Solving the Problem of Time in General Relativity and Cosmology with Phantoms and k – Essence

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Abstract

We show that if the Lagrangean for a scalar field coupled to General Relativity only contains derivatives, then it is possible to completely deparametrise the theory. This means that
1. Physical observables, i.e. functions which Poisson commute with the spatial diffeomorphism and Hamiltonian constraints of General Relativity, can be easily constructed.
2. The physical time evolution of those observables is generated by a *natural* physical Hamiltonian which is (constrained to be) positive.

The mechanism by which this works is due to Brown and Kuchař. In order that the physical Hamiltonian is close to the Hamiltonian of the standard model and the one used in cosmology, the required Lagrangean must be that of a Dirac – Born – Infeld type. Such matter has been independently introduced previously by cosmologists in the context of k – essence due to Armendariz-Picon, Mukhanov and Steinhardt in order to solve the cosmological coincidence (dark energy) problem. We arrive at it by totally unrelated physical considerations originating from quantum gravity.

Our manifestly gauge invariant approach leads to important modifictaions of the interpretation and the the analytical appearance of the standard FRW equations of classical cosmology in the late universe. In particular, our concrete model implies that the universe should recollapse at late times on purely classical grounds.

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1 Introduction

By "the problem of time" in General Relativity (GR) one means that GR is a completely parametrised system. That is, there is no natural notion of time due to the diffeomorphism invariance of the theory and therefore the canonical Hamiltonian which generates time reparametrisations vanishes. In fact, instead of a Hamiltonian there are an infinite number of spatial diffeomorphism and Hamiltonian constraints respectively, of which the canonical Hamiltonian is a linear combination, which generate infinitesimal spacetime diffeomorphisms[1]. Physical observables, sometimes called Dirac observables, are functions on phase space which are gauge invariant, that is, they Poisson commute with all constraints. In particular, they do not evolve with respect to the canonical Hamiltonian. Hence "nothing seems to happen in quantum gravity".

The problem of time is not only of academic interest. One of the motivations for the present article actually comes from cosmology and can be phrased as the following question:

Why is it that the FRW equations describe the physical time evolution which is actually observed for instance through red shift experiments, of physical, that is observable, quantities such as the scale parameter?

The puzzle here is that these observed quantities are mathematically described by functions on the phase space which *do not Poisson commute with the constraints!* Hence they are not gauge invariant and therefore should not be observable in obvious contradiction to reality. Moreover, the time evolution described by the FRW equations is obtained from the Hamiltonian equations of motion generated by the Hamiltonian constraint and not by an actual Hamiltonian. This is due to the fact that the "Hamiltonian" used to derive the FRW equations is actually constrained to vanish by one of the Einstein equations. The "evolution equations" generated by a constraint must therefore be interpreted as gauge transformations and those, by the very definition of gauge transformations, are also not observable, again in sharp contradiction to observation. Thus we arrive at the following devastating conclusion:

Either the mathematical formalism, which has been tested experimetally so excellently in other gauge theories such as QED, is inappropriate or we are missing some new physics.

We will show in this article that the problem of time and the above puzzle can be solved in the canonical approach to GR if one manages to deparametrise the theory. By this we mean that it is possible to write the Hamiltonian constraints in the form $C(x) = \pi(x) + H(x)$ where π is the momentum conjugate to a scalar field ϕ and where $H(x)$ is a positive function on phase space[2] which depends on neither ϕ or π. In this situation it is possible to construct explicitly physical observables and the function $H := \int_\sigma d^3x \; H(x)$ is the *natural* physical Hamiltonian which generates the time evolution of those observables. We will show explicitly that the scalar matter Lagrangean can be chosen in such a way that the physical Hamiltonian is close to the Hamiltonian of the standard model and the one used in cosmology and that the gauge invariant physical observables are closely related to the "non – observables" mentioned above. [3]

[1] When the equations of motion hold.

[2] As always in the canonical approach we assume that the spacetime manifold is diffeomorphic to $\mathbb{R} \times \sigma$ where σ is an arbitrary three manifold and x are local coordinates on σ.

[3] Of course it is conceivable that other than scalar matter can induce deparametrisation while the corresponding

The missing physics could therefore be a scalar matter component which in a precise sense is pure gauge. We therefore call it a *phantom field* because it is not directly observable. This is phenomenologically appealing because scalar matter has not yet been observed in nature. Its main effect is that it provides a notion of physical time evolution, it is a *perfect physical clock*. Although it is pure gauge, its presence has further observable consequences: The physical Hamiltonian deviates slightly from the usual Hamiltonian that one uses in the standard model or cosmology and therefore changes the dynamics slightly. The associated modified dynamics of observable quantities can be used in principle in order to test a given, deparametrisation generating, model experimentally. In fact, the modified evolution equations generated by the physical *Hamiltonian* rather than the Hamiltonian *Constraint* can be recasted into FRW form, however, at least for the concrete realisation of deparametrisation that we consider here, now the FRW equations adopt additional terms which are *dynamically generated*. There are two types of modifications. The first type is expected: In the standard interpretation of the FRW equations, these can be interpreted as matter terms which at early times statisfy the equation of state of dust $w = 0$ while at late times it becomes a cosmological constant $w = -1$. However, the energy of the scalar field is negative which requires that there be positive energy matter with those equations of state in order to have overall positive matter energy. Thus the model could be able to explain dynamically why there must be dark matter and dark energy. The second type of modifications are deviations from the FRW form itself. At very late times, where "late" depends on the parameters of the model, the FRW interpretation breaks down and the universe evolves *drastically differently* with respect to the physical Hamiltonian. In fact, our concrete model suggests that the universe should recollapse on purely classical grounds. Therefore, if we really observe evolution with respect to the physical time parameter corresponding to the physical Hamiltonian induced by our scalar field then the FRW equations are an approximation to the true evolution of the universe, which is valid at sufficiently early times of the universe only. Of course, the parameters of the model and its dynamical constants of motion can be tuned such that the FRW equations are still valid today. Let us finish this paragraph with the following provocative lesson:

All textbooks on classical GR incorrectly describe the Friedmann equations as physical evolution equations rather than what they really are, namely gauge transformation equations. The true evolution equations acquire possibly observable modifications to the gauge transformation equations whose magnitude depends on the physical clock that one uses to deparametrise the gauge transformation equations.

Both types of modifications just mentioned will of course not only happen in homogeneous cosmology but also in full GR. Notice that we do not exclude observable scalar matter such as an inflaton in the Lagrangean, rather we propose that whatever scalar or other *observable* matter is present in nature, there is in addition our negative energy scalar field which is actually the reason for why that other matter can mathematically be related to gauge invariant, i.e. observable, quantities. In a sense, the mathematical formalism (gauge theory) together with the experimental evidence (e.g. the experimental verification of the FRW equations) inavoidably force us to conclude that there is something like a negative energy matter field which therefore could be called a *prediction*[4].

In this paper we show that it is possible to find a whole class of scalar field Lagrangeans with the

Hamiltonian has the properties mentioned. In particular, it would be desirable to find a scalar mode among the gravitational degrees of freedom leading to deparametrisation. However, this has proved to be impossible.

[4]Of course, there may be other realisations of deparametrisation, different from a scalar field. However, the conclusion that there is a matter component of which we are unaware when we treat the FRW equations as if they came from a true Hamiltonian rather than the Hamiltonian constraint, remains.

required properties. The mechanism which leads to deparametrisation rests on a observation due to Brown and Kuchař made in their seminal work [1] which enabled them to reformulate the Hamiltonian constraints of GR such that they Poisson commute among each other, which is a necessary condition for deparametrisation as we will see. The only requirement is that the covariant scalar field Lagrangean depends only on the first derivatives of the scalar field. However, it may nevertheless self – interact due to a non – polynomial Lagrangean similar to quintessence fields [3] and more generally as in in k – essence models[5] due to Armendariz-Picon, Mukhanov and Steinhardt [4]. All possible mutually Poisson commuting Hamiltonian constraints have been found in [5], but only a subclass of them originate from a covariant Lagrangean which we will provide in this paper. A, possibly unique, two parameter family within that class leads to physical Hamiltonians which approach the standard model Hamiltonian when the scalar field is close to being spatially homogeneous and that of standard cosmology at sufficiently early times. That it is spatially homogeneous (i.e. a constant) turns out to be a *natural* requirement in order that the scalar field defines a good (i.e. synchronised everywhere on σ) clock.

