the field satisfies the Poisson equation. The function f depends on time t, the spatial point $x \in \mathbb{R}^3$ and the velocity $v \in \mathbb{R}^3$. It is natural to pose an initial value problem with f being prescribed at $t = 0$. For an initial datum which is C^1 and has compact support, it is known that there exists a unique corresponding C^1 solution, globally in time [15,18]. The support of f is compact at each fixed time t, and an important diagnostic quantity is $P(t)$, the supremum of $|v|$ over the support of f at time t. Estimates are known for $P(t)$ [12]; these imply estimates for $\|\rho(t)\|_{L^\infty}$ where ρ, the spatial density of particles, is given by $\rho(t,x) = \int f(t,x,v)\mathrm{d}v$. We have $P(t) \leqq C(1+t)\log(2+t)$. Unfortunately these estimates seem far from optimal. They are the same for the plasma physics and stellar dynamics cases. Intuitively, it is to be expected that the optimal estimates differ in these two cases. In the stellar dynamics case, there exist time-independent solutions so that $\|\rho(t)\|_{L^\infty}$ does not decay in general. In the plasma physics case, decay estimates for integral norms of ρ are established in [13] and [17]. The pointwise estimates can also be improved to give a bound for $P(t)$ of the form $C(1+t)^{2/3}$ [21].

It is possible to consider the analogue of the Vlasov–Poisson system in higher dimensions. It is, however, known that global existence fails in four space dimensions [11]. An explicit example of singularity formation and information on the asymptotics of solutions near a singularity were obtained in [14].

There is a case where much more is known about the long-time asymptotics of solutions of the Vlasov–Poisson system, namely that of small initial data. The first global existence theorem for that case due to Bardos and Degond [3] naturally comes with decay estimates. They show that

$$\|\rho(t)\|_{L^\infty} \leqq C(1+t)^{-3}.$$

If the data are sufficiently differentiable, then it would be natural to expect decay estimates of the form

$$\|D^k\rho(t)\|_{L^\infty} \leqq C(1+t)^{-3-k}, \quad k = 1, 2, \dots \tag{1.1}$$

To date, however, these estimates have not been derived in the literature. In this paper we apply new techniques for this problem to obtain estimates of this form for solutions with small initial data. Furthermore, we obtain asymptotic expansions for these solutions.

Note that there are a number of generalizations of the results of [3] in the literature. The fully relativistic generalization of the plasma physics problem is given by the Vlasov–Maxwell system. An analogue of the result of [3] in that case was proved in [9]. In the stellar dynamics problem the fully relativistic generalization is the Einstein–Vlasov system [24], which is much more complicated. A small data global existence theorem in the spherically symmetric case was obtained in [22]. A related system which is physically incorrect but mathematically interesting is the Vlasov–Nordström system, for which there is a global existence theorem [5]. Surprisingly, it seems that no analogue of the asymptotic result of [3] has been proved for this system. There are generalizations of the results for solutions of the Vlasov–Poisson and Vlasov–Maxwell systems with small data to almost spherically symmetric data [19,25]. There are also results for solutions of the Vlasov–Poisson system with non-standard boundary conditions which are relevant to cosmology [20,23]. Global existence has also been proved for some cosmological solutions of the Einstein–Vlasov system with symmetry; see, for instance, [26]. It would be interesting to extend the results of this paper to some of the cases mentioned in this paragraph.

This paper was motivated by the wish to prove a small data global existence theorem for the Einstein–Vlasov system which does not require any symmetry assumptions. To understand the difficulty of this problem, note first that even the vacuum Einstein equations, from the present point of view the Einstein–Vlasov system with $f = 0$, are very hard to handle mathematically. The landmark work of Christodoulou and Klainerman on small data global existence for the vacuum Einstein equations [6] is so complicated as to discourage any attempts to incorporate matter. The more recent alternative proof of Lindblad and Rodnianski [16] looks much more promising. Nevertheless, it seems to require good decay estimates for higher derivatives, that is, estimates similar to those proved for the Vlasov–Poisson system here.

The Vlasov–Poisson system in N dimensions reads

$$f_t + v \cdot \nabla_x f + \gamma \nabla_x \phi \cdot \nabla_v f = 0, \quad x \in \mathbb{R}^N, \quad t > 0 \tag{1.2}$$

$$\Delta\phi = \int_{\mathbb{R}^N} f \mathrm{d}v \equiv \rho(x,t), \quad x \in \mathbb{R}^N, \quad t > 0, \tag{1.3}$$

where $f = f(x, v, t)$. In the following we assume that $f(x, v, 0) = f_0(x, v)$ has finite L^1 norm and $N \geqq 3$. The sign $\gamma = \pm 1$ corresponds to the plasma physics and gravitational problems, respectively. Since the results of this paper apply equally to both cases, we will restrict our analysis to the case $\gamma = 1$. No sign condition on f_0 is needed. However, some additional decay properties for $f_0(x, v)$ will be assumed.

Global existence and decay estimates for the Vlasov–Poisson system were studied in [3] in three spatial dimensions under suitable smallness and regularity assumptions for f_0. These estimates are optimal in the rate of decay for the density ρ since, for small compactly supported initial data, the volume of the support of ρ can be bounded by $C(1+t)^3$ so that if the decay in L^∞ was stronger than $(1+t)^{-3}$, the total number of particles (that is the L^1 norm of ρ) would decay, leading to a contradiction due to the conservation of that quantity. However, they do not provide the optimal rate of decay for the derivatives that could be expected on dimensional grounds.

For small initial data the dynamics of the Vlasov–Poisson system might be expected to be dominated by the free streaming part of the equation

$$f_t + v \cdot \nabla_x f = 0,$$

because the term $\nabla_x \phi \cdot \nabla_v f$ is quadratic in the density. (Actually this is a consequence of the Bardos–Degond analysis). If we assume that the dynamics of the problem is dominated by the free streaming regime as $t \to \infty$ and the initial density of particles is, say, compactly supported (fast enough decay works similarly), the velocities of the particles would be bounded by a number of order one. Therefore, the support of the density ρ would spread linearly. The field $\nabla\phi$ generated by a particle density with finite mass spread over a region of order t decreases as $\frac{1}{t^2}$ as can be easily seen by means of a rescaling argument. Notice that a posteriori this provides a justification for the assumption that was made before concerning the finiteness of the deviation of the velocities of the particles due to the interaction of the field.

The main contribution of this paper is the development of a technique that allows us to obtain optimal decay estimates for the solutions of the VP system in N-dimensional space. More precisely, the rescaling argument sketched above suggests that the particles spread into a region of volume t^N in the x-coordinate. Since the total mass of the particles is of order one, it would be natural to expect the following estimates for the density

$$\begin{aligned}
|\rho| &\leqq \frac{C}{(t+1)^N} \\
|\nabla\rho| &\leqq \frac{C}{(t+1)^{N+1}} \\
\left|\nabla^2\rho\right| &\leqq \frac{C}{(t+1)^{N+2}} \\
&\cdots \\
\left|\nabla^k\rho\right| &\leqq \frac{C}{(t+1)^{N+k}}.
\end{aligned}$$

The first estimate was obtained by Bardos–Degond for the case $N = 3$ and can be similarly extended to the case $N > 3$. Our method allows us to obtain the corresponding estimates for the derivatives for small initial data.

The basic idea of the method is as follows. It is easy to see self-similar behaviour for the density (and the derivatives) in the free streaming case. Indeed, in that case, integration along characteristics yields

$$f(x, v, t) = f_0(x - vt, v)$$

whence:

$$\rho(x, t) = \int f(x, v, t)\,\mathrm{d}v = \int f_0(x - vt, v)\,\mathrm{d}v.$$

In order to obtain self-similar behaviour, we make the change of variables

$$\begin{aligned}
x_0 &= x - vt, \\
\mathrm{d}v &= \left|\det\left(\frac{\partial v}{\partial x_0}\right)\right| \mathrm{d}x_0 = \frac{1}{t^N}\mathrm{d}x_0,
\end{aligned}$$

whence

$$\rho(x, t) = \frac{1}{t^N}\int f_0\left(x_0, \frac{x - x_0}{t}\right)\mathrm{d}x_0.$$

In the limit $t \to \infty$ this formula yields self-similar behaviour in the region where $|x|$ is of order t

$$\rho(x, t) \sim \frac{1}{t^N}\int f_0\left(x_0, \frac{x}{t}\right)\mathrm{d}x_0 = \frac{1}{t^N}\rho_{fs}\left(\frac{x}{t}\right). \tag{1.4}$$

Here the asymptotic free streaming density ρ_{fs} is given by

$$\rho_{fs}(y) \equiv \int f_0(x_0, y)\,\mathrm{d}x_0.$$

Notice that (1.4), at least formally, provides the desired estimates for the derivatives of ρ. The key idea of our argument is a method for generalizing this method to the full VP system with small initial data. The main point is the following. Suppose that the characteristics starting at x_0, v_0 reach the points x, v at time t. Assuming suitable invertibility conditions, any pair of variables in the set (x_0, v_0, x, v) can be used as a set of independent variables in order to represent the others. The previous argument for the free streaming case suggests using $x,\ x_0$ as independent variables. However, in order to determine the functions that provide $v_0,\ v$ in terms of $x,\ x_0$, it turns out to be necessary to solve a boundary value problem for the characteristic equations. The main argument of this paper consists in proving that such a boundary problem can be solved for small initial densities and that the corresponding solutions of such a boundary value problem satisfy suitable regularity and decay estimates.

Using a similar method, it is possible to obtain not only estimates for the derivatives of the density, but also convergence of the solutions of the VP system to a self-similar solution. More precisely, we rewrite the problem using the self-similar variables $y = \frac{x}{(t+1)},\ \ v = v,\ \ \tau = \log(t+1),\ \ f = \frac{1}{(t+1)^N} g$ and, after integrating the resulting equations along characteristics, we replace the variables (y, v) by (y, y_0). This change of variables requires the solution of a boundary value problem for the characteristic equations analogous to the one described above. Such a boundary value problem can be analyzed in detail as $\tau \to \infty$, and this provides the asymptotic behaviour of the density ρ as $t \to \infty$. One of the relevant results of the analysis is the fact that, although the asymptotics of the solutions are self-similar, the precise function describing the asymptotics of the density depends in a very sensitive manner on the choice of the initial data $f_0(x, v)$. This analysis is done in the last section of the paper (Theorem 4). Notice that this Theorem shows that the estimates (1.1) are optimal for the class of data for which Theorem 4 applies.

The paper is organized as follows. In Section 2, we derive estimates for the density and its derivatives. In Section 3, we prove convergence to the self-similar solution. Throughout the paper, $C > 0$ will denote a generic constant that may change from line to line and is independent of t, ε_0, f_0 and any of the independent variables involved.

Throughout this paper we will use the following notation for the derivatives. The notations $\nabla_x^\ell,\ \frac{\partial^\ell}{\partial x^\ell}$ are both used for the derivatives with respect to the x variables. The x-derivatives of functions depending only on x, t will be denoted as ∇^ℓ. The notations $\nabla_v^\ell,\ \frac{\partial^\ell}{\partial v^\ell}$ are both used with respect to the variables v. In the case of the derivatives with respect to the auxiliary variables x_0 we will use $\frac{\partial^\ell}{\partial x_0^\ell}$.

2. Estimating ρ and its derivatives

2.1. The main result

We will use the following function spaces extensively

$$X_{k,\alpha} = \left\{\rho \in L^\infty\left(\mathbb{R}^N \times \mathbb{R}^+\right) : \|\rho\|_{X_{k,\alpha}} < \infty\right\},$$
$$Y_{k,\alpha} = \left\{\phi \in L^\infty\left(\mathbb{R}^N \times \mathbb{R}^+\right) : \|\phi\|_{Y_{k,\alpha}} < \infty\right\},$$

where $0 < \alpha < 1$ and k is an integer $\geqq 0$ and where we use the norms

$$\|\rho\|_{X_{k,\alpha}} = \sup_{t \geqq 0} \left\{ \int |\rho(x,t)| \, \mathrm{d}^N x + (t+1)^N \sum_{\ell=0}^{k} (t+1)^{\ell} \left\| \nabla^{\ell} \rho(\cdot,t) \right\|_{L^{\infty}(\mathbb{R}^N)} \right. \\ \left. + (t+1)^{N+k+\alpha} \sup_{x,y \in \mathbb{R}^N} \frac{\left| \nabla^k \rho(x,t) - \nabla^k \rho(y,t) \right|}{|x-y|^{\alpha}} \right\}, \tag{2.1}$$

$$\|\phi\|_{Y_{k,\alpha}} = \sup_{t \geqq 0} \left\{ (t+1)^{N-2} \sum_{\ell=1}^{k+2} (t+1)^{\ell} \left\| \nabla^{\ell} \phi(\cdot,t) \right\|_{L^{\infty}(\mathbb{R}^N)} \right. \\ \left. + (t+1)^{N+k+\alpha} \sup_{x,x' \in \mathbb{R}^N} \frac{\left| \nabla^{k+2} \phi(x,t) - \nabla^{k+2} \phi(x',t) \right|}{|x-x'|^{\alpha}} \right\}. \tag{2.2}$$

The main result of the paper is the following Theorem.

Theorem 1. *Let $k \geqq 1$ be an integer. Suppose that $f_0(x,v) \in C^{k,\alpha} \cap L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ satisfies the following assumptions*

$$\sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \left| \frac{\partial^{\ell} f_0}{\partial x^m \partial v^{\ell-m}} \right| \leqq \frac{\delta_0}{(1+|x|)^K (1+|v|)^K}, \tag{2.3}$$

$$\sum_{m=0}^{k} \sup_{x,x' \in \mathbb{R}^N} \frac{\left| \frac{\partial^k f_0}{\partial x^m \partial v^{k-m}}(x,v) - \frac{\partial^k f_0}{\partial x^m \partial v^{k-m}}(x',v) \right|}{|x-x'|^{\alpha}} \leqq \frac{\delta_0}{(1+|v|)^K}, \quad 0 < \alpha < 1,$$

$$\sum_{m=0}^{k} \sup_{|v'-v| \leqq 1} \frac{\left| \frac{\partial^k f_0}{\partial x^m \partial v^{k-m}}(x,v) - \frac{\partial^k f_0}{\partial x^m \partial v^{k-m}}(x,v') \right|}{|v-v'|^{\alpha}} \leqq \frac{\delta_0}{(1+|x|)^K (1+|v|)^K}, \\ 0 < \alpha < 1,$$

for some suitable $K > N$ and $\delta_0 > 0$ small enough. Then there exists a unique corresponding solution $f(\cdot,\cdot,t)$ in $C^{k,\alpha} \cap L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ of the Vlasov–Poisson system with

$$\|\rho\|_{X_{k,\alpha}} \leqq C\delta_0. \tag{2.4}$$

Remark 1. The rate of decay in (2.4) is the best one that can be obtained for a large class of small initial data satisfying (2.3). This is a consequence of Theorem 4 proved in Section 3, where the explicit self-similar behaviour for a large class of solutions, not necessarily compactly supported, will be obtained. The rate of decay for the derivatives computed there shows that no better decay estimates for the derivatives than (2.4) can be expected for such solutions.

A result analogous to Theorem 1 was proved in the case $k = 2, \ N = 3$ under slightly different assumptions on f_0, by BARDOS–DEGOND (see [3]). The ideas in [3] can be adapted to study the case $k = 2, \ N > 3$. The main contribution of this paper is to derive the optimal decay estimates for the derivatives of ρ.

2.2. A basic boundary value problem for the characteristic curves

We introduce some basic notation. Suppose that the characteristics starting at (x_0, v_0) at time $t = 0$ reach the point (x, v) at time t. The basic idea in the paper is to use $x,\ x_0$ as independent variables to describe the values of v and v_0. More precisely, we will write

$$v = w\,(t, x, x_0), \tag{2.5}$$

$$v_0 = w_0\,(t, x, x_0). \tag{2.6}$$

The existence of the functions w, w_0 will be shown later. By changing the variable v to x_0 in the integral with x and t fixed, it then follows, using (2.5) that

$$\mathrm{d}v = \left|\det\left(\frac{\partial w\,(t, x, x_0)}{\partial x_0}\right)\right| \mathrm{d}x_0$$

whence

$$\begin{aligned}
&\rho\,(x, t)\\
&= \int_{\mathbb{R}^N} f\,(x, v, t)\,\mathrm{d}v\\
&= \int f_0\,(X\,(0; x, w\,(t, x, x_0)\,, t)\,, V\,(0; x, w\,(t, x, x_0), t))\,\mathrm{d}v\\
&= \int f_0\,(X\,(0; x, w\,(t, x, x_0)\,, t), V\,(0; x, w\,(t, x, x_0), t)) \left|\det \frac{\partial w\,(t, x, x_0)}{\partial x_0}\right| \mathrm{d}x_0\\
&= \int f_0\,(x_0, V\,(0; x, w\,(t, x, x_0), t)) \left|\det \frac{\partial w\,(t, x, x_0)}{\partial x_0}\right| \mathrm{d}x_0.
\end{aligned}$$

We now formulate the following auxiliary boundary value problem that describes the evolution of the characteristics starting at the spatial point x_0 at the initial time and reaching the point x at time t

$$\frac{\mathrm{d}X\,(s)}{\mathrm{d}s} = V\,(s),\ \frac{\mathrm{d}V\,(s)}{\mathrm{d}s} = \nabla\phi\,(X\,(s), s),\ \ X\,(t) = x,\ \ X\,(0) = x_0. \tag{2.7}$$

Notice that the functions $X\,(s),\ \ V\,(s)$ depend also on the variables $x,\ x_0,\ t$. However, for simplicity, we will not write the dependence on these variables explicitly unless it is needed. We then rewrite the above characteristics as a perturbation from those associated to the free streaming case as follows:

$$\begin{aligned}
\frac{\mathrm{d}X\,(s)}{\mathrm{d}s} &= V\,(s) = \frac{x - x_0}{t} + \varphi\,(s),\\
\frac{\mathrm{d}\varphi}{\mathrm{d}s} &= \nabla\phi\,(X\,(s), s), X\,(t) = x, X\,(0) = x_0,
\end{aligned} \tag{2.8}$$

where $\varphi\,(s) = \varphi\,(s; x, x_0, t)$ is the perturbed value of the velocity with respect to the free streaming case. We then define precisely the functions in (2.5), (2.6) as:

$$\begin{aligned}
w\,(t, x, x_0) &= \frac{x - x_0}{t} + \varphi\,(t; x, x_0, t),\\
w_0\,(t, x, x_0) &= \frac{x - x_0}{t} + \varphi\,(0; x, x_0, t)
\end{aligned} \tag{2.9}$$

Notice that in the limit of zero density $\rho \equiv 0$, the field ϕ vanishes and $\varphi\,(s) \equiv 0$.

We examine the derivatives of the function $\varphi(s)$ with respect to the variables x, x_0 in order to derive suitable estimates for ρ. The density function ρ can be represented as

$$\rho(x,t) = \int f(x,v,t)\,\mathrm{d}v$$
$$= \int f_0\left(x_0, V\left(0; x, w\left(t,x,x_0\right), t\right)\right) \left|\det\left(\frac{\partial w}{\partial x_0}\right)\right| \mathrm{d}x_0. \quad (2.10)$$

Along the characteristics, we have

$$\frac{\partial w}{\partial x_0} = \frac{\partial V}{\partial x_0}(t) = -\frac{1}{t} I_N + \frac{\partial \varphi}{\partial x_0}(t),$$

where I_N is the N-dimensional identity matrix.

On the other hand, we wish to obtain estimates for the derivatives of ρ. Suppose for the moment that we restrict our attention to the first derivative of ρ with respect to x. Such a derivative is given by

$$\frac{\partial \rho}{\partial x}(x,t) = \int \frac{\partial f_0}{\partial v}\left(x_0, V\left(0; x, w\left(t,x,x_0\right), t\right)\right) \frac{\partial V}{\partial x}(0) \left|\det\left(\frac{\partial w}{\partial x_0}\right)\right| \mathrm{d}x_0,$$
$$+ \int f_0\left(x_0, V\left(0; x, w\left(t,x,x_0\right), t\right)\right) \frac{\partial}{\partial x}\left[\left|\det\left(\frac{\partial w}{\partial x_0}\right)\right|\right] \mathrm{d}x_0.$$

Estimating the first derivative of ρ reduces to deriving estimates for

$$\frac{\partial V}{\partial x}(s=0), \; \frac{\partial V}{\partial x_0}(s=t), \; \frac{\partial^2 V}{\partial x \partial x_0}(s=t).$$

Equivalently,

$$\frac{\partial \varphi}{\partial x}(0), \; \frac{\partial \varphi}{\partial x_0}(t), \; \frac{\partial^2 \varphi}{\partial x \partial x_0}(t).$$

Notice that the equation of the characteristics (2.8) indicates that in order to obtain bounds for two derivatives with respect to x, x_0 of the characteristic curves we need to estimate three derivatives of the potential ϕ. This is the exact number of derivatives that can be expected to be estimated from the Poisson equation under the assumption that $\frac{\partial \rho}{\partial x}$ is bounded. Nevertheless, in order to avoid the standard problems that arise in the regularity estimates for the Poisson equation in the C^k spaces, it is necessary to work with the Hölder spaces $C^{k,\alpha}$.

2.3. Estimates on the regularity and the rate of decay of $\varphi(s; x, x_0, t)$ in terms of the properties of the potential ϕ

We present a key a priori estimate for φ in terms of ϕ in the following Proposition. We define two norms with respect to the spatial variable x.

Definition 1. Given any function For $u(\cdot) \in L^\infty\left(\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^+\right)$, we define a function in $L^\infty\left(\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^+\right)$ by means of

$$\|u\|_{L^\infty_{(x)}}(s;x_0,t)=\|u\|_{L^\infty_{(x)}}\equiv \operatorname{ess}\sup_{x\in\mathbb{R}^N}|u(s;x,x_0,t)|.$$

We also define functions $[u]_{0,\alpha,(x)}\in L^\infty\left(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^+\right)$ as

$$[u]_{0,\alpha,(x)}(s;x_0,t)=[u]_{0,\alpha,(x)}\equiv\sup_{x_1,x_2\in\mathbb{R}^N}\frac{|u(s;x_1,x_0,t)-u(s;x_2,x_0,t)|}{|x_1-x_2|^\alpha},\ 0<\alpha<1,$$

as well as $\|u\|_{k,\alpha}\in L^\infty\left(\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^+\right)$

$$\|u\|_{k,\alpha}(s;x_0,t)=\|u\|_{k,\alpha}\equiv\sum_{\ell=0}^{k}\left\|\nabla^\ell u\right\|_{L^\infty_{(x)}}+\left[\nabla^k u\right]_{0,\alpha,(x)},$$

for $k=0,1,2,3,\ldots$. We will assume that $\nabla^0 u=u$.

For notational simplicity, in the following, we will use $\|u(s)\|_{L^\infty_{(x)}}$, $[u(s)]_{0,\alpha,(x)}$ instead of $\|u(s;\cdot,x_0,t)\|_{L^\infty_{(x)}}$, $[u(s;\cdot,x_0,t)]_{0,\alpha,(x)}$, which in fact depend on s, x_0, and t. For example, $\left[\frac{\partial\varphi}{\partial x}(s)\right]_{0,\alpha,(x)}$ will denote $\left[\frac{\partial\varphi}{\partial x}(s;\cdot,x_0,t)\right]_{0,\alpha,(x)}$.

Proposition 1. *Consider a solution of the characteristic system for the Vlasov equation with potential ϕ. Suppose that $t\geqq 1$ and that*

$$\|\phi\|_{Y_{1,\alpha}}\leqq\varepsilon_0.$$

for a suitable $\varepsilon_0>0$ sufficiently small. Then, there exists a unique function φ defined by means of (2.8). Moreover, the derivative $\frac{\partial\varphi}{\partial x_0}(s)$ is defined for $0\leqq s\leqq t$. Moreover, the functions $\varphi(s)$ and $\frac{\partial\varphi}{\partial x_0}(s)$ belong to the space $C^{1,\alpha}$ with respect to the variable x and the following a priori estimates hold

$$t\sup_{0\leqq s\leqq t}\left\|\frac{\partial\varphi}{\partial x}(s)\right\|_{L^\infty_{(x)}}+t^{1+\alpha}\sup_{0\leqq s\leqq t}\left[\frac{\partial\varphi}{\partial x}(s)\right]_{0,\alpha,(x)}\leqq C\|\phi\|_{Y_{1,\alpha}},$$

$$\begin{aligned}&\int_0^t\left\|\frac{\partial\varphi}{\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s+t^\alpha\int_0^t\left[\frac{\partial\varphi}{\partial x_0}(s)\right]_{0,\alpha,(x)}\mathrm{d}s\\&\quad+t\int_0^t\mathrm{d}s\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right\|_{L^\infty_{(x)}}+t^{1+\alpha}\int_0^t\left[\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right]_{0,\alpha,(x)}\mathrm{d}s\\&\quad\leqq C\|\phi\|_{Y_{1,\alpha}},\end{aligned}$$

$$\begin{aligned}&t\left\|\frac{\partial\varphi}{\partial x_0}(t)\right\|_{L^\infty_{(x)}}+t^{1+\alpha}\left[\frac{\partial\varphi}{\partial x_0}(t)\right]_{0,\alpha,(x)}\\&\quad+t^2\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(t)\right\|_{L^\infty_{(x)}}+t^{2+\alpha}\left[\frac{\partial^2\varphi}{\partial x\partial x_0}(t)\right]_{0,\alpha,(x)}\\&\quad\leqq C\|\phi\|_{Y_{1,\alpha}}\end{aligned}$$

for some suitable constant $C>0$ independent of $t,\ \varepsilon_0$.

Proposition 1 is the main new technical result of the paper. This estimate provides optimal decay properties for the derivatives of the characteristics in terms of the decay properties of the derivatives of the potential ϕ.

Furthermore, we derive a generalization of Proposition 1 under additional regularity and decay assumptions for the potential ϕ.

Proposition 2. *Consider a solution of the characteristic system for the Vlasov equation with potential ϕ and let φ be defined by (2.8). Suppose that*

$$\|\phi\|_{Y_{\ell,\alpha}} \leqq \varepsilon_0, \quad \ell \geqq 2,$$

for a suitable $\varepsilon_0 > 0$ sufficiently small. Suppose that $t \geqq 1$. Then, the derivative $\frac{\partial \varphi}{\partial x_0}(s)$ is defined for $0 \leqq s \leqq t$. Moreover, the functions $\varphi(s)$ and $\frac{\partial \varphi}{\partial x_0}(s)$ belong to the space $C^{\ell,\alpha}$ with respect to the variable x and the following estimates hold

$$\sum_{k=1}^{\ell} t^k \sup_{0 \leqq s \leqq t} \left\| \frac{\partial^k \varphi}{\partial x^k}(s) \right\|_{L^\infty_{(x)}} + t^{\ell+\alpha} \sup_{0 \leqq s \leqq t} \left[\frac{\partial^\ell \varphi}{\partial x^\ell}(s) \right]_{0,\alpha,(x)} \leqq C \|\phi\|_{Y_{\ell,\alpha}}, \tag{2.11}$$

$$\sum_{k=1}^{\ell} t^k \int_0^t \left\| \frac{\partial^{k+1} \varphi}{\partial x^k \partial x_0}(s) \right\|_{L^\infty_{(x)}} \mathrm{d}s + t^{\ell+\alpha} \int_0^t \left[\frac{\partial^{\ell+1} \varphi}{\partial x^\ell \partial x_0}(s) \right]_{0,\alpha,(x)} \mathrm{d}s \leqq C \|\phi\|_{Y_{\ell,\alpha}}, \tag{2.12}$$

$$\sum_{k=1}^{\ell} t^{k+1} \left\| \frac{\partial^{k+1} \varphi}{\partial x^k \partial x_0}(s) \right\|_{L^\infty_{(x)}} + t^{\ell+1+\alpha} \left[\frac{\partial^{\ell+1} \varphi}{\partial x^\ell \partial x_0}(t) \right]_{0,\alpha,(x)} \leqq C \|\phi\|_{Y_{\ell,\alpha}}, \tag{2.13}$$

for some constant $C > 0$ independent of t, ε_0.

2.3.1. Preliminary results: Integral equation satisfied by $\varphi(s, x, x_0, t)$ A key point in deriving information about the evolution of the characteristic curves associated to (2.7) is to study the properties of the function φ whose heuristic definition has been given in (2.8). More precisely, this function will be considered as a function of the variables $(s; x, x_0, t)$. Its precise definition will be given in the following Lemma.

Lemma 1. *For any $t \geqq 1$, $x \in \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$ and $0 \leqq s \leqq t$, let us assume that $\varphi(s) = \varphi(s; x, x_0, t)$ satisfies*

$$\varphi(s) = -\int_s^t G_\varphi(\xi)\, \mathrm{d}\xi + \frac{1}{t} \int_0^t \xi G_\varphi(\xi)\, \mathrm{d}\xi, \qquad \varphi(\cdot; x, x_0, t) \in C[0, t], \tag{2.14}$$

where

$$G_\varphi(\xi) \equiv \nabla \phi(X(\xi), \xi), \tag{2.15}$$

$$X(\xi) = x_0 + \frac{x - x_0}{t} \xi + \int_0^\xi \varphi(\bar{s})\, \mathrm{d}\bar{s}. \tag{2.16}$$

and where $\|\phi\|_{Y_{0,\alpha}} \leqq 1$. *Then, the functions* $X(s) = X(s; x, x_0, t)$, $V(s) = V(s; x, x_0, t)$ *given by* (2.16) *and*

$$V(s; x, x_0, t) = \frac{x - x_0}{t} + \varphi(s; x, x_0, t) \tag{2.17}$$

satisfy (2.8).

Proof. Differentiating (2.14), (2.17) and using (2.15) we obtain

$$\frac{\mathrm{d}V(s)}{\mathrm{d}s} = \frac{\partial\varphi(s; x, x_0, t)}{\partial s} = G_{\varphi}(s) = \nabla\phi(X(s), s). \tag{2.18}$$

On the other hand, integrating (2.14) and using Fubini's Theorem we obtain

$$\int_0^t \varphi(s)\,\mathrm{d}s = 0. \tag{2.19}$$

It then follows from (2.16) that

$$X(0) = x_0 \quad X(t) = x, \tag{2.20}$$

Differentiating (2.16) and using (2.17) we obtain:

$$\frac{\mathrm{d}X}{\mathrm{d}s}(s) = V(s). \tag{2.21}$$

Combining (2.18), (2.20), (2.21) we obtain (2.8). The regularity assumptions made for ϕ, φ allowed us to make all the previous manipulations in a rigorous way. □

The integral equations (2.14), (2.15), (2.16) are the key ingredients that will be used to derive optimal regularity and decay estimates for the functions $w(t, x, x_0)$, $w_0(t, x, x_0)$ whose existence has been suggested in (2.5), (2.6). As a first step we prove that the solutions of (2.14), (2.15), (2.16) are well defined if ε_0 is small enough.

Lemma 2. (Solvability) *There exists* $\varepsilon_0 > 0$ *such that, for any* $t \geqq 1$, $x \in \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$ *and any function* ϕ *satisfying*

$$\|\phi\|_{Y_{0,\alpha}} \leqq \varepsilon_0,$$

there exists a unique solution $\varphi(\cdot) = \varphi(\cdot; x, x_0, t) \in C([0, t])$ *of* (2.14), (2.15), (2.16). *Moreover,*

$$\|\varphi\|_{L^\infty_{(x)}} \leqq C\,\|\phi\|_{Y_{0,\alpha}}. \tag{2.22}$$

Proof. Let the space of functions for $\varphi(s)$ be

$$\mathcal{X} \equiv \left\{ \varphi \in C([0,t]) : \sup_{0 \leqq s \leqq t} |\varphi(s)| \leqq 1 \right\}.$$

Let

$$\mathcal{J}(\varphi)(s) \equiv -\int_s^t G_\varphi(\xi)\,\mathrm{d}\xi + \frac{1}{t}\int_0^t \xi G_\varphi(\xi)\,\mathrm{d}\xi, \tag{2.23}$$

where

$$G_\varphi(\xi) = \nabla\phi\left(x_0 + \frac{x - x_0}{t}\xi + \int_0^\xi \varphi(\bar{s})\,\mathrm{d}\bar{s}, \xi\right). \tag{2.24}$$

We first show that $\mathcal{J}$ is a well-defined operator in the space $\mathcal{X}$. By definition, we have

$$\left|G_\varphi(\xi)\right| \leqq \|\nabla\phi(\cdot,\xi)\|_{L^\infty_{(x)}} \leqq \frac{\|\phi\|_{Y_{k,\alpha}}}{(\xi+1)^{N-1}}.$$

This yields

$$\begin{aligned}
|\mathcal{J}(\varphi)(s)| &\leqq \int_s^t \left|G_\varphi(\xi)\right|\mathrm{d}\xi + \frac{1}{t}\int_0^t \xi\left|G_\varphi(\xi)\right|\mathrm{d}\xi \\
&\leqq \|\phi\|_{Y_{k,\alpha}}\left[\int_0^t \frac{\mathrm{d}\xi}{(1+\xi)^{N-1}} + \frac{1}{t}\int_0^t \frac{\xi}{(1+\xi)^{N-1}}\mathrm{d}\xi\right] \\
&\leqq \|\phi\|_{Y_{k,\alpha}}\left[1 + \frac{\log(1+t)}{t}\right] \leqq C\|\phi\|_{Y_{k,\alpha}},
\end{aligned} \tag{2.25}$$

where we have estimated the last integral term using the fact that $N \geqq 3$. This idea will be used repeatedly in the following. Thus $\mathcal{J}(\varphi)$ is bounded for all t and we choose $\varepsilon_0 > 0$ small enough such that $C\varepsilon_0 < 1$ in the above inequality so that $\mathcal{J}$ is well-defined in the space $\mathcal{X}$. Next we show that the operator $\mathcal{J}$ is contractive. Taking the difference of $\mathcal{J}(\varphi_1)$ and $\mathcal{J}(\varphi_1)$ yields

$$\begin{aligned}
[\mathcal{J}(\varphi_1) - \mathcal{J}(\varphi_2)](s) = &-\int_s^t \left[G_{\varphi_1}(\xi) - G_{\varphi_2}(\xi)\right]\mathrm{d}\xi \\
&+\frac{1}{t}\int_0^t \xi\left[G_{\varphi_1}(\xi) - G_{\varphi_2}(\xi)\right]\mathrm{d}\xi.
\end{aligned}$$

Using the definition of $\|\phi\|_{Y_{k,\alpha}}$, we have

$$\begin{aligned}
&\sup_{0 \leqq s \leqq t} |[\mathcal{J}(\varphi_1) - \mathcal{J}(\varphi_2)](s)| \\
&\leqq \int_s^t \left\|\nabla^2\phi(\cdot,\xi)\right\|_{L^\infty_{(x)}}\left[\int_0^\xi |\varphi_1(\bar{s}) - \varphi_2(\bar{s})|\,\mathrm{d}\bar{s}\right]\mathrm{d}\xi \\
&\quad+\frac{1}{t}\int_0^t \xi\left\|\nabla^2\phi(\cdot,\xi)\right\|_{L^\infty_{(x)}}\left[\int_0^\xi |\varphi_1(\bar{s}) - \varphi_2(\bar{s})|\,\mathrm{d}\bar{s}\right]\mathrm{d}\xi
\end{aligned}$$

$$\leqq \|\phi\|_{Y_{k,\alpha}} \left[\int_s^t \frac{1}{(\xi+1)^N} \left[\int_0^\xi |\varphi_1(\bar{s}) - \varphi_2(\bar{s})| \, \mathrm{d}\bar{s} \right] \mathrm{d}\xi \right.$$
$$\left. + \frac{1}{t} \int_0^t \frac{\xi}{(\xi+1)^N} \left[\int_0^\xi |\varphi_1(\bar{s}) - \varphi_2(\bar{s})| \, \mathrm{d}\bar{s} \right] \mathrm{d}\xi \right]$$
$$\leqq C\varepsilon_0 \sup_{0 \leqq s \leqq t} |(\varphi_1 - \varphi_2)(s)| + C\varepsilon_0 \frac{\log(t+1)}{t} \sup_{0 \leqq s \leqq t} |(\varphi_1 - \varphi_2)(s)|$$
$$\leqq C\varepsilon_0 \sup_{0 \leqq s \leqq t} |(\varphi_1 - \varphi_2)(s)| .$$

Thus we again choose ε_0 small enough such that $C\varepsilon_0 < 1$ in the above inequality and conclude that $\mathcal{J}$ is contractive. Notice that C is independent of $t,\ x_0, x$ and ε_0 can be chosen independently of these variables. Then, by the Banach fixed point theorem, we deduce the existence and uniqueness for φ satisfying (2.14) in the space $\mathcal{X}$. The estimate (2.22) is then a consequence of (2.25). □

In the following arguments we will estimate the derivatives of the function φ with respect to $x,\ x_0$. In order to ensure that the function φ has the required differentiability properties we will use the Implicit Function Theorem as follows.

Proposition 3. *Let $k \geqq 0$ be an integer. Suppose that $\|\phi\|_{Y_{k,\alpha}} \leqq \varepsilon_0$ for some $\varepsilon_0 > 0$ sufficiently small, $t \geqq 1,\ x \in \mathbb{R}^N,\ x_0 \in \mathbb{R}^N$. Then, the function $\varphi(s, t; x, x_0)$ defined by means of Lemma 2 has $(k + 1 + \alpha)$ derivatives for each $0 < s \leqq t$.*

Proof. Let us fix $t \geqq 1,\ \bar{x} \in \mathbb{R}^N,\ \bar{x}_0 \in \mathbb{R}^N$. Given $v_0 \in \mathbb{R}^N,\ x_0 \in \mathbb{R}^N$ we define $\bar{X}(s; x_0, v_0),\ \bar{V}(s; x_0, v_0)$ by means of the ODE problem

$$\frac{\mathrm{d}\bar{X}}{\mathrm{d}s}(s; x_0, v_0) = \bar{V}(s; x_0, v_0), \quad \frac{\mathrm{d}\bar{V}}{\mathrm{d}s}(s; x_0, v_0) = \nabla\phi\left(\bar{X}(s; x_0, v_0), s\right),$$
$$\bar{X}(0; x_0, v_0) = x_0, \quad \bar{V}(0; x_0, v_0) = v_0. \tag{2.26}$$

Suppose that $\bar{X}(t; \bar{x}_0, v_0) = \bar{x}$. Let us define $\bar{v}_0 = \frac{\bar{x} - \bar{x}_0}{t}$. Lemma 1 implies that

$$\bar{X}(s; \bar{x}_0, v_0) = \bar{x}_0 + \frac{\bar{x} - \bar{x}_0}{t} s + \int_0^s \varphi(\bar{s}; \bar{x}, \bar{x}_0, t) \, \mathrm{d}\bar{s},$$
$$\bar{V}(s; \bar{x}_0, v_0) = \bar{v}_0 + \varphi(s; \bar{x}, \bar{x}_0, t). \tag{2.27}$$

We now claim that there exists $\delta_0 > 0$ sufficiently small, perhaps depending on $\bar{x}_0,\ \bar{v}_0,\ t$ and a function $w_0(t; x, x_0)$ such that, for $|x_0 - \bar{x}_0| + |v_0 - \bar{v}_0| < \delta_0$,

$$\bar{X}(t; x_0, w_0(t; x, x_0)) = x. \tag{2.28}$$

The existence of the function $w_0(t; x, x_0)$ is a consequence of the Implicit Function Theorem. In order to apply it, we need to check that $\frac{\partial \bar{X}(t; \bar{x}_0, \bar{v}_0)}{\partial v_0}$ is invertible. Notice that since $\phi \in Y_{k,\alpha}$, it follows that the functions $\bar{X}(s; x_0, v_0)$ and $\bar{V}(s; x_0, v_0)$ have $(k + 1 + \alpha)$ derivatives with respect to the variables (x_0, v_0) for

$|x_0 - \bar{x}_0| + |v_0 - \bar{v}_0| < \delta_0$ if δ_0 is chosen sufficiently small. On the other hand, differentiating (2.26) with respect to v_0 we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \bar{X}}{\partial v_0}\right) = \frac{\partial \bar{V}}{\partial v_0}, \quad \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial \bar{V}}{\partial v_0}\right) = \nabla^2 \phi\left(\bar{X}(s), s\right)\frac{\partial \bar{X}}{\partial v_0},$$
$$\frac{\partial \bar{X}}{\partial v_0}(0; \bar{x}_0, \bar{v}_0) = 0, \quad \frac{\partial \bar{V}}{\partial v_0}(0; \bar{x}_0, \bar{v}_0) = I,$$

where I is the identity matrix and we write $\bar{X}(s) = \bar{X}(s; \bar{x}_0, \bar{v}_0)$ for shortness. Since $\|\phi\|_{Y_{k,\alpha}} \leqq \varepsilon_0$ it follows that

$$\left|\frac{\partial \bar{V}}{\partial v_0}(s; \bar{x}_0, \bar{v}_0) - I\right| \leqq C\varepsilon_0 \int_0^s \frac{1}{(\xi+1)^N}\left|\frac{\partial \bar{X}}{\partial v_0}(\xi; \bar{x}_0, \bar{v}_0)\right| \mathrm{d}\xi,$$

whence

$$\left|\frac{\partial \bar{X}}{\partial v_0}(t; \bar{x}_0, \bar{v}_0) - tI\right| \leqq C\varepsilon_0 \int_0^t \left(\int_0^s \frac{1}{(\xi+1)^N}\left|\frac{\partial \bar{X}}{\partial v_0}(\xi; \bar{x}_0, \bar{v}_0)\right| \mathrm{d}\xi\right) \mathrm{d}s. \tag{2.29}$$

As long as $\left|\frac{\partial \bar{X}}{\partial v_0}(\xi; \bar{x}_0, \bar{v}_0)\right| \leqq 2|I|\xi$ we can estimate the right-hand side of (2.29), after exchanging the order of integration, by

$$C\varepsilon_0 t \int_0^t \frac{1}{(\xi+1)^N}\left|\frac{\partial \bar{X}}{\partial v_0}(\xi; \bar{x}_0, \bar{v}_0)\right| \mathrm{d}\xi \leqq C\varepsilon_0 t \int_0^t \frac{\xi}{(\xi+1)^N} \mathrm{d}\xi \leqq C\varepsilon_0 t,$$

where the constant C might change from line to line, but it is independent of ε_0 and t. It then follows that for ε_0 sufficiently small, the estimate $\left|\frac{\partial \bar{X}}{\partial v_0}(s; \bar{x}_0, \bar{v}_0)\right| \leqq 2|I|s$ holds for $0 \leqq s \leqq t$. Moreover, we have also

$$\left|\frac{\partial \bar{X}}{\partial v_0}(t; \bar{x}_0, \bar{v}_0) - tI\right| \leqq C\varepsilon_0 t$$

and for ε_0 sufficiently small the invertibility of $\frac{\partial \bar{X}}{\partial v_0}(t; x_0, v_0)$ follows for any $t > 0$. Therefore the function $w_0(t; x, x_0)$ is well defined and it has $(k+1+\alpha)$ derivatives with respect to the variables (x, x_0) for $|x_0 - \bar{x}_0| + |v_0 - \bar{v}_0| < \delta_0$.