Curiously, as we will see, this family of scalar field Lagrangeans, to which we are driven naturally by physical and mathematical considerations, has been considered before by cosmologists [6] for entirely different reasons. Its physical properties agree with what cosmologists call a phantom field[6]. It turns out that our family of Lagrangeans are necessarily of Dirac – Born – Infeld type with a constant potential. Notice again that this phantom field is not directly observable. However, we can, and probably must in order to have a positive energy budget, add further k – essence matter. Such observable k – essence matter is being discussed as a candidate for dark energy and inflation by cosmologists.

We should mention that the deparametrisation technique is a special, very simple case of the more general "relational" approach due to Rovelli, see [7] and references therein. The mathematical implementation of this idea has been much improved recently [8] (see also [9]). It consists in choosing an infinite number of gauge fixing conditions called "clocks" and the afore mentioned physical observables are the gauge invariant extensions, off the associated gauge cut, of non – invariant "partial observables". The analytical formulae are very complicated power series in general and there are unsolved mathematical issues such as convergence of the series. In contrast to the deparametrisation case, in the more general case the associated physical observables Poisson commute only weakly with the constraints, that is, when the constraints hold, they are weak Dirac observables. Observables coming from the deparametrised theory are strong Dirac observables which is mathematically much more convenient. Fortunately, the much more complicated partial observable machinery is *not needed* in order to arrive at the results of the present paper. All the results that we claim in this paper will be proved by elementary methods, the paper is self – contained in that respect.

We emphasise that the formalism developed in this paper is *exact and non – perturbative*. On the other hand, it is purely classical only so far. This is true for almost all the available literature on relational physics. In order to apply quantum theory to it, operator ordering issues have to be solved for the power series. This is a difficult issue in the general relational framework, however, under natural mathematical assumptions, we can actually solve the operator ordering problem as we will sketch in section 7. Yet, it may be necessary to develop a perturbative scheme just like in S – matrix theory. This is a good approximation as long as the (kinematical) states with which we probe these observables are strongly peaked, at physical time τ, at the phantom field value $\phi = \tau$ which

[5]Basically, a k – essence field is a scalar field Φ which depends non – linearly on the kinetic term $g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}$.

[6]The generally accepted rough definition of a phantom field seems to be that in a cosmological setting the first order term in $g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}$ of the Lagrangean comes with a coefficient which has a sign opposite to the sign in the Klein – Gordon Lagrangean.

explains why the phantom field should be close to spatially homogeneous. Hence, for a sufficiently short period period of physical time τ, the approximation should be quite good.

With the formalism developed in this paper, a natural platform for carrying out cosmological quantum field theory[7] within the framework of Loop Quantum Gravity (LQG) [10] is launched. See [11] for a corresponding proposal.

The article is organised as follows:

As this article is intended for both cosmologists and quantum geometers, in section two we state the results of our analysis without proofs. The proofs will be supplied in the remaining sections. Readers just interested in the results can therefore skip all the rest of the paper except for section seven.

In section three we review the Brown – Kuchař mechanism to generate mutually commuting Hamiltonian constraints.

In section four we define the physical observables of the theory as well as the physical Hamiltonian originating from a general phantom field Lagrangean.

In section five we show that physical and mathematical considerations naturally lead to a Dirac – Born – Infeld scalar field Lagrangean which for certain parameter range has the interpretation of what cosmologists call a phantom. The associated physical selection principle is that the corresponding physical Hamiltonian is positive and close to that of the standard model (when the metric is flat).

In section six we derive the consequences of the gauge invariance principle for cosmology by computing and interpreting the modified FRW equations.

In section seven we conclude and outline what we plan to do with our formalism in the future, in particular in *quantum cosmology.*

2 Summary

The scalar field Lagrangean which leads to deparametrisation, induces a positive Hamiltonian which is close to that used in the standard model when the metric is flat and which leads to physical equations of motion which are in agreement with the cosmological FRW equations is given by

$$L = -\beta + \alpha\sqrt{|\det(g)|}\sqrt{1 + g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}} \tag{2.1}$$

Here α, β are constants of dimension[8] cm^{-2}. We must have necessarily $\alpha > 0$ as we will see below. The sign of β is unconstrained. A natural value for β would be $\beta = \alpha$ so that for small $(\nabla\Phi)^2$ the Lagrangean becomes $\alpha(\nabla\Phi)^2/2$ which up to the positive constant α is the massless Klein – Gordon field Lagrangean with the wrong sign, i.e. it is a phantom. The other natural value for β is $\beta = 0$ because β could always be absorbed into a cosmological term. Let us choose $\beta = 0$ for concreteness in this preliminary discussion. Lagrangeans of the form (2.1) are being discussed in k – essence [4], albeit there with non – trivial potential. For our purposes, non – trivial potentials are forbidden.

[7] By this we mean Quantum Gravity in the sector whose classical limit is classical cosmology. This should not be confused with quantum cosmology which is just a quantum mechanical toy model of the actual quantum gravitational field theory.

[8] We assume signature $(-,+,+,+)$ and choose units for which $8\pi G_{\text{Newton}} = 1$. Moreover, we assume that spatial coordinates are dimensionless while time coordinates have dimension cm and $ds^2 = g_{\mu\nu}dX^\mu dX^\nu$ has dimension cm^2. We take Φ to have dimension cm so that the argument of the root in (2.1) is dimensionfree.

We arrive at the model (2.1) by a totally independent mathematical and physical reasoning, namely deparametrisation, hence the fact that we stumble on k – essence is rather curious.

The canonical formulation leads to the following spatial diffeomorphism and Hamiltonian constraints respectively

$$\begin{aligned} D_a^{\text{tot}} &= D_a + \pi\phi_{,a} \\ C^{\text{tot}} &= C - \sqrt{[1 + q^{ab}\phi_{,a}\phi_{,b}][\pi^2 + \alpha^2 \det(q)]} \end{aligned} \tag{2.2}$$

Here D_a, C respectively are the contributions to the spatial diffeomorphism constraint and the Hamiltonian constraint of the gravitational and non – phantom matter degrees of freedom, π is the momentum conjugate to ϕ and q_{ab} is the metric intrinsic to the spatial slices with inverse q^{ab}. Clearly $a, b, .. = 1, 2, 3$ while $\mu, \nu, .. = 0, 1, 2, 3$. From (2.2) we see that C is constrained to be positive. This will be important for what follows. If we had chosen the other sign for α then the root in (2.2) would come with the opposite sign and C would be constrained to be negative. One can also not reverse the sign in front of $(\nabla\Phi)^2$ in (2.1) because this would lead to a non – definite argument of the root in (2.2).

The interpretation of (2.2) is that these are constraints, i.e. they must vanish. The canonical transformations on phase space that they generate are therefore not evolutions but gauge transformations. In fact, one can show [12] that when the Einstein equations hold, the canonical transformations that they generate precisely coincide with spacetime diffeomorphisms. Any object which has non – vanishing Poisson brackets with the constraints is therefore not observable because only gauge invariant objects have physical meaning. The problem of time is therefore that we do not have a priori a Hamiltonian which generates physical time evolution of gauge invariant objects.

The Brown – Kuchař mechanism [1] consists in the crucial observation that

$$\pi^2 q^{ab}\phi_{,a}\phi_{,b} = q^{ab} D_a D_b =: D \tag{2.3}$$

when $D_a^{\text{tot}} = 0$. Thus, using (2.3) we can solve $C^{\text{tot}} = 0$ for π and obtain a different Hamiltonian constraint

$$C'^{\text{tot}}(x) = \pi(x) + \left[\sqrt{\frac{1}{2}[C^2 - D - \alpha^2 Q] + \sqrt{\frac{1}{4}[C^2 - D - \alpha^2 Q]^2 - \alpha^2 DQ}}\right](x) =: \pi(x) + H(x) \tag{2.4}$$

where $Q := \det(q)$. Together with the $D_a^{\text{tot}}(x)$ it defines the same constraint surface[9] as the system (2.2) and is also first class. By virtue of the Brown – Kuchař mechanism, the new Hamiltonmian constraints even mutually Poisson commute among each other. The arguments of the roots in (2.4) are constrained to be non – negative as the derivation of that expression reveals.