We now define, for $|x_0 - \bar{x}_0| + |v_0 - \bar{v}_0| < \delta_0,\ t \geqq 1$,

$$\bar{\varphi}(s; x, x_0, t) = \bar{V}(s; x_0, w_0(t; x, x_0)) - \frac{x - x_0}{t}, \quad 0 \leqq s \leqq t.$$

We now claim that $\bar{\varphi}(s; x, x_0, t) = \varphi(s; \bar{x}, \bar{x}_0, t)$ for $|x_0 - \bar{x}_0| + |v_0 - \bar{v}_0| < \delta_0,\ 0 \leqq s \leqq t$. Indeed, integrating the first two equations in (2.26) with $v_0 = w_0(t; x, x_0)$, it follows that $\bar{\varphi}(s; x, x_0, t) = \bar{\varphi}(s)$ solves

$$\bar{\varphi}(s) = \bar{\varphi}(0) + \int_0^s G_{\bar{\varphi}}(\xi)\,\mathrm{d}\xi,$$
$$G_{\bar{\varphi}}(\xi) \equiv \nabla\phi\left(\bar{X}(\xi), \xi\right), \quad \bar{X}(s) = x_0 + \frac{x - x_0}{t}s + \int_0^s \bar{\varphi}(\bar{s})\,\mathrm{d}\bar{s}. \tag{2.30}$$

Using (2.28) it follows that $\bar{X}(t) = \bar{x}$, whence $\int_0^t \bar{\varphi}(\bar{s})\,\mathrm{d}\bar{s} = 0$. Therefore, applying Fubini

$$\bar{\varphi}(t) = -\frac{1}{t}\int_0^t \xi G_{\bar{\varphi}}(\xi)\,\mathrm{d}\xi.$$

It then follows that $\bar{\varphi}(s)$ solves (2.14). The uniqueness of $\bar{\varphi}$ obtained in Lemma 2 then implies that

$$\bar{\varphi}(s; x, x_0, t) = \varphi(s; \bar{x}, \bar{x}_0, t),$$

for $|x_0 - \bar{x}_0| + |v_0 - \bar{v}_0| < \delta_0,\ 0 \leqq s \leqq t$. Since $\bar{\varphi}$ has $(k+1+\alpha)$ derivatives with respect to the variables (x, x_0) and $\bar{x}, \bar{x}_0$ are arbitrary, Proposition 3 follows. $\square$

We now turn to decay estimates for the density function ρ and its derivatives. Before we proceed, we state some basic properties of the Hölder norms.

Lemma 3.

$$[fg]_{0,\alpha,(x)} \leqq C\left\{\|f\|_{L^\infty}[g]_{0,\alpha,(x)} + \|g\|_{L^\infty}[f]_{0,\alpha,(x)}\right\},$$

for any $f, g \in L^\infty \cap C^{0,\alpha}$,

$$[f]_{0,\alpha,(x)} \leqq C\,\|f\|_{L^\infty}^{1-\alpha}\,\|\nabla f\|_{L^\infty}^{\alpha},$$

for any $f \in W^{1,\infty}$,

$$[F \circ u]_{0,\alpha,(x)} \leqq [F]_{0,\alpha,(x)}\,\|\nabla u\|_{L^\infty}^{\alpha},$$

for any $F \in C^{0,\alpha}$ *and any* $u \in W^{1,\infty}$.

Proof. The results in the lemma are standard estimates for Hölder norms (see [7]). $\square$

2.3.2. The proof of Proposition 1 The proof of Proposition 1 follows from a sequence of lemmas. There are three ideas that will appear repeatedly in all the remaining arguments of this paper. Estimating terms like $\frac{\partial\varphi}{\partial x_0}$ and its derivatives with respect to x, it is not possible to obtain bounds for the rate of decay suggested by dimensional considerations for all the values of $s \in [0, t]$. It is, however, possible to obtain such optimal decay estimates for the integrals of such terms in the interval $[0, t]$ as well as for the time $s = t$ which is the only one where such optimal estimates are really needed. The second idea is that it is convenient to obtain, before deriving pointwise estimates, integral estimates for terms like $\int_0^t \left\|\frac{\partial\varphi}{\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s$. The third idea is that the estimates for terms that do not contain derivatives with respect x_0 are more easily obtained by directly estimating the supremum over the interval $[0, t]$ and using Gronwall-type arguments, without any need for estimating integrals over the interval $[0, t]$.

Lemma 4. *There exists ε_0 small such that for $t > 1$ and any function ϕ satisfying*

$$\|\phi\|_{Y_{0,\alpha}} \leqq \varepsilon_0,$$

we have

$$\int_0^t \left\| \frac{\partial \varphi}{\partial x_0}(s) \right\|_{L^\infty_{(x)}} \mathrm{d}s \leqq C \|\phi\|_{Y_{0,\alpha}}, \tag{2.31}$$

$$\left\| \frac{\partial \varphi}{\partial x_0}(t) \right\|_{L^\infty_{(x)}} \leqq C \frac{\|\phi\|_{Y_{0,\alpha}}}{t}. \tag{2.32}$$

Proof. Differentiating (2.14) with respect to x_0 yields

$$\frac{\partial \varphi}{\partial x_0}(s) = -\int_s^t \frac{\partial}{\partial x_0} G(\xi)\, \mathrm{d}\xi + \frac{1}{t}\int_0^t \xi \frac{\partial}{\partial x_0} G(\xi)\, \mathrm{d}\xi. \tag{2.33}$$

where, for simplicity we will write from now on $G_\varphi = G$. Notice that the existence of the derivative $\frac{\partial \varphi}{\partial x_0}$ as well as all the other derivatives appearing in the rest of the paper is a consequence of Corollary 3. We now take $\frac{\partial}{\partial x_0}$ of (2.15)–(2.16) to get

$$\frac{\partial}{\partial x_0} G(\xi) = \nabla^2 \phi(X(\xi), \xi) \frac{\partial}{\partial x_0} X(\xi) = \nabla^2 \phi(X(\xi), \xi)\left[\left(1 - \frac{\xi}{t}\right) I + \int_0^\xi \frac{\partial \varphi}{\partial x_0}\right]$$

and since $\frac{\xi}{t} \leqq 1$, we have

$$\begin{aligned}\left\| \frac{\partial}{\partial x_0} G(\xi) \right\|_{L^\infty_{(x)}} &\leqq \frac{\|\phi\|_{Y_{0,\alpha}}}{(\xi+1)^N} \left\| \left(1 - \frac{\xi}{t}\right) I + \int_0^\xi \frac{\partial \varphi}{\partial x_0}(\bar{s})\, \mathrm{d}\bar{s} \right\|_{L^\infty_{(x)}} \\ &\leqq \frac{\|\phi\|_{Y_{0,\alpha}}}{(\xi+1)^N}\left[1 + \int_0^\xi \left\| \frac{\partial \varphi}{\partial x_0}(\bar{s}) \right\|_{L^\infty_{(x)}} \mathrm{d}\bar{s}\right].\end{aligned}$$

Putting the above into (2.33) yields

$$\begin{aligned}\left\| \frac{\partial \varphi}{\partial x_0}(s) \right\|_{L^\infty_{(x)}} &\leqq C\|\phi\|_{Y_{0,\alpha}} \int_s^t \frac{1}{(\xi+1)^N}\left[1 + \int_0^\xi \left\| \frac{\partial \varphi}{\partial x_0}(\bar{s}) \right\|_{L^\infty_{(x)}} \mathrm{d}\bar{s}\right] \mathrm{d}\xi \\ &\quad + C\|\phi\|_{Y_{0,\alpha}} \frac{1}{t}\int_0^t \frac{\xi}{(\xi+1)^N}\left[1 + \int_0^\xi \left\| \frac{\partial \varphi}{\partial x_0}(\bar{s}) \right\|_{L^\infty_{(x)}} \mathrm{d}\bar{s}\right] \mathrm{d}\xi.\end{aligned} \tag{2.34}$$

By integrating the above from 0 to t and by the assumption, we obtain

$$\begin{aligned}&\int_0^t \left\| \frac{\partial \varphi}{\partial x_0}(s) \right\|_{L^\infty_{(x)}} \mathrm{d}s \\ &\quad \leqq C\|\phi\|_{Y_{0,\alpha}} \int_0^t \left[\int_s^t \frac{1}{(\xi+1)^N}\left[1 + \int_0^\xi \left\| \frac{\partial \varphi}{\partial x_0}(\bar{s}) \right\|_{L^\infty_{(x)}} \mathrm{d}\bar{s}\right] \mathrm{d}\xi\right] \mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+C\,\|\phi\|_{Y_{0,\alpha}}\int_0^t\frac{\xi}{(\xi+1)^N}\left[1+\int_0^\xi\left\|\frac{\partial\varphi}{\partial x_0}(\bar{s})\right\|_{L^\infty_{(x)}}\mathrm{d}\bar{s}\right]\mathrm{d}\xi\\
&\leqq C\,\|\phi\|_{Y_{0,\alpha}}+C\,\|\phi\|_{Y_{0,\alpha}}\int_0^t\left\|\frac{\partial\varphi}{\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s\\
&\leqq C\,\|\phi\|_{Y_{0,\alpha}}+C\varepsilon_0\int_0^t\left\|\frac{\partial\varphi}{\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s,
\end{aligned}
$$

where we have used the estimate $\int_0^\xi\left\|\frac{\partial\varphi}{\partial x_0}(\bar{s})\right\|_{L^\infty_{(x)}}\mathrm{d}\bar{s}\leqq\int_0^t\left\|\frac{\partial\varphi}{\partial x_0}(\bar{s})\right\|_{L^\infty_{(x)}}\mathrm{d}\bar{s}$ and the fact that $\int_0^t\left[\int_s^t\frac{\mathrm{d}\xi}{(\xi+1)^3}\right]\mathrm{d}s=\int_0^t\frac{1}{(\xi+1)^3}\left[\int_0^\xi\mathrm{d}s\right]\mathrm{d}\xi\leqq C$. Thus if ε_0 is small enough so that $C\varepsilon_0\leqq 1/2$, we get

$$
\int_0^t\left\|\frac{\partial\varphi}{\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s\leqq C\,\|\phi\|_{Y_{0,\alpha}}\,. \tag{2.35}
$$

Putting $s=t$ in (2.34) and using (2.35) yields

$$
\begin{aligned}
\left\|\frac{\partial\varphi}{\partial x_0}(t)\right\|_{L^\infty_{(x)}}&\leqq C\,\|\phi\|_{Y_{0,\alpha}}\frac{1}{t}\int_0^t\frac{\xi}{(\xi+1)^N}\left[1+\int_0^\xi\left\|\frac{\partial\varphi}{\partial x_0}(\bar{s})\right\|_{L^\infty_{(x)}}\mathrm{d}\bar{s}\right]\mathrm{d}\xi\\
&\leqq C\frac{\|\phi\|_{Y_{0,\alpha}}}{t}\left(1+\|\phi\|_{Y_{0,\alpha}}\right)\leqq C\frac{\|\phi\|_{Y_{0,\alpha}}}{t}.
\end{aligned}
$$

Thus we obtain (2.31) and (2.32). This completes the proof of the lemma. □

Lemma 5. *There exists ε_0 small such that for $\|\phi\|_{Y_{1,\alpha}}\leqq\varepsilon_0$ and $t>1$ we have the following decay estimates*

$$
\sup_{0\leqq s\leqq t}\left\|\frac{\partial\varphi}{\partial x}(s)\right\|_{L^\infty_{(x)}}\leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t}, \tag{2.36}
$$

$$
\int_0^t\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s\leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t}, \tag{2.37}
$$

$$
\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(t)\right\|_{L^\infty_{(x)}}\leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t^2}. \tag{2.38}
$$

Proof. Differentiating (2.15)–(2.16) with respect to x we get

$$
\frac{\partial}{\partial x}G(\xi)=\nabla^2\phi(X(\xi),\xi)\frac{\partial}{\partial x}X(\xi)=\nabla^2\phi(X(\xi),\xi)\left[\frac{\xi}{t}I+\int_0^\xi\frac{\partial\varphi}{\partial x}(\bar{s})\,\mathrm{d}\bar{s}\right]
$$

and thus

$$\begin{aligned}\left\| \frac{\partial}{\partial x} G(\xi) \right\|_{L^\infty_{(x)}} &\leqq \frac{\|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^N} \left\| \frac{\xi}{t} I + \int_0^\xi \frac{\partial \varphi}{\partial x}(\bar{s})\, d\bar{s} \right\|_{L^\infty_{(x)}} \\ &\leqq \frac{\|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^{N-1}} \left[\frac{\xi}{(\xi+1)t} + \frac{1}{(\xi+1)} \int_0^\xi \left\| \frac{\partial \varphi}{\partial x}(\bar{s}) \right\|_{L^\infty_{(x)}} d\bar{s} \right] \\ &\leqq \frac{\|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^{N-1}} \left[\frac{1}{t} + \frac{1}{(\xi+1)} \int_0^\xi \left\| \frac{\partial \varphi}{\partial x}(\bar{s}) \right\|_{L^\infty_{(x)}} d\bar{s} \right].\end{aligned}$$

Differentiating (2.14) with respect to x and using the estimate above, we obtain

$$\begin{aligned}&\left\| \frac{\partial \varphi}{\partial x}(s) \right\|_{L^\infty_{(x)}} \\ &\quad \leqq C \|\phi\|_{Y_{1,\alpha}} \int_s^t \frac{1}{(\xi+1)^{N-1}} \left[\frac{1}{t} + \frac{1}{(\xi+1)} \int_0^\xi \left\| \frac{\partial \varphi}{\partial x}(\bar{s}) \right\|_{L^\infty_{(x)}} d\bar{s} \right] d\xi \\ &\qquad + C \|\phi\|_{Y_{1,\alpha}} \frac{1}{t} \int_0^t \frac{\xi}{(\xi+1)^{N-1}} \left[\frac{1}{t} + \frac{1}{(\xi+1)} \int_0^\xi \left\| \frac{\partial \varphi}{\partial x}(\bar{s}) \right\|_{L^\infty_{(x)}} d\bar{s} \right] d\xi \\ &\quad \leqq C \frac{\|\phi\|_{Y_{1,\alpha}}}{t} + C \|\phi\|_{Y_{1,\alpha}} \sup_{0 \leqq s \leqq t} \left\| \frac{\partial \varphi}{\partial x}(s) \right\|_{L^\infty_{(x)}} \\ &\qquad + C \|\phi\|_{Y_{1,\alpha}} \frac{\log(t+1)}{t^2} + C \|\phi\|_{Y_{1,\alpha}} \frac{\log(t+1)}{t} \sup_{0 \leqq s \leqq t} \left\| \frac{\partial \varphi}{\partial x}(s) \right\|_{L^\infty_{(x)}}.\end{aligned}$$

It then follows that, for $t > 1$,

$$\sup_{0 \leqq s \leqq t} \left\| \frac{\partial \varphi}{\partial x}(s) \right\|_{L^\infty_{(x)}} \leqq C \frac{\|\phi\|_{Y_{1,\alpha}}}{t} + C \|\phi\|_{Y_{1,\alpha}} \sup_{0 \leqq s \leqq t} \left\| \frac{\partial \varphi}{\partial x}(s) \right\|_{L^\infty_{(x)}}. \tag{2.39}$$

By the assumption, if ε_0 is small enough, then we get

$$\sup_{0 \leqq s \leqq t} \left\| \frac{\partial \varphi}{\partial x}(s) \right\|_{L^\infty_{(x)}} \leqq C \frac{\|\phi\|_{Y_{1,\alpha}}}{t}.$$

and (2.36) follows.

In order to derive (2.37) and (2.38), we compute $\frac{\partial^2 G}{\partial x \partial x_0}$, $\frac{\partial^2 \varphi}{\partial x \partial x_0}$ using (2.15), (2.16) and (2.30),

$$\begin{aligned}\frac{\partial^2}{\partial x \partial x_0} G(\xi) &= \nabla^3 \phi(X(\xi), \xi) \frac{\partial}{\partial x} X(\xi) \frac{\partial}{\partial x_0} X(\xi) \\ &\quad + \nabla^2 \phi(X(\xi), \xi) \frac{\partial^2}{\partial x \partial x_0} X(\xi) \\ &= \nabla^3 \phi(X(\xi), \xi) \left[\frac{\xi}{t} I + \int_0^\xi \frac{\partial \varphi}{\partial x}(\bar{s})\, d\bar{s} \right] \left[\left(1 - \frac{\xi}{t}\right) I \right.\end{aligned}$$

$$+\int_0^{\xi}\frac{\partial\varphi}{\partial x_0}(\bar{s})\,\mathrm{d}\bar{s}\Bigg]+\nabla^2\phi\left(X(\xi),\xi\right)\int_0^{\xi}\frac{\partial^2\varphi}{\partial x\partial x_0}(\bar{s})\,\mathrm{d}\bar{s},\quad (2.40)$$

$$\frac{\partial^2\varphi}{\partial x\partial x_0}(s)=-\int_s^t\frac{\partial^2 G}{\partial x\partial x_0}(\xi)\,\mathrm{d}\xi+\frac{1}{t}\int_0^t\xi\frac{\partial^2 G}{\partial x\partial x_0}(\xi)\,\mathrm{d}\xi. \quad (2.41)$$

Taking the norm $\|\cdot\|_{L^\infty_{(x)}}$ of these equations, integrating the resulting formula with respect to s, using Lemma 4, (2.36), and the definition of $\|\phi\|_{Y_{1,\alpha}}$, we obtain

$$\begin{aligned}
&\int_0^t\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s\\
&\leqq\int_0^t\left[\int_s^t\left\|\frac{\partial^2 G}{\partial x\partial x_0}(\xi)\right\|_{L^\infty_{(x)}}\mathrm{d}\xi\right]\mathrm{d}s+\int_0^t\xi\left\|\frac{\partial^2 G}{\partial x\partial x_0}(\xi)\right\|_{L^\infty_{(x)}}\mathrm{d}\xi\\
&\leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t}\int_0^t\mathrm{d}s\int_s^t\frac{\xi\mathrm{d}\xi}{(\xi+1)^{N+3}}\\
&\quad+C\|\phi\|_{Y_{1,\alpha}}\left(\int_0^t\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s\right)\left(\int_0^t\left[\int_s^t\frac{\mathrm{d}\xi}{(\xi+1)^N}\right]\mathrm{d}s\right)\\
&\leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t}\int_0^t\frac{\mathrm{d}\xi\ \xi^2}{(\xi+1)^{N+1}}\\
&\quad+C\|\phi\|_{Y_{1,\alpha}}\left(\int_0^t\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s\right)\left(\int_0^t\frac{\xi\mathrm{d}\xi}{(\xi+1)^N}\right)\\
&\leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t}+C\|\phi\|_{Y_{1,\alpha}}\int_0^t\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s.
\end{aligned}$$

Thus if ε_0 is small enough, we get

$$\int_0^t\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s\leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t}$$

and (2.37) follows. We now set $s=t$ in (2.41) and use (2.40) to obtain

$$\begin{aligned}
\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(t)\right\|_{L^\infty_{(x)}}&\leqq\frac{1}{t}\int_0^t\xi\left\|\frac{\partial^2 G}{\partial x\partial x_0}(\xi)\right\|_{L^\infty_{(x)}}\mathrm{d}\xi\\
&\leqq\frac{C\|\phi\|_{Y_{1,\alpha}}}{t^2}\int_0^t\frac{\xi^2\mathrm{d}\xi}{(\xi+1)^{N+1}}\\
&\quad+\frac{C\|\phi\|_{Y_{1,\alpha}}}{t}\int_0^t\frac{\xi\mathrm{d}\xi}{(\xi+1)^N}\int_0^{\xi}\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(\bar{s})\right\|_{L^\infty_{(x)}}\mathrm{d}\bar{s}\\
&\leqq\frac{C\|\phi\|_{Y_{1,\alpha}}}{t^2}+\frac{C\|\phi\|^2_{Y_{1,\alpha}}}{t^2}\leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t^2}
\end{aligned}$$

and the proof of Lemma 5 is complete. □

In order to complete the proof of Proposition 1, it only remains to obtain estimates for the Hölder seminorms of $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial x_0}$, $\frac{\partial^2\varphi}{\partial x\partial x_0}$. These bounds are obtained using ideas analogous to those used in the two previous lemmas.