Since H no longer depends π, ϕ, we have managed to deparametrise General Relativity and in fact the quantity

$$H := \int_\sigma d^3x\; H(x) \tag{2.5}$$

[9]More precisely a subset of the full constraint surface. There are altogether four components of the constraint surface corresponding to the four possible combinations of signs in front of the two square roots involved in (2.4). While these components connect in lower dimensional submanifolds, each of them is preserved by the gauge transformations induced by the full Hamiltonian constraint. We therefore restrict from now on once and for all to the subset defined by $C'^{\text{tot}} = 0$.

is a positive Hamiltonian, it is not constrained to vanish and it is gauge invariant, it Poisson commutes with all constraints. Next let for any real number τ

$$H_\tau := \int_\sigma d^3x \; [\tau - \phi(x)] \;\; H(x) \tag{2.6}$$

Let f be any spatially diffeomorphism invariant quantity on phase space which does not depend on ϕ. Such functions are trivial to construct, a simple example is the volume $f = \int_\sigma d^3x \sqrt{\det(q)}$. Then the series

$$O_f(\tau) := f + \{H_\tau, f\} + \frac{1}{2!}\{H_\tau, \{H_\tau, f\}\} + \frac{1}{3!}\{H_\tau, \{H_\tau, \{H_\tau, f\}\}\} + ... \tag{2.7}$$

defines a one parameter family of gauge invariant function on phase space. Moreover, we have

$$\frac{dO_f(\tau)}{d\tau} = \{H, O_f(\tau)\} \tag{2.8}$$

In other words, the map $\tau \mapsto O_f(\tau)$ describes the physical time evolution of gauge invariant objects generated by the Hamiltonian (2.5).

The crucial additional property of H in (2.4) which was used in order to select the model (2.1) is that when the scalar field ϕ is spatially homogeneous, which is natural in order that it defines an everywhere (on σ) synchronised clock $\phi(x) = \tau$, if the spatial diffeomorphism coinstraint holds and if α is sufficiently small then $H(x) \approx |C(x)| = C(x)$. Hence H approximates the standard model Hamiltonian when the spacetime is close to being flat. If we had chosen the other sign for α we would get $H(x) \approx |C(x)| = -C(x)$. Since $C(x) > 0$ for usual matter when space is flat, it would follow that with this sign we cannot have flat space and moreover that all matter contributions to the Hamiltonian come with the wrong (negative) sign. Hence the choice $\alpha > 0$ is the only suitable one for our purposes.

Let us investigate this more closely:
Since $C = C^{\rm grav} + C^{\rm s-matter} + C^{\rm ns-matter} = 0$ where $C^{\rm s-matter} > 0$ is the standard matter energy density, we must have $C = C^{\rm grav} + C^{\rm ns-matter} < 0$ where the latter contribution is from non standard matter such as our scalar field. The gravitational contribution $C^{\rm grav}$ is indefinite, there are positive and negative scalar modes contained in it and therefore it would be desirable to use a negative gravitational mode for deparametrisation. Unfortunately, such a mode does not lead to deparametrisation because C depends on both the corresponding π, ϕ. Hence we consider $C^{\rm ns-matter} \not\equiv 0$. It turns out that we must restrict on the portion of phase space where we have $H \approx \pm|C^{\rm grav} + C^{\rm s-matter}|$, hence a priori both signs in front of the square root in (2.4) are allowed. Hence we should have $C^{\rm ns-matter} < 0$ or $C^{\rm ns-matter} > 0$ respectively in order that $H \approx C^{\rm grav} + C^{\rm s-matter}$ comes with the *positive* sign in front of $C^{\rm s-matter}$ because on flat space this is the energy density of standard matter. However, as we will see, if $C^{\rm ns-matter} > 0$ then the physical evolution equations adopt modifications which lead to a big rip singularity in cosmological applications (the universe reaches infinite size in a finite amount of time). Thus, if we want to avoid this, we are naturally led to scalar matter with negative energy density.

If we would have $H(x) = C(x)$ exactly, then the physical evolution equations derived from H for gauge invariant observables not involving ϕ, π would exactly equal the gauge transformations on non – gauge invariant quantities not involving ϕ, π derived from the canonical "Hamiltonian" (with $N = 1, \; N^a = 0$)

$$H^{\rm canon}(N, \vec{N}) = \int_\sigma d^3x \; [N(x)C(x) + N^a(x)D_a(x)] \tag{2.9}$$

which ignores the phantom field. This would justify why the Hamiltonian constraint integrated against unit lapse is often used as a Hamiltonian. The phantom field, being pure gauge, would have absolutely no visible effect. However, since $H(x)$, $C(x)$ do not exactly coincide, there are important modifications, both technically and conceptually, to which we turn now in a cosmological setting.

Namely, we will see that the FRW equations must be provided with a new and gauge invariant interpretation. The actual physical evolution equations generated by the Hamiltonian leads to drastic modifications in the very late universe while in the early universe (including today) they keep their standard form to a very good approximation, depending on the numerical value of α. In flat, homogeneous and isotropic models the FRW line element takes the form $ds^2 = -dt^2 + a(t)^2 dx^a dx^b \delta_{ab}$ with scale factor a and all constraints are identically satisfied due to the high symmetry of the Ansatz, except for a single Hamiltonian constraint

$$C^{\rm tot} = [-\frac{P^2}{12a} + (\Lambda + \rho_m)a^3] + \rho_{\rm phantom} a^3 =: C + \rho_{\rm phantom} a^3 \tag{2.10}$$

where

$$\rho_{\rm phantom} = -\sqrt{\frac{\pi^2}{a^6} + \alpha^2} =: -\alpha\sqrt{1+x} \tag{2.11}$$

is the negative phantom energy, P is the momentum conjugate to a, Λ is a cosmological constant and ρ_m is the energy density of all non – phantom matter. The important quantity

$$x := \frac{\pi^2}{\alpha^2 a^6} \tag{2.12}$$

will be called the deviation parameter. The phantom pressure is positive

$$p_{\rm phantom} = -\frac{1}{3a^2}\partial(a^3\rho_{\rm phantom})/\partial a = \alpha\frac{1}{\sqrt{1+x}} \tag{2.13}$$

leading to an equation of state and speed of sound respectively

$$w_{\rm phantom} = \frac{p_{\rm phantom}}{\rho_{\rm phantom}} = -\frac{1}{1+x} = -\frac{\partial p_{\rm phantom}/\partial x}{\partial \rho_{\rm phantom}/\partial x} = -c^2_{\rm phantom} \tag{2.14}$$

We can now do two, conceptually very different, things:
1.
First we follow the standard procedure in cosmology. That is, we use the constraint $C^{\rm tot}$ as if it was a Hamiltonian. The associated equations of motions of non – observable quantities such as the scale factor then lead to the usual FRW equations. From the point of view of gauge theory, the interpretation of those FRW equations as evolution equations of observable quantities is, however, completely wrong. That $a(t)$ is not observable, that is, not gauge invariant, can be easily seen from the fact that $da(t)/dt = \{C^{\rm tot}, a\} \neq 0$. The correct interpretation of those equations is that they describe the behaviour of non – observable quantities under the gauge transformations generated by the Hamiltonian constraint.
2.
The second thing that we can do is to compute the gauge invariant functions such as $O_a(\tau)$ using (2.7) with $H_\tau = (\tau - \phi)H$ where the Hamiltonian (2.6) becomes

$$H := \sqrt{C^2 - \alpha^2 a^6} \tag{2.15}$$

and compute their physical evolution equations generated by (2.15).