Lemma 6. *There exists ε_0 small such that for $t > 1$ and $\|\phi\|_{Y_{1,\alpha}} \leqq \varepsilon_0$, we have the following decay estimates.*

$$\sup_{0\leqq s\leqq t}\left[\frac{\partial\varphi}{\partial x}(s)\right]_{0,\alpha,(x)} \leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}}, \tag{2.42}$$

$$\int_0^t\left[\frac{\partial\varphi}{\partial x_0}(s)\right]_{0,\alpha,(x)} \mathrm{d}s \leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t^{\alpha}}, \tag{2.43}$$

$$\left[\frac{\partial\varphi}{\partial x_0}(t)\right]_{0,\alpha,(x)} \leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}}, \tag{2.44}$$

$$\int_0^t\left[\frac{\partial^2\varphi}{\partial x\partial x_0}(s)\right]_{0,\alpha,(x)} \mathrm{d}s \leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}},$$

$$\left[\frac{\partial^2\varphi}{\partial x\partial x_0}(t)\right]_{0,\alpha,(x)} \leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t^{2+\alpha}}. \tag{2.45}$$

Proof. Using Lemma 3 and (2.36), we get

$$\begin{aligned}
&\left[\frac{\partial}{\partial x}G(\xi)\right]_{0,\alpha,(x)} \\
&\leqq C\left\|\nabla^2\phi(X(\xi),\xi)\right\|_{L^\infty_{(x)}}\left[\frac{\xi}{t}I+\int_0^\xi\frac{\partial\varphi}{\partial x}(\bar{s})\,\mathrm{d}\bar{s}\right]_{0,\alpha,(x)} \\
&\quad +C\left[\nabla^2\phi\right]_{0,\alpha,(x)}\left\|\frac{\xi}{t}I+\int_0^\xi\frac{\partial\varphi}{\partial x}(\bar{s})\,\mathrm{d}\bar{s}\right\|^\alpha_{L^\infty_{(x)}}\left\|\frac{\xi}{t}I+\int_0^\xi\frac{\partial\varphi}{\partial x}\right\|_{L^\infty_{(x)}} \\
&\leqq \frac{C\|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^N}\int_0^\xi\left[\frac{\partial\varphi}{\partial x}(\bar{s})\right]_{0,\alpha,(x)}\mathrm{d}\bar{s}+\frac{C\|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^{N+\alpha}}\frac{\xi^\alpha}{t^\alpha}\frac{\xi}{t} \\
&\leqq \frac{C\|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^N}\int_0^\xi\left[\frac{\partial\varphi}{\partial x}(\bar{s})\right]_{0,\alpha,(x)}\mathrm{d}\bar{s}+\frac{C\|\phi\|_{Y_{1,\alpha}}}{(\xi+1)^{N-1}t^{1+\alpha}}.
\end{aligned}$$

where we have used that $\left[\frac{\xi}{t}I+\int_0^\xi\frac{\partial\varphi}{\partial x}(\bar{s})\,\mathrm{d}\bar{s}\right]_{0,\alpha,(x)}=\left[\int_0^\xi\frac{\partial\varphi}{\partial x}(\bar{s})\,\mathrm{d}\bar{s}\right]_{0,\alpha,(x)}$ as well as the fact that, due to (2.36), $\left\|\frac{\xi}{t}I+\int_0^\xi\frac{\partial\varphi}{\partial x}\right\|_{L^\infty_{(x)}}\leqq\frac{2\xi}{t}$. Differentiating (2.14) with respect to x, taking the Hölder norm and using the previous estimate, we get

$$\begin{aligned}
\left[\frac{\partial\varphi}{\partial x}(s)\right]_{0,\alpha,(x)} &\leqq \int_s^t\left[\frac{\partial}{\partial x}G(\xi)\right]_{0,\alpha,(x)}\mathrm{d}\xi+\frac{1}{t}\int_0^t\xi\left[\frac{\partial}{\partial x}G(\xi)\right]_{0,\alpha,(x)}\mathrm{d}\xi \\
&\leqq \frac{C\|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}}\int_s^t\frac{\mathrm{d}\xi}{(\xi+1)^{N-1}}
\end{aligned}$$

$$
\begin{aligned}
&+C\,\|\phi\|_{Y_{1,\alpha}} \sup_{0\leqq s\leqq t}\left[\frac{\partial\varphi}{\partial x}(s)\right]_{0,\alpha,(x)}\int_s^t \frac{\xi\,\mathrm{d}\xi}{(\xi+1)^N}\\
&+\frac{C\,\|\phi\|_{Y_{1,\alpha}}}{t^{2+\alpha}}\int_0^t \frac{\xi\,\mathrm{d}\xi}{(\xi+1)^{N-1}}\\
&+\frac{C\,\|\phi\|_{Y_{1,\alpha}}}{t}\sup_{0\leqq s\leqq t}\left[\frac{\partial\varphi}{\partial x}(s)\right]_{0,\alpha,(x)}\int_s^t \frac{\xi^2\mathrm{d}\xi}{(\xi+1)^N}\\
\leqq\;& \frac{C\,\|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}}+C\,\|\phi\|_{Y_{1,\alpha}}\sup_{0\leqq s\leqq t}\left[\frac{\partial\varphi}{\partial x}(s)\right]_{0,\alpha,(x)}.
\end{aligned}
$$

We then deduce (2.42), provided ε_0 is small enough. We now derive the Hölder estimate of $\frac{\partial\varphi}{\partial x_0}(t)$. By interpolation, the Hölder inequality, Lemmas 4 and 5, we obtain the following two estimates

$$
\begin{aligned}
\int_0^t\left[\frac{\partial\varphi}{\partial x_0}(s)\right]_{0,\alpha,(x)}\mathrm{d}s &\leqq C\int_0^t\left\|\frac{\partial\varphi}{\partial x_0}(s)\right\|_{L^\infty_{(x)}}^{1-\alpha}\left\|\frac{\partial^2\varphi}{\partial x_0\partial x}(s)\right\|_{L^\infty_{(x)}}^{\alpha}\mathrm{d}s\\
&\leqq C\left(\int_0^t\left\|\frac{\partial\varphi}{\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s\right)^{1-\alpha}\left(\int_0^t\left\|\frac{\partial^2\varphi}{\partial x_0\partial x}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s\right)^{\alpha}\\
&\leqq C\,\|\phi\|_{Y_{1,\alpha}}^{1-\alpha}\frac{\|\phi\|_{Y_{1,\alpha}}^{\alpha}}{t^\alpha}\leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t^\alpha}.\\
\left[\frac{\partial\varphi}{\partial x_0}(t)\right]_{0,\alpha,(x)} &\leqq C\left\|\frac{\partial\varphi}{\partial x_0}(t)\right\|_{L^\infty_{(x)}}^{1-\alpha}\left\|\frac{\partial^2\varphi}{\partial x_0\partial x}(t)\right\|_{L^\infty_{(x)}}^{\alpha}\\
&\leqq C\frac{\|\phi\|_{Y_{1,\alpha}}^{1-\alpha}}{t^{1-\alpha}}\frac{\|\phi\|_{Y_{1,\alpha}}^{\alpha}}{t^{2\alpha}}\leqq C\frac{\|\phi\|_{Y_{1,\alpha}}}{t^{1+\alpha}}.
\end{aligned}
$$

Therefore, (2.43) and (2.44) follow.

We now use Lemmas 3, 4, 5, and (2.42)–(2.44) to get

$$
\begin{aligned}
&\left[\frac{\partial^2}{\partial x\partial x_0}G(\xi)\right]_{0,\alpha,(x)}\\
&\leqq C\left\|\nabla^3\phi(X(\xi),\xi)\right\|_{L^\infty_{(x)}}\left[\left(1-\frac{\xi}{t}\right)I+\int_0^\xi\frac{\partial\varphi}{\partial x_0}\right]_{0,\alpha,(x)}\left\|\frac{\xi}{t}I+\int_0^\xi\frac{\partial\varphi}{\partial x}\right\|_{L^\infty_{(x)}}\\
&\quad+C\left\|\nabla^3\phi(X(\xi),\xi)\right\|_{L^\infty_{(x)}}\left\|\left(1-\frac{\xi}{t}\right)I+\int_0^\xi\frac{\partial\varphi}{\partial x_0}\right\|_{L^\infty_{(x)}}\left[\frac{\xi}{t}I+\int_0^\xi\frac{\partial\varphi}{\partial x}\right]_{0,\alpha,(x)}\\
&\quad+C\left[\nabla^3\phi\right]_{0,\alpha,(x)}\left\|\frac{\xi}{t}I+\int_0^\xi\frac{\partial\varphi}{\partial x}\right\|_{L^\infty_{(x)}}^{\alpha}\left\|\left(1-\frac{\xi}{t}\right)I\right.\\
&\quad\left.+\int_0^\xi\frac{\partial\varphi}{\partial x_0}\right\|_{L^\infty_{(x)}}\left\|\frac{\xi}{t}I+\int_0^\xi\frac{\partial\varphi}{\partial x}\right\|_{L^\infty_{(x)}}\\
&\quad+C\left[\nabla^2\phi\right]_{0,\alpha,(x)}\left\|\frac{\xi}{t}I+\int_0^\xi\frac{\partial\varphi}{\partial x}\right\|_{L^\infty_{(x)}}^{\alpha}\int_0^\xi\left\|\frac{\partial^2\varphi}{\partial x\partial x_0}(\bar{s})\right\|_{L^\infty_{(x)}}\mathrm{d}\bar{s}
\end{aligned}
$$

$$
\begin{aligned}
&+C\left\|\nabla^{2}\phi\left(X\left(\xi\right),\xi\right)\right\|_{L^{\infty}_{(x)}}\int_{0}^{\xi}\left[\frac{\partial^{2}\varphi}{\partial x\partial x_{0}}\left(\bar{s}\right)\right]_{0,\alpha,(x)}\mathrm{d}\bar{s}\\
&\leqq\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{(\xi+1)^{N+1}}\frac{1}{t^{\alpha}}\frac{\xi}{t}+\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{(\xi+1)^{N+1+\alpha}}\frac{\xi^{\alpha}}{t^{\alpha}}\frac{\xi}{t}+\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{(\xi+1)^{N+\alpha}}\frac{\xi^{\alpha}}{t^{\alpha}}\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{t}\\
&+\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{(\xi+1)^{N}}\int_{0}^{\xi}\left[\frac{\partial^{2}\varphi}{\partial x\partial x_{0}}\left(\bar{s}\right)\right]_{0,\alpha,(x)}\mathrm{d}\bar{s}\\
&\leqq\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{(\xi+1)^{N}}\int_{0}^{\xi}\left[\frac{\partial^{2}\varphi}{\partial x\partial x_{0}}\left(s\right)\right]_{0,\alpha,(x)}\mathrm{d}s+\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{(\xi+1)^{N}t^{1+\alpha}}.
\end{aligned}
$$

Similarly, we get from (2.14) as well as the estimate above

$$
\begin{aligned}
&\left[\frac{\partial^{2}\varphi}{\partial x\partial x_{0}}\left(s\right)\right]_{0,\alpha,(x)}\\
&\quad\leqq\int_{s}^{t}\left[\frac{\partial^{2}}{\partial x\partial x_{0}}G\left(\xi\right)\right]_{0,\alpha,(x)}\mathrm{d}\xi+\frac{1}{t}\int_{0}^{t}\xi\left[\frac{\partial^{2}}{\partial x\partial x_{0}}G\left(\xi\right)\right]_{0,\alpha,(x)}\mathrm{d}\xi\\
&\quad\leqq\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{t^{1+\alpha}}\int_{s}^{t}\frac{\mathrm{d}\xi}{(\xi+1)^{N}}\\
&\quad\quad+C\left\|\phi\right\|_{Y_{1,\alpha}}\int_{s}^{t}\frac{1}{(\xi+1)^{N}}\left[\int_{0}^{\xi}\left[\frac{\partial^{2}\varphi}{\partial x\partial x_{0}}\left(\bar{s}\right)\right]_{0,\alpha,(x)}\mathrm{d}\bar{s}\right]\mathrm{d}\xi\\
&\quad\quad+\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{t^{2+\alpha}}\int_{0}^{t}\frac{\xi\mathrm{d}\xi}{(\xi+1)^{N}}\\
&\quad\quad+\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{t}\int_{0}^{t}\frac{\xi}{(\xi+1)^{N}}\left[\int_{0}^{\xi}\left[\frac{\partial^{2}\varphi}{\partial x\partial x_{0}}\left(\bar{s}\right)\right]_{0,\alpha,(x)}\mathrm{d}\bar{s}\right]\mathrm{d}\xi \qquad (2.46)
\end{aligned}
$$

By integrating (2.46) from $s=0$ to $s=t$, we have

$$
\begin{aligned}
\int_{0}^{t}\left[\frac{\partial^{2}\varphi}{\partial x\partial x_{0}}\left(s\right)\right]_{0,\alpha,(x)}\mathrm{d}s&\leqq\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{t^{1+\alpha}}+C\left\|\phi\right\|_{Y_{1,\alpha}}\int_{0}^{t}\left[\frac{\partial^{2}\varphi}{\partial x\partial x_{0}}\left(s\right)\right]_{0,\alpha,(x)}\mathrm{d}s\\
&\leqq\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{t^{1+\alpha}}+C\left\|\phi\right\|_{Y_{1,\alpha}}\int_{0}^{t}\left[\frac{\partial^{2}\varphi}{\partial x\partial x_{0}}\left(s\right)\right]_{0,\alpha,(x)}\mathrm{d}s.
\end{aligned}
$$

If ε_0 is small enough, then we obtain

$$
\int_{0}^{t}\left[\frac{\partial^{2}\varphi}{\partial x\partial x_{0}}\left(s\right)\right]_{0,\alpha,(x)}\mathrm{d}s\leqq\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{t^{1+\alpha}}. \qquad (2.47)
$$

Now putting (2.47) into (2.46) with $s=t$ yields

$$
\left[\frac{\partial^{2}\varphi}{\partial x\partial x_{0}}\left(t\right)\right]_{0,\alpha,(x)}\leqq\frac{C\left\|\phi\right\|_{Y_{1,\alpha}}}{t^{2+\alpha}}.
$$

and this completes the proof of the lemma. □

2.3.3. The proof of Proposition 2 Now we prove the decay estimates for the higher order derivatives of φ. We prove Proposition 2 by induction on ℓ. The induction hypotheses consist of the following estimates, for $0 \leqq m < \ell$,

$$\sup_{0\leqq s\leqq t}\left\|\frac{\partial^m\varphi}{\partial x^m}(s)\right\|_{L^\infty_{(x)}} \leqq \frac{C\,\|\phi\|_{Y_{m,\alpha}}}{t^m},\quad \sup_{0\leqq s\leqq t}\left[\frac{\partial^m\varphi}{\partial x^m}(s)\right]_{0,\alpha,(x)} \leqq \frac{C\,\|\phi\|_{Y_{m,\alpha}}}{t^{m+\alpha}}, \tag{2.48}$$

$$\int_0^t\left\|\frac{\partial^{m+1}\varphi}{\partial x^m\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s \leqq \frac{C\,\|\phi\|_{Y_{m,\alpha}}}{t^m},\quad \int_0^t\left[\frac{\partial^{m+1}\varphi}{\partial x^m\partial x_0}(s)\right]_{0,\alpha,(x)}\mathrm{d}s \leqq \frac{C\,\|\phi\|_{Y_{m,\alpha}}}{t^{m+\alpha}}, \tag{2.49}$$

$$\left\|\frac{\partial^{m+1}\varphi}{\partial x^m\partial x_0}(t)\right\|_{L^\infty_{(x)}} \leqq \frac{C\,\|\phi\|_{Y_{m,\alpha}}}{t^{m+1}},\quad \left[\frac{\partial^{m+1}\varphi}{\partial x^m\partial x_0}(t)\right]_{0,\alpha,(x)} \leqq \frac{C\,\|\phi\|_{Y_{m,\alpha}}}{t^{m+1+\alpha}}. \tag{2.50}$$

Estimates (2.48)–(2.50) have been already proved for $m = 0, 1$ (see Lemma 2 and Proposition 1 respectively). We begin with the estimates of $G = G_\varphi$ in terms of φ.

Lemma 7. *Let $\ell \geqq 2$ be an integer. Assume the induction hypotheses (2.48)–(2.50). There exists ε_0 such that for $t > 1$ and $\|\phi\|_{Y_{\ell,\alpha}} \leqq \varepsilon_0$, we have the following*

$$\left\|\frac{\partial^\ell G}{\partial x^\ell}(\xi)\right\|_{L^\infty_{(x)}} \leqq \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{t^\ell\,(\xi+1)^{N-1}} + \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}}\sup_{0\leqq s\leqq t}\left\|\frac{\partial^\ell\varphi}{\partial x^\ell}(s)\right\|_{L^\infty_{(x)}}, \tag{2.51}$$

$$\begin{aligned}\left[\frac{\partial^\ell G}{\partial x^\ell}(\xi)\right]_{0,\alpha,(x)} \leqq{}& \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{t^{\ell+\alpha}\,(\xi+1)^{N-1}} + \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{t^\alpha\,(\xi+1)^{N-1}}\sup_{0\leqq s\leqq t}\left\|\frac{\partial^\ell\varphi}{\partial x^\ell}(s)\right\|_{L^\infty_{(x)}}\\ &+\frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}}\sup_{0\leqq s\leqq t}\left[\frac{\partial^\ell\varphi}{\partial x^\ell}(s)\right]_{0,\alpha,(x)},\end{aligned} \tag{2.52}$$

$$\left\|\frac{\partial^{\ell+1}G}{\partial x^\ell\partial x_0}(\xi)\right\|_{L^\infty_{(x)}} \leqq \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{t^\ell\,(\xi+1)^N} + \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^N}\int_0^\xi\left\|\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}(\bar{s})\right\|_{L^\infty_{(x)}}\mathrm{d}\bar{s}, \tag{2.53}$$

$$\begin{aligned}\left[\frac{\partial^{\ell+1}G}{\partial x^\ell\partial x_0}(\xi)\right]_{0,\alpha,(x)} \leqq{}& \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{t^{\ell+\alpha}\,(\xi+1)^N} + \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{t^\alpha\,(\xi+1)^N}\int_0^\xi\left\|\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}(\bar{s})\right\|_{L^\infty_{(x)}}\mathrm{d}\bar{s}\\ &+\frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^N}\int_0^\xi\left[\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}(\bar{s})\right]_{0,\alpha,(x)}\mathrm{d}\bar{s}.\end{aligned} \tag{2.54}$$

Proof. Taking $\frac{\partial^\ell}{\partial x^\ell}$ of G yields

$$\begin{aligned}\frac{\partial^\ell G}{\partial x^\ell}(\xi) &= \sum_{\substack{1\leqq i\leqq \ell,\\ j_1+\cdots+j_i=\ell}} A_{ij_1\cdots j_i}\frac{\partial^i}{\partial x^i}\nabla\phi(X,\xi)\frac{\partial^{j_1}}{\partial x^{j_1}}X\cdots\frac{\partial^{j_i}}{\partial x^{j_i}}X\\ &= \nabla^2\phi(X,\xi)\frac{\partial^\ell X}{\partial x^\ell}+\sum_{\substack{j_m<\ell\\ 1\leqq m\leqq i}}\cdots\\ &= \nabla^2\phi(X,\xi)\int_0^\xi \frac{\partial^\ell}{\partial x^\ell}\varphi(\bar{s})\,\mathrm{d}\bar{s}+\sum_{\substack{j_m<\ell\\ 1\leqq m\leqq i}}\cdots, \qquad (2.55)\end{aligned}$$

where

$$X = X(\xi) = x_0 + \frac{x-x_0}{t}\xi + \int_0^\xi \varphi(\bar{s})\,\mathrm{d}\bar{s},$$

and where $A_{ij_1\cdots j_i}$ are suitable numerical coefficients. Using the induction hypotheses, we bound each term with $j_m < \ell$ for all $1 \leqq m \leqq i$ on the right-hand side of the above identity as

$$\begin{aligned}&\left\|\frac{\partial^i}{\partial x^i}\nabla\phi(X,\xi)\frac{\partial^{j_1}}{\partial x^{j_1}}X\cdots\frac{\partial^{j_i}}{\partial x^{j_i}}X\right\|_{L^\infty_{(x)}}\\ &\quad\leqq \frac{C\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{i+N-1}}\frac{C\|\phi\|_{Y_{\ell,\alpha}}\xi}{t^{j_1}}\cdots\frac{C\|\phi\|_{Y_{\ell,\alpha}}\xi}{t^{j_i}} \leqq \frac{C\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}t^\ell},\\ &\left[\frac{\partial^i}{\partial x^i}\nabla\phi(X,\xi)\frac{\partial^{j_1}}{\partial x^{j_1}}X\cdots\frac{\partial^{j_i}}{\partial x^{j_i}}X\right]_{0,\alpha,(x)}\\ &\quad\leqq \left[\frac{\partial^i}{\partial x^i}\nabla\phi(X,\xi)\right]_{0,\alpha,(x)}\left\|\frac{\partial^{j_1}}{\partial x^{j_1}}X\cdots\frac{\partial^{j_i}}{\partial x^{j_i}}X\right\|_{L^\infty_{(x)}}\\ &\qquad+\sum_{1\leqq m\leqq i}\left\|\frac{\partial^i}{\partial x^i}\nabla\phi(X,\xi)\right\|_{L^\infty_{(x)}}\left[\frac{\partial^{j_m}}{\partial x^{j_m}}X\right]_{0,\alpha,(x)}\prod_{p\neq m}\left\|\frac{\partial^{j_p}}{\partial x^{j_p}}X\right\|_{L^\infty_{(x)}}\\ &\quad\leqq \frac{C\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}t^{\ell+\alpha}}.\end{aligned}$$

Putting the above inequalities into (2.55) yields

$$\begin{aligned}\left\|\frac{\partial^\ell G}{\partial x^\ell}(\xi)\right\|_{L^\infty_{(x)}} &\leqq \frac{C\|\phi\|_{Y_{\ell,\alpha}}}{t^\ell(\xi+1)^{N-1}}+\frac{C\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^N}\int_0^\xi\left\|\frac{\partial^\ell\varphi}{\partial x^\ell}(\bar{s})\right\|_{L^\infty_{(x)}}\mathrm{d}\bar{s}\\ &\leqq \frac{C\|\phi\|_{Y_{\ell,\alpha}}}{t^\ell(\xi+1)^{N-1}}+\frac{C\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}}\sup_{0\leqq s\leqq t}\left\|\frac{\partial^\ell\varphi}{\partial x^\ell}(s)\right\|_{L^\infty_{(x)}},\end{aligned}$$

$$\begin{aligned}\left[\frac{\partial^\ell G}{\partial x^\ell}(\xi)\right]_{0,\alpha,(x)} &\leqq \frac{C\|\phi\|_{Y_{\ell,\alpha}}}{t^{\ell+\alpha}(\xi+1)^{N-1}} + \frac{C\|\phi\|_{Y_{\ell,\alpha}}}{t^{\alpha}(\xi+1)^{N}}\int_0^\xi \left\|\frac{\partial^\ell \varphi}{\partial x^\ell}(\bar{s})\right\|_{L^\infty_{(x)}} \mathrm{d}\bar{s} \\ &\quad + \frac{C\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N}}\int_0^\xi \left[\frac{\partial^\ell \varphi}{\partial x^\ell}(\bar{s})\right]_{0,\alpha,(x)} \mathrm{d}\bar{s} \\ &\leqq \frac{C\|\phi\|_{Y_{\ell,\alpha}}}{t^{\ell+\alpha}(\xi+1)^{N-1}} + \frac{C\|\phi\|_{Y_{\ell,\alpha}}}{t^{\alpha}(\xi+1)^{N-1}}\sup_{0\leqq s\leqq t}\left\|\frac{\partial^\ell \varphi}{\partial x^\ell}(s)\right\|_{L^\infty_{(x)}} \\ &\quad + \frac{C\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}}\sup_{0\leqq s\leqq t}\left[\frac{\partial^\ell \varphi}{\partial x^\ell}(s)\right]_{0,\alpha,(x)}.\end{aligned}$$

In a similar manner, we take $\frac{\partial^{\ell+1}}{\partial x^\ell \partial x_0}$ of G to get

$$\begin{aligned}\frac{\partial^{\ell+1} G}{\partial x^\ell \partial x_0}(\xi) &= \sum_{\substack{1\leqq i\leqq \ell,\\ j_1+\cdots+j_i=\ell}} B_{ij_1\cdots j_i}\frac{\partial^i}{\partial x^i}\nabla\phi(X,\xi)\frac{\partial \partial^{j_1}}{\partial x_0 \partial x^{j_1}}X\cdots\frac{\partial^{j_i}}{\partial x^{j_i}}X \\ &= \nabla^2\phi(X,\xi)\frac{\partial^{\ell+1}X}{\partial x^\ell \partial x_0} + \sum_{\substack{j_m<\ell\\ 1\leqq m\leqq i}}\cdots \\ &= \nabla^2\phi(X,\xi)\int_0^\xi \frac{\partial^{\ell+1}\varphi}{\partial x^\ell \partial x_0}(\bar{s})\,\mathrm{d}\bar{s} + \sum_{\substack{j_m<\ell\\ 1\leqq m\leqq i}}\cdots,\end{aligned}$$

We use the induction hypotheses to bound all the terms with $j_m < \ell$ for all $1 \leqq m \leqq i$ on the right-hand side of the above identity as

$$\begin{aligned}&\left\|\frac{\partial^i}{\partial x^i}\nabla\phi(X,\xi)\frac{\partial \partial^{j_1}}{\partial x_0 \partial x^{j_1}}X\cdots\frac{\partial^{j_i}}{\partial x^{j_i}}X\right\|_{L^\infty_{(x)}} \\ &\quad\leqq \left\|\frac{\partial^i}{\partial x^i}\nabla\phi(\xi)\right\|_{L^\infty_{(x)}}\int_0^\xi \left\|\frac{\partial \partial^{j_1}\varphi}{\partial x_0 \partial x^{j_1}}(\bar{s})\right\|_{L^\infty_{(x)}}\mathrm{d}\bar{s}\cdots\left\|\frac{\partial^{j_i}}{\partial x^{j_i}}X\right\|_{L^\infty_{(x)}} \\ &\quad\leqq \frac{C\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{i+N-1}}\frac{C\|\phi\|_{Y_{\ell,\alpha}}}{t^{j_1}}\frac{C\|\phi\|_{Y_{\ell,\alpha}}\xi}{t^{j_2}}\cdots\frac{C\|\phi\|_{Y_{\ell,\alpha}}\xi}{t^{j_i}} \leqq \frac{C\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N}t^{\ell}}, \\ &\left[\frac{\partial^i}{\partial x^i}\nabla\phi(X,\xi)\frac{\partial \partial^{j_1}}{\partial x_0 \partial x^{j_1}}X\cdots\frac{\partial^{j_i}}{\partial x^{j_i}}X\right]_{0,\alpha,(x)} \\ &\quad\leqq \left[\frac{\partial^i}{\partial x^i}\nabla\phi(X,\xi)\right]_{0,\alpha,(x)}\left(\int_0^\xi \left\|\frac{\partial \partial^{j_1}\varphi}{\partial x_0 \partial x^{j_1}}(\bar{s})\right\|_{L^\infty_{(x)}}\mathrm{d}\bar{s}\right)\left(\prod_{p\neq 1}\left\|\frac{\partial^{j_p}}{\partial x^{j_p}}X\right\|_{L^\infty_{(x)}}\right) \\ &\qquad + \left\|\frac{\partial^i}{\partial x^i}\nabla\phi(X,\xi)\right\|_{L^\infty_{(x)}}\left(\int_0^\xi \left[\frac{\partial \partial^{j_1}\varphi}{\partial x_0 \partial x^{j_1}}(\bar{s})\right]_{0,\alpha,(x)}\mathrm{d}\bar{s}\right)\left(\prod_{p\neq 1}\left\|\frac{\partial^{j_p}}{\partial x^{j_p}}X\right\|_{L^\infty_{(x)}}\right)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{2\leqq m\leqq i}\left\|\frac{\partial^{i}}{\partial x^{i}}\nabla\phi\left(X,\xi\right)\right\|_{L^{\infty}_{(x)}}\left(\int_{0}^{\xi}\left\|\frac{\partial\partial^{j_1}\varphi}{\partial x_0\partial x^{j_1}}\left(\bar{s}\right)\right\|_{L^{\infty}_{(x)}}\mathrm{d}\bar{s}\right)\\
&\times\left(\left[\frac{\partial^{j_m}}{\partial x^{j_m}}X\right]_{0,\alpha,(x)}\right)\left(\prod_{p\neq m,1}\left\|\frac{\partial^{j_p}}{\partial x^{j_p}}X\right\|_{L^{\infty}_{(x)}}\right)\\
\leqq&\ \frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^{i+N-1+\alpha}}\frac{\xi^{\alpha}}{t^{\alpha}}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{t^{j_1}}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_2}}\cdots\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_i}}\\
&+\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^{i+N-1}}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{t^{j_1+\alpha}}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_2}}\cdots\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_i}}\\
&+\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^{i+N-1}}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{t^{j_1}}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_m+\alpha}}\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_2}}\cdots\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}\xi}{t^{j_i}}\\
\leqq&\ \frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N}t^{\ell+\alpha}}.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\left\|\frac{\partial^{\ell+1}G}{\partial x^{\ell}\partial x_0}\left(\xi\right)\right\|_{L^{\infty}_{(x)}}&\leqq\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{t^{\ell}\left(\xi+1\right)^{N}}+\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N}}\int_{0}^{\xi}\left\|\frac{\partial^{\ell+1}\varphi}{\partial x^{\ell}\partial x_0}\left(\bar{s}\right)\right\|_{L^{\infty}_{(x)}}\mathrm{d}\bar{s}\\
\left[\frac{\partial^{\ell+1}G}{\partial x^{\ell}\partial x_0}\left(\xi\right)\right]_{0,\alpha,(x)}&\leqq\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{t^{\ell+\alpha}\left(\xi+1\right)^{N}}+\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{t^{\alpha}\left(\xi+1\right)^{N}}\int_{0}^{\xi}\left\|\frac{\partial^{\ell+1}\varphi}{\partial x^{\ell}\partial x_0}\left(\bar{s}\right)\right\|_{L^{\infty}_{(x)}}\mathrm{d}\bar{s}\\
&\quad+\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N}}\int_{0}^{\xi}\left[\frac{\partial^{\ell+1}\varphi}{\partial x^{\ell}\partial x_0}\left(\bar{s}\right)\right]_{0,\alpha,(x)}\mathrm{d}\bar{s}.
\end{aligned}
$$

This completes the proof. □

We now prove Proposition 2.

Proof of Proposition 2. Taking $\frac{\partial^{\ell}}{\partial x^{\ell}}$ of φ and using (2.14), we get

$$\frac{\partial^{\ell}\varphi}{\partial x^{\ell}}\left(s\right)=-\int_{s}^{t}\frac{\partial^{\ell}G}{\partial x^{\ell}}\left(\xi\right)\mathrm{d}\xi+\frac{1}{t}\int_{0}^{t}\xi\frac{\partial^{\ell}G}{\partial x^{\ell}}\left(\xi\right)\mathrm{d}\xi. \tag{2.56}$$

We first estimate $\left\|\cdot\right\|_{L^{\infty}}$ and get from (2.56),

$$
\begin{aligned}
\left\|\frac{\partial^{\ell}\varphi}{\partial x^{\ell}}\left(s\right)\right\|_{L^{\infty}_{(x)}}&\leqq\int_{s}^{t}\left\|\frac{\partial^{\ell}G}{\partial x^{\ell}}\left(\xi\right)\right\|_{L^{\infty}_{(x)}}\mathrm{d}\xi+\frac{1}{t}\int_{0}^{t}\xi\left\|\frac{\partial^{\ell}G}{\partial x^{\ell}}\left(\xi\right)\right\|_{L^{\infty}_{(x)}}\mathrm{d}\xi\\
&\leqq 2\int_{0}^{t}\left\|\frac{\partial^{\ell}G}{\partial x^{\ell}}\left(\xi\right)\right\|_{L^{\infty}_{(x)}}\mathrm{d}\xi,
\end{aligned}\tag{2.57}
$$

where we used $\frac{\xi}{t}\leqq 1$. Suppose that the induction hypothesis (2.48)–(2.49) is satisfied. By Lemma 7, we then have

$$\left\|\frac{\partial^{\ell}G}{\partial x^{\ell}}\left(\xi\right)\right\|_{L^{\infty}_{(x)}}\leqq\frac{C\left\|\phi\right\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}}\left\{\frac{1}{t^{\ell}}+\sup_{0\leqq s\leqq t}\left\|\frac{\partial^{\ell}\varphi}{\partial x^{\ell}}\left(s\right)\right\|_{L^{\infty}_{(x)}}\right\}. \tag{2.58}$$

Putting (2.58) into (2.57) yields

$$\begin{aligned}\left\|\frac{\partial^\ell\varphi}{\partial x^\ell}(s)\right\|_{L^\infty_{(x)}} &\leqq C\,\|\phi\|_{Y_{\ell,\alpha}}\left(\int_0^t\frac{\mathrm{d}\xi}{(\xi+1)^{N-1}}\right)\left\{\frac{1}{t^\ell}+\sup_{0\leqq s\leqq t}\left\|\frac{\partial^\ell\varphi}{\partial x^\ell}(s)\right\|_{L^\infty_{(x)}}\right\}\\ &\leqq C\,\|\phi\|_{Y_{\ell,\alpha}}\left\{\frac{1}{t^\ell}+\sup_{0\leqq s\leqq t}\left\|\frac{\partial^\ell\varphi}{\partial x^\ell}(s)\right\|_{L^\infty_{(x)}}\right\}.\end{aligned}$$

Thus if ε_0 is small enough, we obtain the first part of (2.11). In a similar way, we have

$$\left\|\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}(s)\right\|_{L^\infty_{(x)}} \leqq \int_s^t\left\|\frac{\partial^{\ell+1}G}{\partial x^\ell\partial x_0}(\xi)\right\|_{L^\infty_{(x)}}\mathrm{d}\xi+\frac{1}{t}\int_0^t\xi\left\|\frac{\partial^{\ell+1}G}{\partial x^\ell\partial x_0}(\xi)\right\|_{L^\infty_{(x)}}\mathrm{d}\xi. \tag{2.59}$$

Using Lemma 7 yields

$$\left\|\frac{\partial^{\ell+1}G}{\partial x^\ell\partial x_0}(\xi)\right\|_{L^\infty_{(x)}} \leqq \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^N}\left\{\frac{1}{t^\ell}+\int_0^\xi\left\|\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}(\bar{s})\right\|_{L^\infty_{(x)}}\mathrm{d}\bar{s}\right\}. \tag{2.60}$$

By putting (2.60) into (2.59) and integrating from $s=0$ to $s=t$, we have

$$\begin{aligned}&\int_0^t\left\|\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s\\ &\leqq C\,\|\phi\|_{Y_{\ell,\alpha}}\left(\int_0^t\int_s^t\frac{\mathrm{d}\xi}{(\xi+1)^N}\mathrm{d}s\right)\left\{\frac{1}{t^\ell}+\left(\int_0^t\left\|\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s\right)\right\}\\ &\quad+C\,\|\phi\|_{Y_{\ell,\alpha}}\left(\int_0^t\frac{\xi\,\mathrm{d}\xi}{(\xi+1)^N}\right)\left\{\frac{1}{t^\ell}+\left(\int_0^t\left\|\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s\right)\right\}\\ &\leqq C\,\|\phi\|_{Y_{\ell,\alpha}}\left\{\frac{1}{t^\ell}+\left(\int_0^t\left\|\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}(s)\right\|_{L^\infty_{(x)}}\mathrm{d}s\right)\right\}.\end{aligned}$$

Thus we obtain the $\|\cdot\|_{L^\infty}$ part of (2.12) provided ε_0 is small. We then substitute (2.60) for (2.12) and put $s=t$ in (2.59) to get

$$\left\|\frac{\partial^{\ell+1}\varphi}{\partial x^\ell\partial x_0}(t)\right\|_{L^\infty_{(x)}} \leqq \frac{1}{t}\frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{t^\ell}\int_0^t\frac{\xi\,\mathrm{d}\xi}{(\xi+1)^N}\leqq\frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{t^{\ell+1}}.$$

Hence we obtain $\|\cdot\|_{L^\infty}$ part of (2.13). We now estimate $[\cdot]_{0,\alpha}$. From (2.56), we get

$$\begin{aligned}\left[\frac{\partial^\ell\varphi}{\partial x^\ell}(s)\right]_{0,\alpha} &\leqq \int_s^t\left[\frac{\partial^\ell\varphi}{\partial x^\ell}(\xi)\right]_{0,\alpha}\mathrm{d}\xi+\frac{1}{t}\int_0^t\xi\left[\frac{\partial^\ell G}{\partial x^\ell}(\xi)\right]_{0,\alpha}\mathrm{d}\xi\\ &\leqq 2\int_0^t\left[\frac{\partial^\ell G}{\partial x^\ell}(\xi)\right]_{0,\alpha}\mathrm{d}\xi,\end{aligned} \tag{2.61}$$

Using the induction hypothesis (2.48)–(2.49) and using (2.11) and Lemma 7, we have

$$\left[\frac{\partial^\ell G}{\partial x^\ell}(\xi)\right]_{0,\alpha} \leqq \frac{C\,\|\phi\|_{Y_{\ell,\alpha}}}{(\xi+1)^{N-1}}\left\{\frac{1}{t^{\ell+\alpha}} + \sup_{0\leqq s\leqq t}\left[\frac{\partial^\ell \varphi}{\partial x^\ell}(s)\right]_{0,\alpha}\right\}. \tag{2.62}$$

Putting (2.62) into (2.61) yields

$$\begin{aligned}\left[\frac{\partial^\ell \varphi}{\partial x^\ell}(s)\right]_{0,\alpha} &\leqq C\,\|\phi\|_{Y_{\ell,\alpha}}\left(\int_0^t \frac{d\xi}{(\xi+1)^{N-1}}\right)\left\{\frac{1}{t^{l+\alpha}} + \sup_{0\leqq s\leqq t}\left[\frac{\partial^\ell \varphi}{\partial x^\ell}(s)\right]_{0,\alpha}\right\}\\ &\leqq C\,\|\phi\|_{Y_{\ell,\alpha}}\left\{\frac{1}{t^{l+\alpha}} + \sup_{0\leqq s\leqq t}\left[\frac{\partial^\ell \varphi}{\partial x^\ell}(s)\right]_{0,\alpha}\right\}.\end{aligned}$$

Thus if ε_0 is small enough, we obtain the $[\cdot]_{0,\alpha}$ part of (2.11). In a similar way, we can deduce the $[\cdot]_{0,\alpha}$ parts of (2.12) and (2.13). Therefore the proof is complete. □

2.4. Estimating the potential ϕ in terms of the density ρ

The following is a standard regularity result for the Poisson equation.

Lemma 8. (Elliptic regularity theory) *Suppose that ϕ solves the Poisson equation (1.3). Let us assume that $\|\rho\|_{X_{k,\alpha}} < \infty$ for some $k \geqq 0$, $0 < \alpha < 1$. Then*

$$\|\phi\|_{Y_{k,\alpha}} \leqq C\,\|\rho\|_{X_{k,\alpha}} \tag{2.63}$$

for some $C > 0$ independent of ρ.

Proof. For any fixed $t > 0$ we define

$$\tilde{\rho}(z,t) = (t+1)^N \rho(z(t+1),t).$$

Notice that

$$\int |\tilde{\rho}(z)|\, d^N z = \int (t+1)^N |\rho(z(t+1),t)|\, d^N z = \int |\rho(x)|\, d^N x.$$

On the other hand,

$$\sum_{\ell=0}^{k}\left\|\nabla_z^\ell \tilde{\rho}(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)} = (t+1)^N \sum_{\ell=0}^{k}(t+1)^\ell \left\|\nabla_x^\ell \rho(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)}$$

$$\sup_{z_1,z_2\in\mathbb{R}^N}\frac{\left|\nabla_z^k \tilde{\rho}(z_1,t) - \nabla_z^k \tilde{\rho}(z_2,t)\right|}{|z_1-z_2|^\alpha} = (t+1)^{N+k+\alpha}\sup_{x_1,x_2\in\mathbb{R}^N}\frac{\left|\nabla_x^k \rho(x_1,t) - \nabla_x^k \rho(x_2,t)\right|}{|x_1-x_2|^\alpha}.$$

Then we have

$$\begin{aligned}&\int |\tilde{\rho}(z)|\, d^N z + \sum_{\ell=0}^{k}\left\|\nabla_z^\ell \tilde{\rho}(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)}\\ &\quad + \sup_{z_1,z_2\in\mathbb{R}^N}\frac{\left|\nabla_z^k \tilde{\rho}(z_1,t) - \nabla_z^k \tilde{\rho}(z_2,t)\right|}{|z_1-z_2|^\alpha} \leqq \|\rho\|.\end{aligned} \tag{2.64}$$

On the other hand, by assumption, we have

$$\Delta_x \phi = \rho.$$

We define

$$\tilde{\phi}(z) = (t+1)^{N-2} \phi(z(t+1), t).$$

Then we have

$$\Delta_z \tilde{\phi} = \tilde{\rho}.$$

We now claim that the following estimate holds.

$$\sum_{\ell=1}^{k+2} \left\| \nabla_z^{\ell} \tilde{\phi}(\cdot, t) \right\|_{L^\infty(\mathbb{R}^N)} + \sup_{z_1, z_2 \in \mathbb{R}^N} \frac{\left| \nabla_z^{k+2} \tilde{\phi}(z_1, t) - \nabla_z^{k+2} \tilde{\phi}(z_2, t) \right|}{|z_1 - z_2|^{\alpha}} \leqq C J,$$

$$J \equiv \left[\int_{\mathbb{R}^N} |\tilde{\rho}(z)| \, \mathrm{d}^N z + \sum_{\ell=0}^{k} \left\| \nabla_z^{\ell} \tilde{\rho}(\cdot, t) \right\|_{L^\infty(\mathbb{R}^N)} \right.$$
$$\left. + \sup_{z_1, z_2 \in \mathbb{R}^N} \frac{\left| \nabla_z^{k} \tilde{\rho}(z_1, t) - \nabla_z^{k} \tilde{\rho}(z_2, t) \right|}{|z_1 - z_2|^{\alpha}} \right].$$

Indeed, a standard interpolation argument yields

$$\|\tilde{\rho}\|_{L^p} \leqq \|\tilde{\rho}\|_{L^\infty}^{\frac{p-1}{p}} \|\tilde{\rho}\|_{L^1}^{\frac{1}{p}} \leqq J, \quad 1 \leqq p \leqq \infty.$$

Then, using the Calderon–Zygmund inequality, it follows that

$$\left\| \nabla_z^2 \tilde{\phi} \right\|_{L^p} \leqq C \|\tilde{\rho}\|_{L^p} \leqq C J, \quad 1 < p < \infty.$$

Therefore, the Sobolev embedding theorem implies that

$$\left\| \tilde{\phi} \right\|_{L^q} \leqq C J, \quad \frac{N}{N-2} < q < \infty.$$

Interior estimates for the Poisson equation in Sobolev spaces give a uniform bound on the $W^{k+2,q}$ norm of the restriction of $\tilde{\phi}$ to any unit ball and hence of the C^{α} norm of this restriction. Using this estimate, (2.63) follows from the inequality

$$\left\| \tilde{\phi} \right\|_{C^{k+2,\alpha}(B_{1/2}(x_0))} \leqq C \left[\left\| \tilde{\phi} \right\|_{C^{0,\alpha}(B_1(x_0))} + \|\tilde{\rho}\|_{C^{k,\alpha}(B_1(x_0))} \right] \leqq C J,$$

that is just a consequence of classical interior estimates for the Poisson equation (see [7]). Using the estimate (2.64) it can be concluded that

$$\sum_{\ell=1}^{k+2} \left\| \nabla_z^{\ell} \tilde{\phi}(\cdot, t) \right\|_{L^\infty(\mathbb{R}^N)} + \sup_{z_1, z_2 \in \mathbb{R}^N} \frac{\left| \nabla_z^{k+2} \tilde{\phi}(z_1, t) - \nabla_z^{k+2} \tilde{\phi}(z_2, t) \right|}{|z_1 - z_2|^{\alpha}} \leqq C \|\rho\|_{X_{k,\alpha}}.$$

Using the definition of $\tilde{\phi}$ as well as the definition of the norm $\|\cdot\|_{Y_{k,\alpha}}$ as in (2.2), it then follows that

$$\|\phi\|_{Y_{k,\alpha}} \leqq C \|\rho\|_{X_{k,\alpha}}.$$

and this completes the proof of the lemma. □

2.5. Conservation of the L^1 norm of f

The following result is standard in the theory of the Vlasov–Poisson equation. For instance, see [8].