Mathematically the two procedures are very similar to each other: In the first approach we compute $da/dt = \{C^{\rm tot}, a\}$ and express P in terms of da/dt. The first FRW equation then results by substituting P in terms of da/dt into the constraint equation $C^{\rm tot}$. Then we compute $dP/dt = \{C^{\rm tot}, P\}$ and insert this into $d^2a/dt^2 = \{C^{\rm tot}, \{C^{\rm tot}, a\}\}$ which results in the second FRW equation. They take the usual form

$$\begin{aligned} 3(\frac{da/dt}{a})^2 &= \Lambda + \rho_m + \rho_{\rm phantom} \\ 3\frac{d^2a/dt^2}{a} &= \Lambda - \frac{1}{2}[\rho_m + 3p_m + \rho_{\rm phantom} + 3p_{\rm phantom}] \end{aligned} \tag{2.16}$$

In the second approach we compute $dO_a(\tau) = \{H, O_a(\tau)\} = O_{\{H,a\}}(\tau)$ and can then solve $O_P(\tau)$ in terms of $dO_a(\tau)$. The first FRW equation then results by expressing $C^{\rm tot}$ in terms of physical observables, that is, computing $O_{C^{\rm tot}}(\tau)$ and imposing $O_{C^{\rm tot}}(\tau) = 0$. That this should hold follows from $dO_{C^{\rm tot}}(\tau)/d\tau = O_{\{H,C^{\rm tot}\}}(\tau) = 0$ since H is an observable, hence $O_{C^{\rm tot}}(\tau) = O_{C^{\rm tot}}(\phi) = C^{\rm tot} = 0$. The second FRW equation then is obtained by computing $dO_P(\tau)/d\tau = O_{\{H,P\}}(\tau)$ and using this in $d^2O_a(\tau)/d\tau^2 = O_{\{H,\{H,a\}\}}(\tau)$. This results in

$$\begin{aligned} 3(\frac{dO_a/d\tau}{O_a})^2 &= [\Lambda + O_{\rho_m} + O_{\rho_{\rm phantom}}](1 + \frac{1}{x}) \\ 3\frac{d^2O_a/d\tau^2}{O_a} &= \Lambda(1 + \frac{4}{x}) - \frac{1}{2}\{[O_{\rho_m} + O_{\rho_{\rm phantom}}](1 - \frac{5}{x}) + 3[O_{p_m} + O_{p_{\rm phantom}}](1 + \frac{1}{x})\} \end{aligned} \tag{2.17}$$

where now

$$x = \frac{E^2}{\alpha^2 O_a(\tau)^6} \tag{2.18}$$

and where $E = H = -\pi$ is a constant of motion, namely the energy of the universe.

Comparing (2.17) and (2.18) reveals:
1.
Although from the point of gauge theory it is incorrect to interpret the FRW equations (2.16) as evolution equations of observable quantities, as long as x is large, the actual physical evolution equations of observables (2.18) generated by the physical Hamiltonian take exactly the same form. All that we have to do is to make the substitution $(t, a(t)) \to (\tau, O_a(\tau))$.
2. When x gets small, the correct equations (2.17) differ drastically from the incorrect equations (2.16). Notice that what we observe in experiment is really a gauge invariant object such as $O_a(\tau)$ and not $a(t)$. Of course, the concrete scenario for deparametrisation that we have proposed here may not be realised in nature, however, we insist that whatever matter is used for deparametrisation, there will be corrections to the standard FRW equations. This should have observable consequences!

Notice that we do not doubt the validity the Einstein equations (2.16). They follow from the fundamental object $C^{\rm tot}$ which we also used in our construction. However, we stress that their interpretation as physical evolution equations of observables is fundamentally wrong. The domain of validity of the interpretation of the usual FRW equations as evolution equations is controlled by the deviation parameter x. It depends on the kinematical model parameter α and the dynamical constant of motion E. The critical value is $x = 1$ and is reached at scale factor $O_a = \sqrt[3]{E/\alpha}$ which can be as large as we want for sufficiently small α. Thus we see that the mathematical formalism together with our concrete model predicts that the universe evolves differently at late times, that is, at large scale factor. We expect similar modifications in other applications of GR such as black hole physics and it is an interesting speculation that the corresponding gauge invariant interpretation of Einstein's equations could predict large scale deviations from Newton's law which then could be in agreement with the measured rotation curves of galaxies. Notice that all of this is a purely classical effect, there is no quantum gravity involved in this although our motivation, deparametrisation, certainly comes from quantum gravity.

The fact that the phantom makes a negative contribution to the energy budget may be disappointing for supporters of k – essence where $\rho_k > 0$ is usually required. However, $\rho_{\text{phantom}} < 0$ is of no concern as long as the remaining matter makes an overall positive contribution[10]. Actually, since the gravitational contribution to the Hamiltonian constraint is negative definite in cosmology, in fact due to $C^{\text{tot}} = 0$ we must have $\rho_m > |\rho_{\text{phantom}}|$. Notice by the equation of state the phantom behaves like dust at small scales $\rho_{\text{phantom}} \to -E/O_a^3$ and as a negative cosmological constant $\rho_{\text{phantom}} \to -\alpha$ at large scales. This can be easily compensated by additional positive energy k – essence matter or simply by ordinary (dark?) matter plus an additional cosmological constant term $\Lambda - \alpha > 0$. In a sense, if we want to explain the observational fact that the FRW equations describe the universe while their mathematical derivation violates the principles of gauge theory, then something like a phantom is needed for deparametrisation and in turn it requires something like k – essence for reasons of total positive matter energy budget. From this point of view, both a phantom and k – essence are a prediction of the mathematical formalism (gauge theory) together with observation (FRW cosmology).

We will see furthermore that in order that the universe does not reach infinite size in finite τ time, it must in fact recollapse which can be achieved by a suitable choice of the parameters. Then the picture of a periodic universe arises if one can establish that Quantum Gravity effects avoid big bang and big crunch singularities. This would imply that the universe evolves through the "would be" singularities in an infinite number of cycles. Notice that recently [13] a simple cosmological toy model has been rigorously quantised by the methods of LQG using precisely the gauge invariant programme suggested in [11] and for which possible classical foundations have been layed out in the present paper for the full theory. In that model, the singularity is indeed quantum mechanically avoided which is a promising hint that in full LQG the singularity is avoided as well.

We close this section with some final remarks:

1.

From the point of view of a cosmologist nothing would be more natural than to use the scale factor itself as a clock: It is a monotonic function of the unphysical time parameter t (until possible recollapse). Why did we not do that immediately (we can do it indirectly, see below)? There are two reasons. First of all, we wanted to provide a universal framework, i.e. to provide a physical notion of time in all possible situations and not only in homogeneous ones. However, in inhomogeneous situations, the notion of a scale factor is void. As a substitute one could consider the volume of (subsets of) σ (the total volume is infinite for non compact topology of σ). However, this does not work for the same second reason for which also the scale factor itself is inappropriate in the homogeneous situation: In order to achieve deparametrisation and to obtain a physical Hamiltonian with all the required properties, the clock variable(s) must be cyclic in the Hamiltonian constraint (at least weakly). This condition is violated for the scale factor and its inhomogeneous relatives due to the universal coupling of matter to gravity.

[10] The usual energy conditions on the energy momentum tensor do not make any restrictions on individual matter species but only on the overall matter content of nature.