Lemma 9. *Suppose that $f(x, v, t)$ solves the problem* (1.2), (1.3). *Then we have*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x, v, t)|\, dv dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f_0(x, v)|\, dv dx, \quad t > 0. \tag{2.65}$$

2.6. Local existence theorem

Later, we will need a local existence result that provides the optimal regularity that can be expected from the choice of initial data. We define the following functional spaces, for any integer $k \geqq 0$ and $0 < \alpha < 1$,

$$X_{k,\alpha}(\delta) = \left\{\rho \in L^\infty\left(\mathbb{R}^N \times \mathbb{R}_+\right) : \|\rho\|_{X_{k,\alpha}(\delta)} < \infty\right\},$$
$$Y_{k,\alpha}(\delta) = \left\{\phi \in L^\infty\left(\mathbb{R}^N \times \mathbb{R}_+\right) : \|\phi\|_{X_{k,\alpha}(\delta)} < \infty\right\},$$

where

$$\|\rho\|_{X_{k,\alpha}(\delta)} = \sup_{0\leqq t\leqq\delta}\left\{\int|\rho(x,t)|\, d^N x + (t+1)^N \sum_{\ell=0}^{k}(t+1)^\ell \left\|\nabla^\ell \rho(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)}\right.$$
$$\left. + (t+1)^{N+k+\alpha}\sup_{x,y\in\mathbb{R}^N}\frac{\left|\nabla^k\rho(x,t) - \nabla^k\rho(y,t)\right|}{|x-y|^\alpha}\right\}. \tag{2.66}$$

$$\|\phi\|_{Y_{k,\alpha}(\delta)} = \sup_{0\leqq t\leqq\delta}\left\{(t+1)^{N-2}\sum_{\ell=1}^{k+2}(t+1)^\ell\left\|\nabla^\ell\phi(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)}\right.$$
$$\left. + (t+1)^{N+k+\alpha}\sup_{x,x'\in\mathbb{R}^N}\frac{\left|\nabla^{k+2}\phi(x,t) - \nabla^{k+2}\phi\left(x',t\right)\right|}{|x-x'|^\alpha}\right\}. \tag{2.67}$$

Theorem 2. *Let $k \geqq 1$ be an integer. Suppose that $f_0(x, v) \in C^{k,\alpha} \cap L^1\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$ satisfies the following assumptions:*

$$\sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\left|\frac{\partial^\ell f_0}{\partial x^m \partial v^{\ell-m}}\right| \leqq \frac{C}{(1+|x-vt^*|)^K(1+|v|)^K}, \tag{2.68}$$

$$\sum_{m=0}^{k}\sup_{x,x'\in\mathbb{R}^N}\frac{\left|\frac{\partial^k f_0}{\partial x^m\partial v^{k-m}}(x,v) - \frac{\partial^k f_0}{\partial x^m\partial v^{k-m}}\left(x',v\right)\right|}{\left|x-x'\right|^\alpha} \leqq \frac{C}{(1+|v|)^K},\ 0<\alpha<1,$$

$$\sum_{m=0}^{k}\sup_{|v'-v|\leqq 1}\frac{\left|\frac{\partial^k f_0}{\partial x^m\partial v^{k-m}}(x,v) - \frac{\partial^k f_0}{\partial x^m\partial v^{k-m}}\left(x,v'\right)\right|}{\left|v-v'\right|^\alpha} \leqq \frac{C}{(1+|x-vt^*|)^K(1+|v|)^K},$$
$$0<\alpha<1,$$

for some suitable $K > N$ *and* $C > 0$ *and* $t^* \in \mathbb{R}$. *Then, there exists a constant* $\delta = \delta(C, K, \|f_0\|_{C^{k,\alpha}\cap L^1(\mathbb{R}^3\times\mathbb{R}^3)}, t^*)$ *and a unique solution* $f(\cdot,\cdot,t)$ *in* $C^{k,\alpha} \cap L^1\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$ *of the Vlasov–Poisson system with*

$$\|\rho\|_{X_{k,\alpha}(\delta)} < \infty$$

defined for $0 \leqq t \leqq \delta$.

Remark 2. Notice that there is a difference between the formulas (2.68) and (2.3), that is, the presence of the factor vt^*. This factor is needed in order to account for the distortion that the function f experiences in the phase plane (x, v) during its evolution.

Proof. Let us restrict our analysis to the case $k = 1$, since in the other cases the analysis is similar. Suppose that $X(s,t;x,v)$, $V(s,t;x,v)$ are as in (2.72). We use a fixed point in the density $\rho(x,t)$. Given $\rho(x,t)$ in $X_{1,\alpha}(\delta)$ we define ϕ solving Poisson's equation. We then solve the characteristic equations (2.72) and we define

$$f(x,v,t) = f_0(X(0,t;x,v), V(0,t;x,v)). \tag{2.69}$$

We can then compute a new $\rho = \tilde{\rho}(x,t)$ by

$$\tilde{\rho}(x,t) = \int f(x,v,t)\,\mathrm{d}v = \int f_0(X(0,t;x,v), V(0,t;x,v))\,\mathrm{d}v.$$

Our goal is to check that the transformation $(\rho \to \tilde{\rho})$ is contractive in the Banach space $X_{1,\alpha}(\delta)$ defined in (2.66). To this end we need to prove that given $\rho \in X_{1,\alpha}(\delta)$ we have $\tilde{\rho} \in X_{1,\alpha}(\delta)$ and also that given $\rho_1, \rho_2 \in X_{1,\alpha}$ we have $\|\tilde{\rho}_1 - \tilde{\rho}_2\|_{1,\alpha} \leqq \theta\,\|\rho_1 - \rho_2\|_{1,\alpha}$. We will prove the second statement in detail, since the first one can be done using similar estimates, but directly estimating functions instead of their differences.

Suppose that we give ρ_1, ρ_2 in $X_{1,\alpha}(\delta)$. Then it can be concluded that $\|\phi_1 - \phi_2\|_{Y_{1,\alpha}(\delta)} \leqq C\,\|\rho_1 - \rho_2\|_{X_{1,\alpha}(\delta)}$. On the other hand, differentiating the characteristic equations we obtain

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}s}\left(\left[\frac{\partial X_1}{\partial x} - \frac{\partial X_2}{\partial x}\right]_{0,\alpha} + \left[\frac{\partial V_1}{\partial x} - \frac{\partial V_2}{\partial x}\right]_{0,\alpha}\right)\right| \\
&\quad \leqq C\,\|\phi_1 - \phi_2\|_{2,\alpha} + C\left[\|\phi_1\|_{3,\alpha} + \|\phi_2\|_{3,\alpha}\right]\left[\frac{\partial X_1}{\partial x} - \frac{\partial X_2}{\partial x}\right]_{0,\alpha}
\end{aligned}$$

and similar estimates for the v derivatives. The key point is that we need to bound three derivatives of ϕ but the "source term" $\|\phi_1 - \phi_2\|_{C^{2,\alpha}}$ contains even fewer derivatives). In any case, Gronwall's Lemma implies that, as long as $\|\phi_1\|_{C^{3,\alpha}} + \|\phi_2\|_{C^{3,\alpha}}$ are bounded, we have estimates of the form

$$\begin{aligned}
&\left\|\frac{\partial x_{0,1}}{\partial x} - \frac{\partial x_{0,2}}{\partial x}\right\|_{L^\infty} + \left\|\frac{\partial v_{0,1}}{\partial x} - \frac{\partial v_{0,2}}{\partial x}\right\|_{L^\infty} \\
&\quad + \left[\frac{\partial x_{0,1}}{\partial x} - \frac{\partial x_{0,2}}{\partial x}\right]_{0,\alpha} + \left[\frac{\partial v_{0,1}}{\partial x} - \frac{\partial v_{0,2}}{\partial x}\right]_{0,\alpha} \\
&\quad \leqq \theta\,\|\phi_1 - \phi_2\|_{2,\alpha} \leqq \theta\,\|\rho_1 - \rho_2\|_{0,\alpha}\,,
\end{aligned} \tag{2.70}$$

where $x_{0,i} = X_i(0,t;x,v)$, $v_{0,i} = V_i(0,t;x,v)$, $i = 1, 2$ and where θ is small if δ is small.

We can then use the formula $\tilde{\rho}(x,t) = \int f_0(X(0,t;x,v), V(0,t;x,v))\,\mathrm{d}v$. Taking the difference $\tilde{\rho}_1 - \tilde{\rho}_2$ and using the $1+\alpha$ regularity of f_0, as well as (2.70), we obtain estimates for $\|\tilde{\rho}_1 - \tilde{\rho}_2\|_{1,\alpha}$:

$$\|\tilde{\rho}_1 - \tilde{\rho}_2\|_{1,\alpha} \leqq \theta\,\|\rho_1 - \rho_2\|_{1,\alpha}\,. \tag{2.71}$$

A standard fixed point argument then yields the result. □

2.7. The proof of Theorem 1

We first prove the following auxiliary result.

Proposition 4. *Suppose that $\|\phi\|_{Y_{0,\alpha}} \leqq \varepsilon_0$ for some $\varepsilon_0 > 0$ sufficiently small. Let $\bar{X}$, $\bar{V}$ be as in (2.26). The mapping $(x_0, v_0) \to \left(\bar{X}(t;\bar{x}_0,\bar{v}_0), \bar{V}(t;\bar{x}_0,\bar{v}_0)\right)$ is a diffeomorphism in $\mathbb{R}^N \times \mathbb{R}^N$. Moreover, suppose that we define $X(s,t;x,v)$, $V(s,t;x,v)$ as the solutions of the equations:*

$$\frac{\mathrm{d}X(s)}{\mathrm{d}s} = V(s), \quad \frac{\mathrm{d}V(s)}{\mathrm{d}s} = \nabla\phi(X(s),x), \quad X(t,t;x,v) = x,$$
$$V(t,t;x,v) = v, \tag{2.72}$$

where we drop some of the dependencies on t, x, v for simplicity. Then, for $t \geqq 1$,

$$X(0,t;x,w(t,x,x_0)) = x_0, \quad V(0,t;x,w(t,x,x_0)) = \frac{x-x_0}{t} + \varphi(0;t,x,x_0). \tag{2.73}$$

Proof. Integrating (2.26) we obtain

$$\bar{X}(t;x_0,v_0) = x_0 + v_0 t + \int_0^t \int_0^s \nabla\phi\left(\bar{X}(\xi;x_0,v_0),\xi\right)\mathrm{d}\xi\,\mathrm{d}s, \tag{2.74}$$

$$\bar{V}(s;x_0,v_0) = v_0 + \int_0^t \nabla\phi\left(\bar{X}(s;x_0,v_0),s\right)\mathrm{d}s. \tag{2.75}$$

The fact that the mapping $(x_0, v_0) \to \left(\bar{X}(t;x_0,v_0), \bar{V}(t;x_0,v_0)\right)$ is a diffeomorphism in $\mathbb{R}^N \times \mathbb{R}^N$ can be proved, arguing as in the Proof of Proposition 3, using the Implicit Function Theorem. To this end we just need to show that $\frac{\partial\left(\bar{X}(t;x_0,v_0),\bar{V}(s;x_0,v_0)\right)}{\partial(x_0,v_0)}$ is invertible for any (x_0, v_0), and this can be made using the assumption $\|\phi\|_{Y_{0,\alpha}} \leqq \varepsilon_0$ as well as (2.2).

Suppose now that $t \geqq 1$. We now prove formulas (2.73) for arbitrary values of (x, x_0). Suppose that we give any $\bar{x} \in \mathbb{R}$, $\bar{x}_0 \in \mathbb{R}$. We compute $w(t,\bar{x},\bar{x}_0)$ by means of (2.9). Since the mapping $(x_0, v_0) \to \left(\bar{X}(t;x_0,v_0), \bar{V}(t;x_0,v_0)\right)$ is a diffeomorphism in $\mathbb{R}^N \times \mathbb{R}^N$, we can find a unique $(x_0, v_0) \in \mathbb{R}^N \times \mathbb{R}^N$ such that

$$\left(\bar{X}(t;x_0,v_0), \bar{V}(t;x_0,v_0)\right) = (\bar{x}, w(t,\bar{x},\bar{x}_0)). \tag{2.76}$$

We then apply (2.27) with $\bar{v}_0 = \frac{\bar{x}-x_0}{t}$. Then:

$$\bar{X}(s;\bar{x}_0,v_0) = x_0 + \frac{\bar{x}-x_0}{t}s + \int_0^s \varphi(\bar{s};\bar{x},x_0,t)\,\mathrm{d}\bar{s},$$
$$\bar{V}(s;\bar{x}_0,v_0) = \bar{v}_0 + \varphi(s;\bar{x},x_0,t)$$

whence, using also the initial values for the problem (2.74), (2.75) and (2.9)

$$\bar{X}(t;\bar{x}_0,v_0) = \bar{x}, \quad \bar{V}(t;x_0,v_0) = \bar{v}_0 + \varphi(t;\bar{x},x_0,t),$$
$$\bar{X}(0;x_0,v_0) = x_0, \quad \bar{V}(0;x_0,v_0) = \bar{v}_0 + \varphi(0;\bar{x},x_0,t) = v_0.$$

Using (2.76) it then follows that

$$\begin{aligned}\frac{\bar{x}-x_0}{t} + \varphi(t;\bar{x},x_0,t) &= \bar{v}_0 + \varphi(t;\bar{x},x_0,t)\\ &= w(t,\bar{x},\bar{x}_0) = \frac{\bar{x}-\bar{x}_0}{t} + \varphi(t;\bar{x},\bar{x}_0,t)\end{aligned}$$

whence

$$x_0 - t\varphi(t;\bar{x},x_0,t) = \bar{x}_0 - t\varphi(t;\bar{x},\bar{x}_0,t).$$

Using (2.32) in Lemma 4 it follows that $x_0 = \bar{x}_0$ if ε_0 is sufficiently small. Therefore

$$\bar{X}(t;\bar{x}_0,v_0) = \bar{x}, \quad \bar{V}(t;\bar{x}_0,v_0) = \bar{v}_0 + \varphi(t;\bar{x},\bar{x}_0,t) = w(t,\bar{x},\bar{x}_0) \tag{2.77}$$
$$\bar{X}(0;\bar{x}_0,v_0) = \bar{x}_0, \quad \bar{V}(0;\bar{x}_0,v_0) = \bar{v}_0 + \varphi(0;\bar{x},\bar{x}_0,t) = v_0. \tag{2.78}$$

Using classical uniqueness results for the initial value problem (2.72), we obtain

$$\begin{aligned}X\left(0,t;\bar{X}(t;\bar{x}_0,v_0),\bar{V}(t;\bar{x}_0,v_0)\right) &= \bar{X}(0;\bar{x}_0,v_0),\\ V\left(0,t;\bar{X}(t;\bar{x}_0,v_0),\bar{V}(t;\bar{x}_0,v_0)\right) &= \bar{V}(0;\bar{x}_0,v_0).\end{aligned} \tag{2.79}$$

Plugging (2.77), (2.78) into (2.79), and using the fact that $\bar{x}_0, \bar{x}$ are dummy variables we obtain (2.73). $\square$

Proof of Theorem 1. We now prove our main Theorem 1. We use a continuation argument as well as the assumptions on f_0 to derive estimates for ρ and to close the argument. More precisely, we will show that given a solution of the Vlasov–Poisson system (1.2)–(1.3) defined in the interval $0 \leqq t \leqq t^*$ and satisfying an estimate of the form

$$\begin{aligned}&\int |\rho(x,t)|\,\mathrm{d}^N x + (t+1)^N \sum_{\ell=0}^{k} (t+1)^\ell \left\|\nabla^\ell \rho(\cdot,t)\right\|_{L^\infty(\mathbb{R}^N)}\\ &\quad + (t+1)^{N+k+\alpha} \sup_{x,y\in\mathbb{R}^N} \frac{\left|\nabla^k \rho(x,t) - \nabla^k \rho(y,t)\right|}{|x-y|^\alpha} \leqq M\delta_0,\end{aligned} \tag{2.80}$$

it is possible to extend it to an interval $0 \leqq t \leqq t^* + \delta(t^*)$, where $\delta(t^*) > 0$. Moreover, such a solution satisfies an estimate of the form

$$\int |\rho(x,t)| \, d^N x + (t+1)^N \sum_{\ell=0}^{k} (t+1)^{\ell} \left\| \nabla^{\ell} \rho(\cdot, t) \right\|_{L^\infty(\mathbb{R}^N)} + (t+1)^{N+k+\alpha} \sup_{x,y \in \mathbb{R}^N} \frac{\left| \nabla^k \rho(x,t) - \nabla^k \rho(y,t) \right|}{|x-y|^{\alpha}} \leqq C\delta_0, \tag{2.81}$$

where C is independent of M if δ_0 is small enough. This result is a consequence of the local existence Theorem 2.

Notice that the main consequence of this result is a global existence theorem for δ_0 small generalizing that of [3] for $N = 3$. Indeed, let t^* be the supremum of the times $t \geqq 0$ for which there exists a solution of (1.2)–(1.3) satisfying (2.80). Suppose that $t^* < \infty$. Then, since f is given by the characteristics formula (2.69) we can derive estimates for $f(x, v, t^*)$ and its derivatives differentiating that formula. Then, using also the estimates (2.3) we obtain

$$\sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \left| \frac{\partial^{\ell} f}{\partial x^m \partial v^{\ell-m}} (x, v, t^*) \right| \leqq \frac{C}{(1+|X(0,t^*;x,v)|)^K (1+|V(0,t^*;x,v)|)^K} \tag{2.82}$$

$$\sum_{m=0}^{k} \sup_{x,x' \in \mathbb{R}^N} \frac{\left| \frac{\partial^k f_0}{\partial x^m \partial v^{k-m}} (x,v) - \frac{\partial^k f_0}{\partial x^m \partial v^{k-m}} (x',v) \right|}{|x-x'|^{\alpha}} \leqq \frac{C}{(1+|V(0,t^*;x,v)|)^K}, \quad 0 < \alpha < 1,$$

$$\sum_{m=0}^{k} \sup_{|v'-v| \leqq 1} \frac{\left| \frac{\partial^k f_0}{\partial x^m \partial v^{k-m}} (x,v) - \frac{\partial^k f_0}{\partial x^m \partial v^{k-m}} (x,v') \right|}{|v-v'|^{\alpha}} \leqq \frac{C}{(1+|X(0,t^*;x,v)|)^K (1+|V(0,t^*;x,v)|)^K}, \quad 0 < \alpha < 1,$$

where C depends on t^*. Notice that in the derivation of this formula we have used estimates for the derivatives of $X(0, t; x, v)$, $V(0, t; x, v)$ that can be derived from (2.72) using classical regularity theory for ODEs with respect to the initial data as well as the fact that (2.80) combined with regularity theory for elliptic equations yields $\left\| \nabla^{k+2} \phi \right\|_{L^\infty} + \left[\nabla^{k+2} \phi \right]_{0,\alpha} \leqq C$.

Moreover, using (2.80) and arguing as in the proof of Lemma 8 we obtain $|\nabla \phi| \leqq \frac{CM\delta_0}{(t+1)^N}$. Using (2.72) we obtain

$$\left| V\left(0, t^*; x, v\right) - v \right| \leqq C\delta_0$$

$$\left| X\left(0, t^*; x, v\right) - \left(x - vt^*\right) \right| \leqq \int_0^{t^*} s \left\| \nabla \phi(s) \right\|_{L^\infty} ds \leqq CM\delta_0$$

Therefore

$$\left| V\left(0, t^*; x, v\right) \right| + 1 \geqq |v| + \frac{1}{2}$$

$$\left| X\left(0, t^*; x, v\right) \right| + 1 \geqq \left| x - t^* v \right| + \frac{1}{2}$$

It follows from (2.82) that $f(\cdot, t^*)$ satisfy the assumptions (2.68). It then follows from Theorem 2 that the solution f can be extended to a time interval $0 \leqq t \leqq t^* + \delta$. This gives a contradiction with the definition of t^*. Therefore $t^* = \infty$, whence the global existence result follows.

We now proceed to prove (2.81). Suppose first that $0 \leqq t^* \leqq 1$. Let (x_0, v_0) denote the starting point for the solution of the characteristic equations reaching the point (x, v) at time t. More precisely,

$$\frac{\mathrm{d}\bar{X}(s)}{\mathrm{d}s} = \bar{V}(s), \quad \frac{\mathrm{d}\bar{V}(s)}{\mathrm{d}s} = \nabla\phi\left(\bar{X}(s), s\right), \quad \bar{X}(t) = x, \quad \bar{V}(t) = v. \tag{2.83}$$

Notice that $\bar{X}(s) = \bar{X}(s; x, v, t)$, $\bar{V}(s) = \bar{V}(s; x, v, t)$ and

$$x_0 = x_0(x, v, t) = \bar{X}(0; x, v, t), \quad v_0 = v_0(x, v, t) = \bar{V}(0; x, v, t).$$

Then

$$|v_0| = |v_0(x, v, t)| \geqq |v| - \int_0^t |\nabla\phi(s)|\,\mathrm{d}s \geqq |v| - C\|\phi\|_{Y_{k,\alpha}} \geqq |v| - \frac{1}{2}, \tag{2.84}$$

if ε_0 is small enough. Since the norms $\|\cdot\|_{Y_{k,\alpha}}$ are defined only for functions globally defined in time, and at this point we know only that the functions ϕ, f, ρ are defined for $0 \leqq t \leqq t^*$, we will understand that all these functions are extended by zero for $t > t^*$.

Taking the derivative $\frac{\partial^\ell}{\partial x^\ell}$ of the formula

$$\rho(x, t) = \int f_0(x_0, v_0)\,\mathrm{d}v$$

yields, for $0 \leqq \ell \leqq k$,

$$\frac{\partial^\ell \rho}{\partial x^\ell}(x, t) = \sum_{\substack{j_1 + \cdots + j_i = \ell, \\ 0 \leqq i \leqq \ell,\ 0 \leqq m \leqq i}} C_{ijpm} \int \frac{\partial^i f_0}{\partial x^{i-m}\partial v^m}(x_0, v_0)\frac{\partial^{j_1} v_0}{\partial x^{j_1}} \cdots \frac{\partial^{j_i} x_0}{\partial x^{j_i}}\mathrm{d}v.$$

Notice that the derivatives $\frac{\partial^{j_1} v_0}{\partial x^{j_1}}, \ldots, \frac{\partial^{j_i} x_0}{\partial x^{j_i}}$ are bounded for $0 \leqq t \leqq 1$ by the expression $C(1 + \|\phi\|_{Y_{\ell,\alpha}})$, as can be seen by differentiating the characteristic equations (2.83) with respect to x and v. It is then straightforward to see that for $0 \leqq \ell \leqq k$, using (2.3) and (2.84) yields

$$\begin{aligned}
\left\|\frac{\partial^\ell \rho}{\partial x^\ell}(t)\right\|_{L^\infty_{(x)}} &\leqq C\left(1 + \|\phi\|_{Y_{\ell,\alpha}}\right) \sum_{0 \leqq i \leqq \ell,\ 0 \leqq m \leqq i} C_{ijpm} \int \left|\frac{\partial^i f_0}{\partial x^{i-m}\partial v^m}(x_0, v_0)\right| \mathrm{d}v \\
&\leqq C\delta_0\left(1 + \|\phi\|_{Y_{\ell,\alpha}}\right) \int \frac{\mathrm{d}v}{(1 + |v|)^K} \\
&\leqq C_\ell \delta_0\left(1 + \|\phi\|_{Y_{\ell,\alpha}}\right), \quad 0 \leqq t \leqq 1,
\end{aligned} \tag{2.85}$$

$$\begin{aligned}
\left[\frac{\partial^\ell \rho}{\partial x^\ell}(t)\right]_{0,\alpha,(x)} &\leqq C\delta_0\left(1 + \|\phi\|_{Y_{\ell,\alpha}}\right) \int \frac{\mathrm{d}v}{(1 + |v|)^K} \\
&\leqq C_\ell \delta_0\left(1 + \|\phi\|_{Y_{\ell,\alpha}}\right), \quad 0 \leqq t \leqq 1.
\end{aligned} \tag{2.86}$$

Next we treat the case $t \geqq 1$. By taking $\frac{\partial^\ell}{\partial x^\ell}$ of $\rho(x,t)$ in (2.10), we get, for $0 \leqq \ell \leqq k$,

$$\frac{\partial^\ell \rho}{\partial x^\ell}(x,t) = \sum_{\substack{j_1+\cdots+j_i+p=\ell \\ 0\leqq i\leqq \ell}} C_{ijp} \int \frac{\partial^i f_0}{\partial v^i}(x_0, V(0)) \left(\frac{\partial}{\partial x}\right)^{j_1} V(0) \cdots \left(\frac{\partial}{\partial x}\right)^{j_i} V(0) \frac{\partial^p}{\partial x^p}\left(\left|\det \frac{\partial w}{\partial x_0}\right|\right) dx_0,$$

where (see (2.73))

$$V(0) = V(0,t,x,w(t,x,x_0)) = \frac{x-x_0}{t} + \varphi(0;t,x,x_0).$$

By Proposition 1, Proposition 2 and the assumption (2.3), it is easy to see that, assuming that $\|\phi\|_{Y_{k,\alpha}} \leqq \varepsilon_0$, for $0 \leqq \ell \leqq k$,

$$\begin{aligned} \left|\frac{\partial^\ell \rho}{\partial x^\ell}(x,t)\right| &\leqq C \sum_{\substack{j_1+\cdots+j_i+p=\ell \\ 0\leqq i\leqq \ell}} \frac{\left(1+C\|\phi\|_{Y_{\ell,\alpha}}\right)}{t^{j_1+\cdots+j_i+p+N}} \int \left|\frac{\partial^i f_0}{\partial v^i}(x_0, V(0))\right| dx_0 \\ &\leqq \frac{C\delta_0\left(1+\|\phi\|_{Y_{\ell,\alpha}}\right)}{t^{\ell+N}} \int \frac{dx_0}{(1+|x_0|)^K} \leqq \frac{C\delta_0\left(1+\|\phi\|_{Y_{\ell,\alpha}}\right)}{t^{\ell+N}}, \quad t>1. \end{aligned} \tag{2.87}$$

Using Lemma 3 with $\ell = k$ yields

$$\begin{aligned} \left[\frac{\partial^k \rho}{\partial x^k}(t)\right]_{0,\alpha,(x)} &\leqq C\frac{\left(1+C\|\phi\|_{Y_{k,\alpha}}\right)}{t^{j_1+\cdots+j_i+p+N+\alpha}} \int \left\{\left|\frac{\partial^i f_0}{\partial v^i}(x_0, V(0))\right| + \left[\frac{\partial^i f_0}{\partial v^i}(x_0,\cdot)\right]_{0,\alpha,(v)}\right\} dx_0 \\ &\leqq \frac{C\delta_0\left(1+\|\phi\|_{Y_{k,\alpha}}\right)}{t^{k+N+\alpha}} \int \frac{dx_0}{(1+|x_0|)^K} \\ &\leqq \frac{C\delta_0\left(1+\|\phi\|_{Y_{k,\alpha}}\right)}{t^{k+N+\alpha}}, \quad t>1 \end{aligned} \tag{2.88}$$

if $\|\phi\|_{Y_{k,\alpha}} \leqq \varepsilon_0$.

On the other hand, combining (2.65) with the decay assumptions for f_0 in (2.3), it follows that

$$\int |\rho(x,t)|\, dx \leqq C\delta_0. \tag{2.89}$$

Notice that due to Lemma 8 we have $\|\phi\|_{Y_{k,\alpha}} \leqq C\delta_0$. Therefore $\|\phi\|_{Y_{k,\alpha}} \leqq \varepsilon_0$ for δ_0 sufficiently small. It then follows from (2.85)–(2.89), as well as from Lemma 8 that

$$\|\rho\|_{X_{k,\alpha}} \leqq C\delta_0 + C\delta_0 \|\phi\|_{Y_{k,\alpha}} \leqq C\delta_0 + C\delta_0 \|\rho\|_{X_{k,\alpha}} .$$

Choosing δ_0 small enough, the existence part of the Theorem 1 follows. The uniqueness result can be obtained by adapting the ideas in [3]. More precisely, the same argument that yields the estimate (2.71) allows us to obtain $\|\rho_1 - \rho_2\|_{1,\alpha} \leqq \theta \|\rho_1 - \rho_2\|_{1,\alpha}$ with $\theta < 1$, whence the desired uniqueness follows. □

3. Convergence to the self-similar behaviour

We define the following set of self-similar variables

$$f(x, v, t) = \frac{1}{(t+1)^N} g(y, v, \tau), \tag{3.1}$$

$$\phi(x, t) = \frac{1}{(t+1)^{N-2}} \Phi(y, \tau), \tag{3.2}$$

where

$$y = \frac{x}{(t+1)}, \quad \tau = \log(t+1). \tag{3.3}$$

A straightforward computation yields the following transformed system

$$g_\tau + (v - y) \cdot \nabla_y g + \mathrm{e}^{-(N-2)\tau} \nabla_y \Phi \cdot \nabla_v g = Ng, \tag{3.4}$$

$$\Delta_y \Phi = \int g(y, v, \tau)\, \mathrm{d}v \equiv \bar{\rho}(y, \tau), \tag{3.5}$$

where $g(x, v, 0) = g_0(x, v) = f_0(x, v) = f(x, v, 0)$.

We define the following function spaces in the new variables.

$$\|\bar{\rho}\|_{\bar{X}_{k,\alpha}} = \sup_{t \geqq 0} \left\{ \int_{\mathbb{R}^N} |\bar{\rho}(y, t)|\, dy + \sum_{\ell=0}^{k} \left\| \nabla^\ell \bar{\rho}(\cdot, t) \right\|_{L^\infty(\mathbb{R}^N)} + \sup_{y, y' \in \mathbb{R}^N} \frac{\left| \nabla^k \bar{\rho}(y, t) - \nabla^k \bar{\rho}\left(y', t\right) \right|}{|y - y'|^\alpha} \right\},$$

$$\|\Phi\|_{\bar{Y}_{k,\alpha}} = \sup_{t \geqq 0} \left\{ \sum_{\ell=1}^{k+2} \left\| \nabla^\ell \Phi(\cdot, t) \right\|_{L^\infty(\mathbb{R}^N)} + \sup_{y, y' \in \mathbb{R}^N} \frac{\left| \nabla^{k+2} \Phi(y, t) - \nabla^{k+2} \Phi\left(y', t\right) \right|}{|y - y'|^\alpha} \right\}.$$

where $0 < \alpha < 1$.

Notice that Lemma 8 is also valid in self-similar variables.

Lemma 10. (Elliptic regularity theory)

$$\|\Phi\|_{\bar{Y}_{k,\alpha}} \leqq C\,\|\bar{\rho}\|_{\bar{X}_{k,\alpha}}, \quad 0<\alpha<1.$$

We reformulate Theorem 1 in self-similar variables.

Theorem 3. *Let $k \geqq 1$ be an integer. Suppose that $g_0(y,v) \in C^{k,\alpha}\left(\mathbb{R}^3 \times \mathbb{R}^3\right) \cap L^1\left(\mathbb{R}^3\right)$ satisfies the following estimates.*

$$\sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\left|\frac{\partial^\ell g_0}{\partial x^m \partial v^{\ell-m}}\right| \leqq \frac{\delta_0}{(1+|x|)^K(1+|v|)^K},$$

$$\sum_{m=0}^{k}\sup_{x,x'\in\mathbb{R}^N}\frac{\left|\frac{\partial^k g_0}{\partial x^m\partial v^{k-m}}(x,v)-\frac{\partial^k g_0}{\partial x^m\partial v^{k-m}}(x',v)\right|}{|x-x'|^\alpha} \leqq \frac{\delta_0}{(1+|v|)^K}, \quad 0<\alpha<1,$$

$$\sum_{m=0}^{k}\sup_{|v'-v|\leqq 1}\frac{\left|\frac{\partial^k g_0}{\partial x^m\partial v^{k-m}}(x,v)-\frac{\partial^k g_0}{\partial x^m\partial v^{k-m}}(x,v')\right|}{|v-v'|^\alpha} \leqq \frac{\delta_0}{(1+|x|)^K(1+|v|)^K}, \quad 0<\alpha<1,$$

where $K>N$ and δ_0 is small enough. Then there exists a corresponding solution $g(\cdot,\cdot,t)$ in $C^{k,\alpha}\cap L^1\left(\mathbb{R}^3\times\mathbb{R}^3\right)$ of the rescaled Vlasov–Poisson system with

$$\|\bar{\rho}\|_{\bar{X}_{k,\alpha}} \leqq C\delta_0.$$

The main theorem that we prove in this Section is the following.

Theorem 4. *Suppose that the assumptions of Theorem 3 are satisfied. Then, there exist $g_\infty(y,y_0)\in C^k_{loc}$, $\bar{\rho}_\infty(y)\in C^k_{loc}\cap L^1\left(\mathbb{R}^N\right)$, $\Phi_\infty(y)\in C^{k+1,\beta}_{loc}$ satisfying*

$$\begin{aligned} e^{-N\tau} g(y,y_0,\tau) &\to g_\infty(y,y_0), && \text{in } C^k_{loc},\\ \bar{\rho}(y,\tau) &\to \bar{\rho}_\infty(y), && \text{in } C^k_{loc},\\ \Phi(y,\tau) &\to \Phi_\infty(y), && \text{in } C^{k+1,\beta}_{loc}, \end{aligned} \tag{3.6}$$

for any $0<\beta<1$, as $\tau\to\infty$. Moreover, we have

$$\|\bar{\rho}_\infty\|_{L^1(\mathbb{R}^N)} = \|g_0\|_{L^1(\mathbb{R}^N\times\mathbb{R}^N)}$$

and we have the following representation formulae for g_∞

$$\begin{aligned} g_\infty(y,y_0) &= g_0(y_0, y+\omega_\infty(0,y,y_0)),\\ \bar{\rho}_\infty(y) &= \int g_\infty(y,y_0)\,J_\infty(y,y_0)\,\mathrm{d}y_0,\\ \Delta_y\Phi_\infty(y) &= \bar{\rho}_\infty(y), \end{aligned} \tag{3.7}$$

as well as the limit formula, as $\tau\to\infty$

$$g(y,v,\tau) \to \int g_\infty(y_0,y)\,\delta(v-y)\,J_\infty(y,y_0)\,\mathrm{d}y_0, \quad \text{in } \mathcal{D}'\left(\mathbb{R}^N\times\mathbb{R}^N\right), \tag{3.8}$$

where $\omega_\infty(s; y, y_0)$ *is the solution of the following integral equation.*

$$\begin{aligned}
&\omega_\infty(s; y, y_0) \\
&= -\int_s^\infty e^{-(N-2)\xi} \nabla_y \Phi\left(y + (y_0 - y)\, e^{-\xi} + \int_0^\xi e^{-(\xi-\eta)} \omega_\infty(\eta; y, y_0)\, d\eta, \xi\right) d\xi,
\end{aligned} \tag{3.9}$$

and where $J_\infty(y, y_0)$ *is given by*

$$J_\infty(y, y_0) = \lim_{\tau\to\infty} \left|\det\left(-I_N + e^\tau \frac{\partial\omega_\infty}{\partial y_0}(\tau; y, y_0)\right)\right|.$$

Remark 3. Notice that (3.6) can be read in the original set of variables as

$$\rho(x, t) \sim \frac{1}{t^N} \bar{\rho}_\infty\left(\frac{x}{t}\right) + o\left(\frac{1}{t^N}\right)$$

as $t \to \infty$, uniformly on sets $|x| \leqq Ct$.

Remark 4. The function $\omega_\infty(s; y, y_0)$ is small for small densities. In particular, the representation formula (3.7) implies that the rescaled density function $\bar{\rho}_\infty(y)$ approaches the one associated to the free streaming case, defined in (1.4) if $\varepsilon_0 \to 0$. Notice that this shows that the particular profile that describes the self-similar behaviour of the solutions depends very sensitively on the initial data g_0. This contrasts with the situation in the one-dimensional case, where the leading self-similar behaviour depends only on the mass of the initial distribution, not on any other information on the initial data g_0 (see [4]). However, notice that it is not possible to obtain a simple closed form expression for g_∞ in terms of g_0 due to the fact that the function $\omega_\infty(s; y, y_0)$ depends on the values of the function Φ for any $t \in (0, \infty)$.

In order to prove Theorem 4, we introduce some changes of variables analogous to the ones used in the previous Section.

Suppose that the characteristics starting at y_0, v_0 reach the points y, v at time τ and we regard $v_0 = w_0(\tau, y, y_0)$, $v = w(\tau, y, y_0)$ as functions of y, y_0, and τ and make the change of variables from v to y_0 to get

$$dv = \left|\det \frac{\partial w}{\partial y_0}\right| dy_0.$$

The existence of the functions $w_0(\tau, y, y_0)$, $w(y, y_0, \tau)$ can be proved arguing as in the Proof of Proposition 3. Moreover, these functions have $(k + 1 + \alpha)$ derivatives with respect to the variables (y, y_0).

$$\begin{aligned}
\bar{\rho}(y, \tau) &= \int_{\mathbb{R}^N} g(y, v, \tau)\, dv = \int e^{N\tau} g_0(y_0, v_0)\, dv \\
&= \int e^{N\tau} g_0(y_0, w_0(\tau, y, y_0)) \left|\det \frac{\partial v}{\partial y_0}\right| dy_0.
\end{aligned}$$

The corresponding boundary value problem in the self similar variables (y, v, τ) reads

$$\begin{aligned}&\frac{\mathrm{d}Y}{\mathrm{d}s} = V - Y,\ \frac{\mathrm{d}V}{\mathrm{d}s} = \mathrm{e}^{-(N-2)s}\nabla_y \Phi\left(Y(s), s\right),\ \frac{\mathrm{d}g}{\mathrm{d}s} = Ng\\ &Y(\tau) = y,\ Y(0) = y_0.\end{aligned}$$

In the absence of the field, we solve

$$\begin{aligned}\frac{\mathrm{d}\tilde{Y}}{\mathrm{d}s} &= \tilde{V} - \tilde{Y}, \quad \frac{\mathrm{d}\tilde{V}}{\mathrm{d}s} = 0,\\ \tilde{Y}(\tau) &= y, \quad \tilde{Y}(0) = y_0,\end{aligned}$$

which yields

$$\tilde{V}(s) = \frac{y - y_0\mathrm{e}^{-\tau}}{1 - \mathrm{e}^{-\tau}}, \quad \tilde{Y}(s) = \frac{y - y_0\mathrm{e}^{-\tau}}{1 - \mathrm{e}^{-\tau}} + \frac{y_0 - y}{1 - \mathrm{e}^{-\tau}}\mathrm{e}^{-s}.$$

As in the previous section, we formulate the above as a perturbed problem from the free streaming one.

$$\begin{aligned}&V \equiv \tilde{V} + \omega,\ Y \equiv \tilde{Y} + \zeta,\\ &\frac{\mathrm{d}\zeta}{\mathrm{d}s} = \omega - \zeta,\ \frac{\mathrm{d}\omega}{\mathrm{d}s} = \mathrm{e}^{-(N-2)s}\nabla_y \Phi\left(\tilde{Y} + \zeta, s\right),\\ &\zeta(\tau) = \zeta(0) = 0.\end{aligned}$$

It is straightforward to see that

$$\begin{aligned}\omega(s) &= -\int_s^\tau \mathrm{e}^{-(N-2)\xi}\nabla_y \Phi\left(Y(\xi), \xi\right)\mathrm{d}\xi\\ &\quad + \frac{\mathrm{e}^{-\tau}}{1 - \mathrm{e}^{-\tau}}\int_0^\tau \mathrm{e}^{-(N-3)\xi}\left(1 - \mathrm{e}^{-\xi}\right)\nabla_y \Phi\left(Y(\xi), \xi\right)\mathrm{d}\xi, \qquad (3.10)\\ \zeta(s) &= \int_0^s \mathrm{e}^{-(s-\xi)}\omega(\xi)\,\mathrm{d}\xi,\end{aligned}$$

where

$$Y(\xi) = \frac{y - y_0\mathrm{e}^{-\tau}}{1 - \mathrm{e}^{-\tau}} + \frac{y_0 - y}{1 - \mathrm{e}^{-\tau}}\mathrm{e}^{-\xi} + \int_0^\xi \mathrm{e}^{-(\xi-\eta)}\omega(\eta)\,\mathrm{d}\eta.$$

Along the characteristics, we have

$$\frac{\partial v}{\partial y_0} = -\left(\frac{1}{1 - \mathrm{e}^{-\tau}}\right)\mathrm{e}^{-\tau} I_N + \frac{\partial \omega}{\partial y_0}(\tau).$$

The following result provides some decay estimates for the derivatives of ω, analogous to the ones derived in Lemma 4.