2.
Of course, *after one has deparametrised the system* and one is only dealing with physical quantities such as the physical scale factor $O_a(\tau)$ (or one of its inhomogeneous relatives) one can use it as a physical clock in place of τ itself which is maybe better geared to what one does in reality. One can then express time dependence of other observables $O_f(\tau)$ in terms of O_a by solving $O_a(\tau)$ for τ. In other words, while we cannot use the scale factor to deparametrise the system, we can still use it as physical clock *after* deparametrisation. The time evolution in terms of the physical scale factor will then be generated by a more complicated physical Hamiltonian.
3.
One could think that what cosmologists usually do in order to describe measurable quantities mathematically is actually precisely correct, that is "relational". For instance the redshift factor

$$z(t_1,t_2) := \frac{\omega_1}{\omega_2} \approx \frac{a(t_2)}{a(t_1)} \tag{2.19}$$

is the ratio between the emission frequency ω_1 of a spectral line (known from a table top experiment on Earth) and the absorption frequency ω_2 observed on Earth is certainly measurable. Formula (2.19) relates this observable quantity to the ratio of the scale factors at unphysical emission time t_1 and absorption time t_2 respectively. We will now show that (2.19) is in fact incorrect:
The reason is that the quantities $a(t)$ are not observable. In order to see what is going on, we have to go through the derivation of the redshift formula. Consider a star at comoving distance r from Earth. For light the geodesic is null and due to $ds^2 = -dt^2 + a(t)^2 dx^a dx^b \delta_{ab}$ we get as an equation of motion $a(t)\dot{r}(t) = 1$. Formula (2.19) then results from the fact that the beginning and the end of the wave travel the same comoving distance $r = \int_{t_1}^{t_2} dt/a(t) = \int_{t_1+T_1}^{t_2+T_2} dt/a(t)$ with $\omega_j = 2\pi/T_j$. This is certainly mathematically correct, however, the quantities $a(t_j)$ are not observable. In order to express z in terms of observale quantities $O_a(\tau)$ we express the line element in terms of τ (see (6.30))

$$ds^2 = -d\tau^2(1+\frac{1}{x}) + O_a(\tau)^2 dx^a dx^b \delta_{ab} \tag{2.20}$$

Notice that τ is no gauge parameter but a physical observable associated with the physical Hamiltonian, hence the factor $1 + 1/x$ cannot be transformed away by a diffeomorphism $\tau \mapsto \varphi(\tau)$ without changing the Hamiltonian. We now obtain the null geodesic equation of motion $O_a(\tau)dO_r(\tau)/d\tau = \sqrt{1+1/x}$. The same argument now leads to the modified redshift factor relation

$$z(\tau_1,\tau_2) = \frac{\omega_1}{\omega_2} = \frac{O_a(\tau_2)}{O_a(\tau_1)} \sqrt{\frac{1+\frac{1}{x(\tau_1)}}{1+\frac{1}{x(\tau_2)}}}, \quad x(\tau) = \frac{E^2}{\alpha^2 O_a(\tau)^6} \tag{2.21}$$

and now all displayed quantities are observable. Hence we see that as long as x is large, (2.21) and (2.19) agree in the following sense: What one incorrectly does in cosmology is to identify the unobservable gauge pair $(t, a(t))$ with the pbservable physical pair $(\tau, O_a(\tau))$. With this interpretation, the wrong relation (2.19) is a good approximation to the correct relation (2.21) as long as x is large. However, there are large deviations especially in the late universe and of course the modification (2.21) may have an observable effect on the interpretation of supernovae type Ia observations (standard candles) which provide evidence for recent accelarated expansion of the universe.

We now proceed to the mathematical and physical details.

3 Review of the Brown – Kuchař Mechanism

3.1 Covariant, Minimally Coupled, Potential Free Scalar Fields

In order to prepare for the explanation of the Brown – Kuchař mechanism we review here the canonical formulation of general scalar field Lagrangeans of a special class.

We consider a general, covariant scalar field Lagrangean minimally coupled to the metric with action

$$S_{\rm phantom} = \int_M d^4X \sqrt{|\det(g)|} L(-g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}/2) \tag{3.1}$$

where L is an arbitrary function of the variable indicated. It will be crucial for the Brown – Kuchař mechanism to work that the scalar field Φ only appears with derivatives, i.e. there is no non – trivial potential term. Obviously, (3.1) is invariant under Diff(M).

As usual we perform a 3+1 split of the action [12] and assume that M is diffeomorphic to $\mathbb{R} \times \sigma$ where σ is a three – manifold of arbitrary topology. Hence, there is a foliation $t \mapsto \Sigma_t = Y_t(\sigma)$ of M by spacelike hypersurfaces which are the images of σ under a one parameter family of embeddings $t \mapsto Y_t$. This way we obtain a diffemorphism $\mathbb{R} \times \sigma \to M$; $(t,x) \mapsto X := Y_t(x)$. We consider the foliation vector field $T(X) := [\partial Y_t(x)/\partial t]_{Y_t(x)=X}$ which can be split as $T(X) = [N(t,x)n(X) + N^a(t,x)\partial Y_t(x)/\partial x^a]_{Y_t(x)=X}$. Here $x^a,\ a = 1,2,3$ are local coordinates of σ while $X^\mu,\ \mu = 0,1,2,3$ are local coordinates of M. The vector field n is everywhere normal to the foliation, that is, $g_{\mu\nu}n^\mu Y^\nu_{t,a} = g_{\mu\nu}n^\mu n^\nu + 1 = 0$. The functions N, N^a respectively are known as lapse and shift functions.

We now pull back (3.1) by the diffeomorphism Y and express everything in terms of $N(t,x),\ N^a(t,x),\ q_{ab}(t,x) = (Y_t^* g)_{ab}(x)$ and $\phi(t,x) = (Y_t^*\Phi)(x)$. It is not difficult to check that in the embedding coordinates the components of the metric tensor read $g_{tt} = -N^2 + N^a N^b q_{ab},\ g_{ta} = q_{ab}N^b,\ g_{ab} = q_{ab}$ and for the inverse $g^{tt} = -1/N^2,\ g^{ta} = N^a/N^2,\ g^{ab} = q^{ab} - N^a N^b/N^2$ where $q^{ac}q_{cb} = \delta^a_b$. It follows that (the lapse is assumed to be everywhere non – negative)

$$\sqrt{|\det(g)|} = N\sqrt{\det(q)},\ \ I := -g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} = (\nabla_n\phi)^2 - q^{ab}\phi_{,a}\phi_{,b} \tag{3.2}$$

where $n = (T - N^a Y_{t,a})/N$ so that $\nabla_n\phi = (\dot\phi - N^a\phi_{,a})/N$.

We are now in position to perform the Legendre transform. We have

$$\pi(t,x) := \delta S_{\rm phantom}/\delta\dot\phi(t,x) = \sqrt{\det(q)}[\nabla_n\phi]L'(I/2) \tag{3.3}$$

where the prime denotes the derivative with repspect to $I/2$. From (3.3) we infer

$$K := [\frac{\pi}{\sqrt{\det(q)}}]^2 = [L'(I/2)]^2(I+V),\ \ V := q^{ab}\phi_{,a}\phi_{,b} \tag{3.4}$$

We assume that L is such that (3.4) can be solved uniquely for $I = J(K,V)$. Then (3.3) can be solved for $\nabla_n\phi$

$$\nabla_n\phi = p/L'(J/2),\ p := \pi/\sqrt{\det(q)} \tag{3.5}$$

We can now complete the Legendre transform

$$S_{\rm phantom} = \int_{\mathbb{R}} dt \int_\sigma d^3x(\pi\dot\phi - [N^a\pi\phi_{,a} + N\sqrt{\det(q)}\{p^2/L'(J/2) - L(J/2)\}] \tag{3.6}$$

From (3.6) we read off the contributions of the scalar field to the spatial diffeomorphism and Hamiltonian constraint respectively

$$\begin{aligned} D_a^{\rm phantom} &= \pi\phi_{,a} \\ C^{\rm phantom} &= \sqrt{\det(q)}[\frac{p^2}{L'(J/2)} - L(J/2)] \end{aligned} \tag{3.7}$$

3.2 The Brown – Kuchař Mechanism

Let us denote by D_a, C respectively the contribution to the spatial diffeomorphism and Hamiltonian constraint respectively of the gravitational field and all other matter fields (say of the standard model or one of its supersymmetric extensions). Then the spatial diffeomorphism constraint is given by $D_a^{\rm tot} = D_a + D_a^{\rm phantom}$ and the Hamiltonian constraint by $C^{\rm tot} = C + C^{\rm phantom}$. The simple, but crucial observation due to Brown and Kuchař is that we may use the spatial diffeomorphism constraint in order to remove the dependence of $C^{\rm tot}$ on ϕ altogether, thus making it a function of p, the gravitational field and all the other matter fields only. Namely, we have, when $D_a^{\rm tot} = 0$

$$V = q^{ab}\phi_{,a}\phi_{,b} = \frac{q^{ab}D_a^{\rm phantom}D_b^{\rm phantom}}{\pi^2} = \frac{q^{ab}D_a D_b}{\pi^2} = \frac{1}{p^2}\frac{q^{ab}D_a D_b}{\det(q)} =: d/K \tag{3.8}$$

This is the Brown – Kuchar Mechanism: The field ϕ, which appears only in the combination V within the Hamiltonain constraint, has been eleminated. This would not work if the Lagrangean also would depend on ϕ explicitly (potential term), not only through the combination I which involves only derivatives of ϕ.

Consider now the function $\tilde{J}(K,d) := J(K, V = d/K)$. Then, the Hamiltonian constraint can be equivalently described by the function

$$\tilde{C}^{\rm tot} = \sqrt{\det(q)}[c + [\frac{K}{L'(\tilde{J}/2)} - L(\tilde{J}/2)] \tag{3.9}$$

where $C = \sqrt{\det(q)}c$. Since the constraints form a first class system, also the new constraints do and they define the same constraint surface.

Notice that (3.9) depends on p only through K. We will now assume that we may solve (3.9) for K algebraically (possibly with several branches)

$$K = G(c,d) \tag{3.10}$$

Notice that by construction G is (constrained to be) non – negative. We may therefore write the Hamiltonian constraint in the still equivalent form

$$C'^{\rm tot} = \pi + \sqrt{\det(q)}\sqrt{G(c,d)} \tag{3.11}$$

The other sign is also possible but the above choice leads to a positive physical Hamiltonian close to that of the standard model when the metric is flat (as mentioned in section (2.2) we will restrict to the subset of the constraint surface defined by (3.11) in what follows).

What is remarkable about the functions $[G(c,d)](x)$ is that they mutually Poisson commute among each other. The formal proof is as follows: The constraints (3.11) form a first class system, hence their mutual Poisson brackets is a linear combination of the constraints $C'^{\rm tot} = \pi + \sqrt{G(c,d)}, D_a^{\rm tot}$ with structure functions. However, since the π Poisson commute among themselves as well as with

$H=\sqrt{\det(q)G}$ because G does not depend on ϕ, it follows that $\{C'^{\rm tot}(x),C'^{\rm tot}(y)\}=\{H(x),H(y)\}$ does not depend on π,ϕ any more. Thus, the linear combination of constraints with structure functions, which are non – singular on the constraint surface, must conspire in such a way that the dependence of the bracket on π,ϕ drops out completely. Suppose then that

$$\{C'^{\rm tot}(N),C'^{\rm tot}(N')\}=\int d^3x[f_{N,N'}(x)C'^{\rm tot}(x)+f^a_{N,N'}(x)D^{\rm tot}_a(x)] \tag{3.12}$$

where N,N' are test functions and $C(N)=\int d^3xN(x)C(x)$. Since (3.12) does not depend on π,ϕ we may choose π such that (3.11) vanishes. Then only the second term in (3.12) survives and must no longer depend on ϕ. It follows that

$$\int d^3x[f^a_{N,N'}(x)D^{\rm tot}_a(x)]_{\pi=-\sqrt{\det(q)G}}=\int d^3x[f^a_{N,N'}(x)]_{\pi=-\sqrt{\det(q)G}}(D_a-\sqrt{\det(q)G}\phi_{,a})(x) \tag{3.13}$$

We can now expand $g^a=(f^a_{N,N'})_{\pi=-\sqrt{\det(q)G}}$ in powers of $\phi_{,a}$, that is

$$g^a=g^a_0+\sum_{n=1}^\infty g^{ab_1..b_n}_n\phi_{,b_1}..\phi_{,b_n} \tag{3.14}$$

where the coefficients are supposed to be independent of π,ϕ. The resulting recursion relation is then given by ($\pi=-\sqrt{\det(q)G}$ being understood)

$$D_ag^{ab_1..b_n}_n+\pi g^{b_1..b_n}_{n-1}=0 \tag{3.15}$$

and can be solved for instance by

$$g^{ab_1..b_n}_n=(-\frac{\pi}{h^cD_c})^{n-1}h^ah^{b_1}..h^{b_{n-1}}g^{b_n}_0 \tag{3.16}$$

where h^a is an arbitrary function such that $h^aD_a\neq 0$ and g^a_0 is also arbitrary. However, then

$$g^a=g^a_0-\frac{h^a\pi g^b_0\phi_{,b}}{h^c[D_c+\pi\phi_{,c}]} \tag{3.17}$$

is singular on the constraint surface and in fact $g^a(D_a+\pi\phi_{,a})=g^a_0D_a$ is not a linear combination of constraints.

We do not need to rely on such a formal argument: The rigorous proof is by actually computing the Poisson bracket. In [5] we find the necessary and sufficient condition for expressions of the form $H=\sqrt{\det(q)G(c,d)}$ to be mutually Poisson commuting: Consider functions of the form $H_n(Q,c,d):=Q^{n/2}h_n(c,d)$ where $Q=\det(q)$. Then, using the well known Poisson algebra generated by the Hamiltonian and spatial diffeomorphism constraints [12] one can compute the Poisson brackets between the smeared functions $H_n(N):=\int d^3xN(x)H_n(x)$ and ask for the condition on h_n such that $\{H_n(N),H_n(N')\}=0$. This leads to the following first order partial differential equation

$$\frac{n}{2}h_n\frac{\partial h_n}{\partial d}=d[\frac{\partial h_n}{\partial d}]^2-\frac{1}{4}[\frac{\partial h_n}{\partial c}]^2 \tag{3.18}$$

Dividing this equation by h_n^2 we get a PDE for $\ln(h_n)$ which one can solve by the method of separation of variables. The general solution, also called complete integral in the theory of first order PDE's, is given by the two parameter family

$$\ln[h_n(c,D;a,b)]=b+\frac{n}{4}\ln(d)+2\epsilon ac+\frac{\delta}{4}[2s+n\ln(\frac{s-n}{s+n})] \tag{3.19}$$

where $s=\sqrt{n^2+16a^2d}$ and $\epsilon,\delta=\pm 1$. The so called general integral, which depends on an arbitrary function g, is obtained by solving the equation

$$d\ln[h_n(c,d;a,b=g(a))]/da = g'(a)+2\epsilon c+\delta\frac{s(a)}{2a}=0 \tag{3.20}$$

for a in terms of c,d and to reinsert the solution $a=A(c,d)$ into h_n, that is, $h_n^g(c,d):=h_n(c,d;a=A(c,d),b=g(A(c,d)))$.

Of course, in our case we are interested in the case $n=1$ and all we have to do is to check whether $h_1(c,d)=\sqrt{G(c,d)}$ solves (3.18). The variety of solutions h_1^g is certainly bigger than those that come from a covariant Lagrangean, that is, from a given function L.

The reason for why we mention the variety h_1^g is that we will be interested in solutions h_1^g to (3.18) with special properties and instead of guessing a suitable Lagrangean it may be more constructive to select a candidate in the family h_1^g and to ask from which Lagrangean, if any, it derives. In order to answer this question, suppose we have selected a solution $h_1^g(c,d)=\sqrt{G(c,d)}$ of (3.18). Then we solve (3.10) $K=G(c,d)$ algebraically for $c=\tilde{c}(K,V)$ using again the key identity $d=V/K$. From this we infer $-\tilde{c}(K,V)=K/L'(J/2)-L(J/2)$ where $I=J(K,V)$ is the solution of $K=L'(I/2)^2(I+V)$. Hence we get

$$\tilde{c}(L'(I/2)^2(I+V),V)=L'(I/2)(I+V)-L(I/2) \tag{3.21}$$

In order that this be an ordinary first order differential equation for L the explicit dependence of (3.21) on V must cancel. In order to achieve this, one performs algebraic manipulations to (3.21) and obtains an equation which is a polynomial in V. The coefficients of all powers of V in that polynomial then have to vanish. This leads in general to more than one ordinary differential equation for L which are contradictory if L does not exist and which are equivalent if L exists. This is the necessary and sufficient condition for a given solution $\sqrt{G(c,d)}$ of (3.18) to come from a covariant Lagrangean of the form $L(I/2)$.