Lemma 11. *There exists ε_0 small such that for any $\tau \geqq 1$ and any function Φ satisfying*

$$\|\Phi\|_{\bar{Y}_{0,\alpha}} \leqq \varepsilon_0,$$

we have

$$\int_0^\tau \left\| \frac{\partial \omega}{\partial y_0}(s) \right\|_{L^\infty_{(y)}} \mathrm{d}s \leqq C \|\Phi\|_{\bar{Y}_{0,\alpha}}, \quad \left\| \frac{\partial \omega}{\partial y_0}(\tau) \right\|_{L^\infty_{(y)}} \leqq C\mathrm{e}^{-\tau} \|\Phi\|_{\bar{Y}_{0,\alpha}}, \tag{3.11}$$

$$\int_0^\tau \left[\frac{\partial \omega}{\partial y_0}(s) \right]_{0,\alpha,(y)} \mathrm{d}s \leqq C \|\Phi\|_{\bar{Y}_{\ell,\alpha}}, \quad \left[\frac{\partial \omega}{\partial y_0}(\tau) \right]_{0,\alpha,(y)} \leqq C\mathrm{e}^{-\tau} \|\Phi\|_{\bar{Y}_{\ell,\alpha}}. \tag{3.12}$$

Proof. The method of proof is similar to the one used in the proof of Lemma 4. We take $\frac{\partial}{\partial y_0}$ of (3.10) to get

$$\begin{aligned} \frac{\partial \omega}{\partial y_0}(s) = &-\int_s^\tau \mathrm{e}^{-(N-2)\xi} \nabla_y^2 \Phi(\xi) \left\{ \frac{\mathrm{e}^{-\xi} - \mathrm{e}^{-\tau}}{1-\mathrm{e}^{-\tau}} I_N + \int_0^\xi \mathrm{e}^{-(\xi-\eta)} \frac{\partial \omega}{\partial y_0}(\eta)\, \mathrm{d}\eta \right\} \mathrm{d}\xi \\ &+ \frac{\mathrm{e}^{-\tau}}{1-\mathrm{e}^{-\tau}} \int_0^\tau \mathrm{e}^{-(N-3)\xi} \left(1-\mathrm{e}^{-\xi}\right) \nabla_y^2 \Phi(Y(\xi), \xi) \left\{ \frac{\mathrm{e}^{-\xi} - \mathrm{e}^{-\tau}}{1-\mathrm{e}^{-\tau}} I_N \right. \\ &\left. + \int_0^\xi \mathrm{e}^{-(\xi-\eta)} \frac{\partial \omega}{\partial y_0}(\eta)\, \mathrm{d}\eta \right\} \mathrm{d}\xi. \end{aligned} \tag{3.13}$$

Taking the $L^\infty_{(y)}$ norm yields

$$\begin{aligned} &\left\| \frac{\partial \omega}{\partial y_0}(s) \right\|_{L^\infty_{(y)}} \\ &\leqq C \|\Phi\|_{\bar{Y}_{0,\alpha}} \int_s^\tau \mathrm{e}^{-(N-2)\xi} \left\{ \mathrm{e}^{-\xi} + \mathrm{e}^{-\tau} + \int_0^\xi \mathrm{e}^{-(\xi-\eta)} \left\| \frac{\partial \omega}{\partial y_0}(\eta) \right\|_{L^\infty_{(y)}} \mathrm{d}\eta \right\} \mathrm{d}\xi \\ &\quad + C \|\Phi\|_{\bar{Y}_{0,\alpha}} \mathrm{e}^{-\tau} \int_0^\tau \mathrm{e}^{-(N-3)\xi} \left\{ \mathrm{e}^{-\xi} + \mathrm{e}^{-\tau} + \int_0^\xi \mathrm{e}^{-(\xi-\eta)} \left\| \frac{\partial \omega}{\partial y_0}(\eta) \right\|_{L^\infty_{(y)}} \mathrm{d}\eta \right\} \mathrm{d}\xi. \end{aligned}$$

Integrating the above inequality from $s = 0$ to $s = \tau$ and using $\mathrm{e}^{-(\xi-\eta)} \leqq 1$, $\mathrm{e}^{-\tau} \leqq \mathrm{e}^{-\xi}$ for $\eta \leqq \xi$, $\xi \leqq \tau$ and $N \geqq 3$ yield

$$\begin{aligned} &\int_0^\tau \left\| \frac{\partial \omega}{\partial y_0}(s) \right\|_{L^\infty_{(y)}} \mathrm{d}s \\ &\leqq C \|\Phi\|_{\bar{Y}_{0,\alpha}} \int_0^\tau \mathrm{e}^{-2s} \mathrm{d}s + C \|\Phi\|_{\bar{Y}_{0,\alpha}} \left(\int_0^\tau \mathrm{e}^{-s} \mathrm{d}s \right) \left(\int_0^\tau \left\| \frac{\partial \omega}{\partial y_0}(\eta) \right\|_{L^\infty_{(y)}} \mathrm{d}\eta \right) \\ &\quad + C \|\Phi\|_{\bar{Y}_{0,\alpha}} \mathrm{e}^{-\tau} \tau + C \|\Phi\|_{\bar{Y}_{0,\alpha}} \mathrm{e}^{-\tau} \tau^2 \left(\int_0^\tau \left\| \frac{\partial \omega}{\partial y_0}(\eta) \right\|_{L^\infty_{(y)}} \mathrm{d}\eta \right). \end{aligned}$$

Thus we have

$$\int_0^\tau \left\| \frac{\partial \omega}{\partial y_0}(s) \right\|_{L^\infty_{(y)}} \mathrm{d}s \leqq C \left\| \Phi \right\|_{\bar{Y}_{0,\alpha}}, \tag{3.14}$$

provided ε_0 is small enough. We now specialize to $s = \tau$ and use (3.14) as well as the fact that $N \geqq 3$ to get

$$\begin{aligned}
\left\| \frac{\partial \omega}{\partial y_0}(\tau) \right\|_{L^\infty_{(y)}} &\leqq C \left\| \Phi \right\|_{\bar{Y}_{0,\alpha}} \mathrm{e}^{-\tau} \int_0^\tau \left\{ \mathrm{e}^{-\xi} + \int_0^\xi \mathrm{e}^{-(\xi-\eta)} \left\| \frac{\partial \omega}{\partial y_0}(\eta) \right\|_{L^\infty_{(y)}} \mathrm{d}\eta \right\} \mathrm{d}\xi \\
&\leqq C \left\| \Phi \right\|_{\bar{Y}_{0,\alpha}} \mathrm{e}^{-\tau} + C \left\| \Phi \right\|_{\bar{Y}_{0,\alpha}} \mathrm{e}^{-\tau} \int_0^\tau \left\| \frac{\partial \omega}{\partial y_0}(\eta) \right\|_{L^\infty_{(y)}} \left(\int_\eta^\tau \mathrm{e}^{-(\xi-\eta)} \mathrm{d}\xi \right) \mathrm{d}\eta \\
&\leqq C \left\| \Phi \right\|_{\bar{Y}_{0,\alpha}} \mathrm{e}^{-\tau},
\end{aligned}$$

where we changed the order of integration. We thus obtain (3.11). Using (3.11), we deduce (3.12) in a similar way. Thus we complete the proof. □

We also obtain the following estimates for the derivative of ω with respect to y.

Lemma 12. *There exists ε_0 small such that for any $\tau \geqq 1$ and any function Φ satisfying*

$$\left\| \Phi \right\|_{\bar{Y}_{0,\alpha}} \leqq \varepsilon_0,$$

we have

$$\sup_{0 \leqq s \leqq \tau} \left\| \frac{\partial \omega}{\partial y}(s) \right\|_{L^\infty_{(y)}} \leqq C \left\| \Phi \right\|_{\bar{Y}_{0,\alpha}}, \quad \sup_{0 \leqq s \leqq \tau} \left[\frac{\partial \omega}{\partial y}(s) \right]_{0,\alpha,(y)} \leqq C \left\| \Phi \right\|_{\bar{Y}_{0,\alpha}}.$$

Proof. We take $\frac{\partial}{\partial y}$ of (3.10) to get

$$\begin{aligned}
&\frac{\partial \omega}{\partial y}(s) \\
&= -\int_s^\tau \mathrm{e}^{-(N-2)\xi} \nabla_y^2 \Phi(Y(\xi), \xi) \left\{ \frac{1-\mathrm{e}^{-\xi}}{1-\mathrm{e}^{-\tau}} I_N + \int_0^\xi \mathrm{e}^{-(\xi-\eta)} \frac{\partial \omega}{\partial y}(\eta)\, \mathrm{d}\eta \right\} \mathrm{d}\xi \\
&\quad + \frac{\mathrm{e}^{-\tau}}{1-\mathrm{e}^{-\tau}} \int_0^\tau \mathrm{e}^{-(N-3)\xi} \left(1-\mathrm{e}^{-\xi}\right) \nabla_y^2 \Phi(Y(\xi), \xi) \left\{ \frac{1-\mathrm{e}^{-\xi}}{1-\mathrm{e}^{-\tau}} I_N \right. \\
&\quad \left. + \int_0^\xi \mathrm{e}^{-(\xi-\eta)} \frac{\partial \omega}{\partial y}(\eta)\, \mathrm{d}\eta \right\} \mathrm{d}\xi.
\end{aligned}$$

Since $N \geqq 3$, we have

$$
\begin{aligned}
&\sup_{0\leqq s\leqq\tau}\left\|\frac{\partial\omega}{\partial y}(s)\right\|_{L^\infty_{(y)}}\\
&\leqq C\,\|\Phi\|_{\bar{Y}_{0,\alpha}}\int_s^\tau \mathrm{e}^{-(N-2)\xi}\left\{1+\left(\sup_{0\leqq s\leqq\tau}\left\|\frac{\partial\omega}{\partial y}(s)\right\|_{L^\infty_{(y)}}\right)\int_0^\xi \mathrm{e}^{-(\xi-\eta)}\mathrm{d}\eta\right\}\mathrm{d}\xi\\
&\quad +C\,\|\Phi\|_{\bar{Y}_{0,\alpha}}\,\mathrm{e}^{-\tau}\int_0^\tau\left\{1+\left(\sup_{0\leqq s\leqq\tau}\left\|\frac{\partial\omega}{\partial y}(s)\right\|_{L^\infty_{(y)}}\right)\int_0^\xi \mathrm{e}^{-(\xi-\eta)}\mathrm{d}\eta\right\}\mathrm{d}\xi\\
&\leqq C\,\|\Phi\|_{\bar{Y}_{0,\alpha}}+C\,\|\Phi\|_{\bar{Y}_{0,\alpha}}\left(\sup_{0\leqq s\leqq\tau}\left\|\frac{\partial\omega}{\partial y}(s)\right\|_{L^\infty_{(y)}}\right),
\end{aligned}
$$

Thus we have

$$
\sup_{0\leqq s\leqq\tau}\left\|\frac{\partial\omega}{\partial y}(s)\right\|_{L^\infty_{(y)}}\leqq C\,\|\Phi\|_{\bar{Y}_{0,\alpha}}\,, \tag{3.15}
$$

provided ε_0 is small enough. In a similar manner, using (3.15) we obtain

$$
\begin{aligned}
&\sup_{0\leqq s\leqq\tau}\left[\frac{\partial\omega}{\partial y}(s)\right]_{0,\alpha,(y)}\\
&\leqq C\,\|\Phi\|_{\bar{Y}_{0,\alpha}}\int_s^\tau \mathrm{e}^{-(N-2)\xi}\left\{1+\left(\sup_{0\leqq s\leqq\tau}\left[\frac{\partial\omega}{\partial y}(s)\right]_{0,\alpha,(y)}\right)\int_0^\xi \mathrm{e}^{-(\xi-\eta)}\mathrm{d}\eta\right\}\mathrm{d}\xi\\
&\quad +C\,\|\Phi\|_{\bar{Y}_{0,\alpha}}\,\mathrm{e}^{-\tau}\int_0^\tau\left\{1+\left(\sup_{0\leqq s\leqq\tau}\left[\frac{\partial\omega}{\partial y}(s)\right]_{0,\alpha,(y)}\right)\int_0^\xi \mathrm{e}^{-(\xi-\eta)}\mathrm{d}\eta\right\}\mathrm{d}\xi\\
&\leqq C\,\|\Phi\|_{\bar{Y}_{0,\alpha}}+C\,\|\Phi\|_{\bar{Y}_{0,\alpha}}\left(\sup_{0\leqq s\leqq\tau}\left[\frac{\partial\omega}{\partial y}(s)\right]_{0,\alpha,(y)}\right).
\end{aligned}
$$

This yields the Hölder estimate of $\frac{\partial\omega}{\partial y}$ and completes the proof. □

We present the following estimates for higher-order derivatives similar to Theorems in the previous section.

Lemma 13. *Let $\ell \geqq 1$ be an integer. There exists ε_0 small such that for any $\tau \geqq 1$ and any function Φ satisfying*

$$
\|\Phi\|_{\bar{Y}_{\ell,\alpha}} \leqq \varepsilon_0,
$$

we have the following

$$\sup_{0\leqq s\leqq\tau}\left\|\frac{\partial^\ell\omega}{\partial y^\ell}(s)\right\|_{L^\infty_{(y)}}\leqq C\|\Phi\|_{\bar{Y}_{\ell,\alpha}},\quad \sup_{0\leqq s\leqq\tau}\left[\frac{\partial^\ell\omega}{\partial y^\ell}(s)\right]_{0,\alpha,(y)}\leqq C\|\Phi\|_{\bar{Y}_{\ell,\alpha}},$$
$$\int_0^\tau\left\|\frac{\partial^{\ell+1}\omega}{\partial y^\ell\partial y_0}(s)\right\|_{L^\infty_{(y)}}\mathrm{d}s\leqq C\|\Phi\|_{\bar{Y}_{\ell,\alpha}},\quad \int_0^\tau\left[\frac{\partial^{\ell+1}\omega}{\partial y^\ell\partial y_0}(s)\right]_{0,\alpha,(y)}\mathrm{d}s\leqq C\|\Phi\|_{\bar{Y}_{\ell,\alpha}},$$
$$\left\|\frac{\partial^{\ell+1}\omega}{\partial y^\ell\partial y_0}(\tau)\right\|_{L^\infty_{(y)}}\leqq C\mathrm{e}^{-\tau}\|\Phi\|_{\bar{Y}_{\ell,\alpha}},\quad \left[\frac{\partial^{\ell+1}\omega}{\partial y^\ell\partial y_0}(\tau)\right]_{0,\alpha,(y)}\leqq C\mathrm{e}^{-\tau}\|\Phi\|_{\bar{Y}_{\ell,\alpha}}.$$

As a consequence of Theorem 3, we have, for any $k\geqq 0$ integer and $0<\alpha<1$,

$$\|\rho\|_{C^{k,\alpha}(\mathbb{R}^N)}\leqq C\delta_0,\quad \|\Phi\|_{C^{k+2,\alpha}(\mathbb{R}^N)}\leqq C\delta_0\leqq\varepsilon_0. \tag{3.16}$$

if δ_0 is chosen small enough.

We now study the limit behaviour of the self-similar system (3.1)–(3.3). Indeed, the limit behaviour is asymptotically equivalent to the free streaming case.

3.1. Proof of Theorem 4

Proof. We begin with, for $s\leqq\tau$,

$$\begin{aligned}\omega(s;y,y_0,\tau)=&-\int_s^\tau \mathrm{e}^{-(N-2)\xi}\nabla_y\Phi\left(Y(\xi;y,y_0,\tau),\xi\right)\mathrm{d}\xi\\ &+\frac{\mathrm{e}^{-\tau}}{1-\mathrm{e}^{-\tau}}\int_0^\tau \mathrm{e}^{-(N-3)\xi}\left(1-\mathrm{e}^{-\xi}\right)\nabla_y\Phi\left(Y(\xi;y,y_0,\tau),\xi\right)\mathrm{d}\xi.\end{aligned} \tag{3.17}$$

By using (3.16) and by the dominated convergence theorem, as $\tau\to\infty$, in C^{k+1},

$$\omega(s;y,y_0,\tau)\to\omega(s;y,y_0,\infty)\equiv\omega_\infty(s,y,y_0).$$

In particular, we have, as $\tau\to\infty$, in C^{k+1},

$$\begin{aligned}w_0(\tau,y,y_0)&=v_0(y,v(y,y_0,\tau))=\tilde{V}(0)+\omega(0;y,y_0,\tau)\\ &\to y+\omega_\infty(0,y,y_0).\end{aligned}$$

Thus, we have, as $\tau\to\infty$, in C^k_{loc},

$$\begin{aligned}\mathrm{e}^{-N\tau}g(y,v,\tau)&=g_0(y_0,w_0(\tau,y,y_0))\\ &\to g_0(y_0,y+\omega_\infty(0,y,y_0))\equiv g_\infty(y,y_0).\end{aligned}$$

Next, since

$$\mathrm{e}^\tau\frac{\partial w}{\partial y_0}=-\left(\frac{1}{1-\mathrm{e}^{-\tau}}\right)I_N+\mathrm{e}^\tau\frac{\partial\omega}{\partial y_0}(\tau),$$

using Lemmas 11–13 yields, as $\tau \to \infty$, in C^k,

$$e^{N\tau} \left| \det \frac{\partial w}{\partial y_0} \right| \to J_\infty (y, y_0) \simeq 1 + \mathcal{O}(\delta_0). \tag{3.18}$$

In order to prove (3.18) we just need to prove the existence of the limit $\lim_{\tau\to\infty} \left[e^\tau \frac{\partial \omega}{\partial y_0} (\tau) \right]$. To this end, we take the value $s = \tau$ in (3.13). Then

$$\begin{aligned} e^\tau \frac{\partial \omega}{\partial y_0} (s) = & \frac{1}{1 - e^{-\tau}} \int_0^\tau e^{-(N-3)\xi} \left(1 - e^{-\xi}\right) \nabla_y^2 \Phi (Y(\xi), \xi) \\ & \times \left\{ \frac{e^{-\xi} - e^{-\tau}}{1 - e^{-\tau}} I_N + \int_0^\xi e^{-(\xi-\eta)} \frac{\partial \omega}{\partial y_0} (\eta) \, d\eta \right\} d\xi. \end{aligned} \tag{3.19}$$

Using the fact that $e^{-(N-3)\xi} \leqq 1$ and $\left| \nabla_y^2 \Phi (Y(\xi), \xi) \right| \leqq C\delta_0$, it follows that the right-hand side of (3.19) might be estimated as

$$C\delta_0 \int_0^\tau e^{-\xi} d\xi + C\delta_0 \int_0^\tau \int_0^\xi e^{-(\xi-\eta)} \frac{\partial \omega}{\partial y_0} (\eta) \, d\eta d\xi.$$

Changing the order of integration in the second term and using (3.11), it follows that this quantity is uniformly bounded. Therefore, we can apply Lebesgue's Theorem to the right-hand side of (3.19) to show that the limit $\lim_{\tau\to\infty} [e^\tau \frac{\partial \omega}{\partial y_0} (\tau)]$ exists. Therefore (3.18) follows.

We then apply the dominated convergence theorem to get, as $\tau \to \infty$, in C^k_{loc},

$$\begin{aligned} \bar{\rho} (y, \tau) &= \int g_0 (y_0, y + \omega (0; y, y_0, \tau)) \, e^{N\tau} \det \frac{\partial w}{\partial y_0} dy_0 \\ &\to \int g_0 (y_0, y + \omega_\infty (0, y, y_0)) \, J_\infty (y, y_0) \, dy_0 \\ &\equiv \bar{\rho}_\infty (y). \end{aligned}$$

Using the elliptic regularity theory from the equation

$$\Delta_y \Phi = \bar{\rho}, \tag{3.20}$$

there exists $\Phi_\infty (y) \in C^{k+1,\beta}_{loc}$, for any $0 < \beta < 1$, such that

$$\Delta_y \Phi_\infty = \bar{\rho}_\infty.$$

Taking the limit in (3.20) as $\tau \to \infty$ yields the last formula in (3.6). Finally, notice that, given a test function $\psi (y, v)$

$$\begin{aligned} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g (y, v, \tau) \, \psi (y, v) \, dy dv &= \int e^{N\tau} g_0 (y_0, v_0) \, \psi (y, v) \, dy dv \\ &= \int e^{N\tau} g_0 (y_0, w_0 (\tau, y, y_0)) \, \psi (y, w (\tau, y, y_0)) \det \frac{\partial w}{\partial y_0} dy dy_0 \end{aligned}$$

and taking the limit $\tau \to \infty$ we obtain

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} g(y,v,\tau)\,\psi(y,v)\,\mathrm{d}y\mathrm{d}v \to \int g_0(y_0, y+\omega_\infty(0,y,y_0))\,\psi(y,y)\,J_\infty(y,y_0)\,\mathrm{d}y_0\mathrm{d}y$$

which can be written in the sense of distributions as

$$g(y,v,\tau) \to \int g_\infty(y_0,y)\,\delta(v-y)\,J_\infty(y,y_0)\,\mathrm{d}y_0 \quad \text{as} \quad \tau\to\infty.$$

This yields (3.8), whence the proof is complete.

Notice that in the limit case $\varepsilon_0 \to 0$ (3.8) reduces to

$$g(y,v,\tau) \to \left[\int g_\infty(y_0,y)\,\mathrm{d}y_0\right]\delta(y-v) = \left[\int g_0(y_0,y)\,\mathrm{d}y_0\right]\delta(y-v) \quad \text{as} \quad \tau\to\infty.$$

□

Acknowledgments Hyung Ju Hwang and Juan Velázquez acknowledge the hospitality and support of the Max Planck Institute for Gravitational Physics where this work was begun. Hyung Ju Hwang is supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (2010-0008127). Juan Velázquez acknowledges also the support of the Humboldt Foundation, the Max Planck Institute for Mathematics in the Natural Sciences and DGES Grant MTM2010-16467.

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(*Received June 19, 2006 / Accepted January 6, 2011*)
Published online February 16, 2011 – © Springer-Verlag (2011)