4 Deparametrisation of General Relativity

The two ingredients that we need here are the following properties of the functions $H(x)=\sqrt{QG(c,d)}(x)$ derived in the previous section:
1. they are mutually Poisson commuting.
2. they do not depend on π,ϕ.
3. they are densities of weight $n=1$.
This is all we need in order to show that the following object

$$O_f(\tau):=[\alpha_M(f)]_{M=\tau-\phi},\;\; \alpha_M(f)=\sum_{n=0}^{\infty}\frac{1}{n!}\{H(M),f\}_{(n)} \tag{4.1}$$

where $H(M)=\int_\sigma d^3x M(x)H(x)$ for some smearing function M, is a one parameter family of strong Dirac observables, i.e. it Poisson commutes with both the spatial diffeomorphism constraint and the Hamiltonian constraint. Due to the Poisson commutativity of $\phi(x),H(y)$, (4.1) can also be written as

$$O_f(\tau)=\alpha_{\tau-\phi}(f)=\sum_{n=0}^{\infty}\frac{1}{n!}\{H_\tau,f\}_{(n)},\;\; H_\tau:=\int_\sigma d^3x[\tau-\phi(x)]H(x) \tag{4.2}$$

The multiple Poisson bracket appearing in (4.1), (4.2) is defined iteratively by $\{H_\tau, f\}_{(0)} = f$, $\{H_\tau, f\}_{(n+1)} = \{H_\tau, \{H_\tau, f\}_{(n)}\}$.
The requirement on f is that
A. it is already spatially diffeomorphism invariant[11] and that
B. it does not depend on π, ϕ.

For the expert the invariance of $O_f(\tau)$ under the gauge motions of GR should be already obvious: First of all, by inspection, H_τ is spatially diffeomorphism invariant because $[(\tau-\phi)H](x)$ is a scalar density of weight one. Hence $O_f(\tau)$ is spatially diffeomorphism invariant. Next, let us consider the Poisson automorphism [12] $\beta_M(f) := \exp(X_{C^{\rm tot}(M)}) \cdot f$ where X_F denotes the Hamiltonian vector field of some phase space function F and $C^{\rm tot}(M) = \int_\sigma d^3x M(x) C^{\rm tot}(x)$ for some test function M. Then we have

$$\begin{aligned} \beta_M(O_f(\tau)) &= \beta_M(\alpha_{\tau-\phi}(f)) && (4.3)\\ &= \alpha_{\beta_M(\tau-\phi)}(\beta_M(f)) \\ &= \alpha_{\tau-\phi-M}(\beta_M(f)) \\ &= \alpha_{\tau-\phi-M}(\alpha_M(f)) \\ &= \alpha_{\tau-\phi}(f) = O_f(\tau) && (4.4) \end{aligned}$$

where in the second step we used $\exp(X)\exp(Y) = \exp([\exp(X) Y \exp(-Y)]\exp(Y)$ for Hamiltonian vector fields X, Y, in the third we evaluated $\beta_M(H_\tau) = H_\tau - H(M)$, in the fourth we noticed that $\beta_M = \alpha_M$ on functions which do not depend on ϕ and in the last we noticed that $H_\tau, H(M)$ Poisson commute so that $\alpha_{\tau-\phi-M} = \alpha_{\tau-\phi} \circ \alpha_M^{-1}$.

The next two subsections can therefore be skipped by the expert, for the non – expert we will supply more details about symplectic geometry and will carry out the calculations in a less elegant but more pedestrian way which has the advantage of explicitly displaying the mechanism due to which all of this works.

4.1 Invariance under Spatial Diffeomorphisms

In order to establish spatial diffeomorphism invariance, let us write (4.1) more explicitly as

$$O_f(\tau) = f + \sum_{n=1}^\infty \frac{1}{n!} \int_\sigma d^3x_1 .. \int_\sigma d^3x_n \ (\tau - \phi(x_1))..(\tau - \phi(x_n)) \{H(x_1), ..\{H(x_n), f\}..\} \qquad (4.5)$$

where we have made use of the fact that $\phi(x)$ Poisson commutes with $H(y), f$ by assumption.

We will establish a result which is stronger than mere Poisson commutativity of $O_f(\tau)$ with $D^{\rm tot}(u) = \int_\sigma d^3x u^a(x) D_a^{\rm tot}(x)$ for arbitrary vector fields u on σ: Let $s \mapsto \varphi^u_s$ be the one parameter family of spatial diffeomorphisms defined by the integral curves $c^u_x(s)$ which are the unique solutions of the system of ordinary differential equations $c^u_x(0) = x$, $\dot{c}^u_x(s) = u(c^u_x(s))$, that is: $\varphi^u_s(x) := c^u_x(s)$. We define for arbitrary functions f on phase space

$$\alpha_{\varphi^u_s}(f) := \sum_{n=0}^\infty \frac{s^n}{n!} \{D^{\rm tot}(u), f\}_{(n)} \qquad (4.6)$$

[11]Such functions are easy to construct, any integral of a scalar density of weight one constructed from the canonical variables is spatially diffeomorphism invariant.

[12]For completeness, let us mention [8, 9] that if f does depend on either π or ϕ then (4.1) must be generalised to $O_t(\tau) := [\beta_M(f)]_{M=\tau-\phi}$. For f independent of π, ϕ we have $\beta_M(f) = \alpha_M(f)$, hence we arrive at (4.1).

Obviously $(\frac{d}{ds})_{s=0}\alpha_{\varphi^u_s}(f) = \{D^{\rm tot}(u), f\}$. It follows that $s \mapsto \alpha_{\varphi^u_s}$ is a one parameter family of Poisson automorphisms[13] with the spatial diffeomorphism constraint as generator and one easily checks that $\alpha_{\varphi^u_s}(\tau - \phi(x)) = \tau - \phi(\varphi^u_s(x))$ and $\alpha_{\varphi^u_s}(H(x)) = |\det(\partial\varphi^u_s(x)/\partial x)|H(\varphi^u_s(x))$. Hence, the automorphisms generate transformations on phase space which are in agreement with the fact that under a spatial diffeomorphism $\varphi \in {\rm Diff}(\sigma)$ the function $\tau - \phi(x) \mapsto \tau - \phi(\varphi(x))$ transforms as a scalar *because* τ *is a constant*. This would not hold if τ would be a non − trivial phase space independent function. This will be important for our discussion below where we derive the relational origin of $O_f(\tau)$. On the other hand $H(x) \mapsto |\det(\partial\varphi/\partial x)|H(\varphi(x))$ transforms as a scalar density of weight one.

Since f Poisson commutes with $D^{\rm tot}(u)$ by assumption we have $\alpha_{\varphi^u_s}(f) = f$. Since $\alpha_{\varphi^u_s}$ is a Poisson automorphism it follows that

$$\begin{aligned} & \alpha_{\varphi^u_s}([\tau - \phi(x_1)]..[\tau - \phi(x_n)]\{H(x_1), ..\{H(x_n), f\}..\}) \\ = \; & \alpha_{\varphi^u_s}(\tau - \phi(x_1))..\alpha_{\varphi^u_s}(\tau - \phi(x_n))\{\alpha_{\varphi^u_s}(H(x_1)), ..\{\alpha_{\varphi^u_s}(H(x_n)), \alpha_{\varphi^u_s}(f)\}..\}) \\ = \; & |\det(\partial\varphi^u_s(x_1)/\partial x_1)|..|\det(\partial\varphi^u_s(x_n)/\partial x_n)|..[\tau - \phi(\varphi^u_s(x_1))]..[\tau - \phi(\varphi^u_s(x_n))] \times \\ & \times\{H(\varphi^u_s(x_1)), ..\{H(\varphi^u_s(x_n)), f\}..\} \end{aligned} \tag{4.7}$$

Here we have used that the Jacobean $|\det(\partial\varphi/\partial x)|$ commutes with Poisson brackets. Now invariance of (4.5) trivially follows since the Jacobean allows us to change variables under the integrals.

This holds for the *infinitesimal* diffeomorphisms which are generated by $D^{\rm tot}_a$ and which allow us to explore the component of the identity of ${\rm Diff}(\sigma)$. However, the invariance certainly extends to the full diffeomorphism group if ϕ, H transform as scalars of density weight zero and one respectively.

4.2 Invariance under the Hamiltonian Constraint

Consider the smeared Hamiltonian constraint $C'^{\rm tot}(M) = \int_\sigma d^3x M(x) C'^{\rm tot}(x)$ where $C'^{\rm tot}(x) = \pi(x) + H(x)$ and M is an arbitrary test function. Using the explicit expression for $O_f(\tau)$ in the form (4.5) we notice that there are contributions to $C'^{\rm tot}(M), O_f(\tau)\}$ from $\{C'^{\rm tot}(M), \phi(x)\} = M(x)$ and

$$\{C'^{\rm tot}(M), \{H(x_1), ..\{H(x_n), f\}..\}\} = \int d^3x M(x)\{H(x), \{H(x_1), ..\{H(x_n), f\}..\}\} \tag{4.8}$$

where we have used the fact that $f, H(x)$ and therefore $\{H(x_1), ..\{H(x_n), f\}..\}$ do not depend on ϕ. We compute

$$\begin{aligned} & \{C'^{\rm tot}(M), O_f(\tau)\} = \{H(M), f\} + \sum_{n=1}^\infty \frac{1}{n!} \int_\sigma d^3x_1 .. \int_\sigma d^3x_n \times \\ & \times[-\sum_{k=1}^n [\tau - \phi(x_1)]..[\tau - \phi(x_{k-1})] M(x_k) [\tau - \phi(x_{k+1}]..[\tau - \phi(x_n)]\{H(x_1), ..\{H(x_n), f\}..\} \\ & + \int_\sigma d^3x_0 M(x_0)[\tau - \phi(x_1)]..[\tau - \phi(x_n)]\{H(x_0), ..\{H(x_n), f\}..\}] \end{aligned} \tag{4.9}$$

The crucial Poisson commutativity of the $H(x_k)$ implies that the function $(x_1, .., x_n) \mapsto \{H(x_1), ..\{H(x_n), f\}..\}$ is completely symmetric in its arguments[14]. Thus, relabelling

[13] That is, $\{\alpha(f), \alpha(g)\} = \alpha(\{f, g\})$ and $\alpha(f + g) = \alpha(f) + \alpha(g), \alpha(fg) = \alpha(f)\alpha(g),\ \alpha(\bar{f}) = \overline{\alpha(f)}$ for arbitrary, possibly complex valued, phase space functions f, g.

[14] Proof: We have $\{H(M_1), ..\{H(M_n), f\}..\} = X_{H(M_1)} \cdot X_{H(M_2)} \cdot .. \cdot X_{H(M_n)} \cdot f$ where $H(M) = \int d^3x M(x) H(x)$ with arbitrary test functions M_k and $X_{H(M)}$ is the Hamiltonian vector field of $H(M)$ which acts on functions by

$x'_0 := x_k,\ x'_1 := x_1,\ ..,\ x'_{k-1} := x_{k-1},\ x'_k := x_{k+1},\ ..,x'_{n-1} := x_n$ we can write (4.9) in the form

$$\begin{aligned} & \{C'^{\rm tot}(M), O_f(\tau)\} = \{H(M), f\} \\ & -\sum_{n=1}^\infty \frac{1}{(n-1)!} \int_\sigma d^3x_0 .. \int_\sigma d^3x_{n-1} M(x_0)[\tau-\phi(x_1)]..[\tau-\phi(x_{n-1})]\{H(x_0), ..\{H(x_{n-1}), f\}..\} \\ & +\sum_{n=1}^\infty \frac{1}{n!} \int_\sigma d^3x_0 .. \int_\sigma M(x_0)[\tau-\phi(x_1)]..[\tau-\phi(x_n)]\{H(x_0), ..\{H(x_n), f\}..\}] \\ = \ & 0 \end{aligned} \tag{4.10}$$

which finishes the proof.

4.3 Physical Hamiltonian

We claim that

$$H := \int_\sigma d^3x H(x) \tag{4.11}$$

is a physical Hamiltonian, that is, a physical observable which generates the time evolution $\tau \mapsto O_f(\tau)$ of all physical observables.

That H is spatially diffeomorphism invariant is trivial because it is the integral of a scalar density of weight one. That it Poisson commutes with all the $C'^{\rm tot}(M)$ is also trivial because the $H(x)$ mutually Poisson commute among each other. Hence H is a physical observable.

To see that H generates the τ evolution we compute

$$\begin{aligned} & \frac{d}{d\tau} O_f(\tau) = \sum_{n=1}^\infty \frac{1}{n!} \sum_{k=1}^n \int_\sigma d^3x_1 .. \int_\sigma d^3x_n [\tau-\phi(x_1)]..[\tau-\phi(x_{k-1})][\tau-\phi(x_{k+1})]..[\tau-\phi(x_n)] \times \\ & \times\{H(x_1), ..\{H(x_n), f\}..\} \\ = \ & \sum_{n=1}^\infty \frac{1}{(n-1)!} \int_\sigma d^3x_0 .. \int_\sigma d^3x_{n-1} [\tau-\phi(x_1)]..[\tau-\phi(x_{n-1})]\{H(x_0), \{H(x-1), ..\{H(x_n), f\}..\} \\ = \ & \sum_{n=0}^\infty \frac{1}{n!} \{H, \int_\sigma d^3x_1 .. \int_\sigma d^3x_n [\tau-\phi(x_1)]..[\tau-\phi(x_n)]\{H(x-1), ..\{H(x_n), f\}..\}\} \\ = \ & \{H, O_f(\tau)\} \end{aligned} \tag{4.12}$$

where we have made use of the same manipulations as in the previous section and in the last step we used again that $\{H(x), \phi(y)\} = 0$. From (4.12) it follows in particular that

$$O_f(\tau) = \alpha_\tau(O_f),\ O_f := O_f(0),\ \alpha_\tau(F) := \sum_{n=0}^\infty \frac{\tau^n}{n!} \{H, F\}_{(n)} \tag{4.13}$$

which can be checked explicitly by expanding $O_f(\tau)$ in powers of τ. In practice one therefore computes $O_f(\tau)$ by solving $df(\tau)/d\tau = \{H, f(\tau)\}$ and then chooses the "constant" of integration to be such that $f(\tau = \phi) = f$.

$X_{H(M)} \cdot f = \{H(M), f\}$. The relation $[X_{H(M)}, X_{H(M')}] = X_{\{H(M), H(M')\}}$ holds between the Lie bracket of Hamitonian vector fields and the associated Poisson brackets. The claim now follows since $\{H(M), H(M')\} = 0$.

Finally, let us note the the identity

$$\{O_f(\tau), O_{f'}(\tau)\} = O_{\{f,f'\}}(\tau) \tag{4.14}$$

which immediately follows from the fact that $O_f(\tau)$ is the Hamiltonian flow of f generated by H_τ. Hence the physical observables satisfy a simple Poisson algebra if the f, f' do. Mathematically speaking, $f \mapsto O_f(\tau)$ is a one – parameter family of Poisson homomorphisms.

4.4 Relational Origin of the Formalism

The following section unveils the relational origin of our formalism and can be skipped by the reader merely interested in the physical application of the phantom field. On the other hand, one learns in this section why $\tau = const.$ rather than $\tau(x)$ is natural and why H is a natural Hamiltonian. We also finish this section with some cautionary remarks which list some assumptions of the formalism which were not yet explicitly mentioned.

Given a system of first class constraints C_I possibly with structure functions, suppose we find functions T_I such that the matrix A with entries $A_{IJ} := \{C_I, T_J\}$ is invertible and let $B = A^{-1}$. Consider the functions $C'_I := \sum_J B_{IJ} C_J$, fix real values τ_I in the range of T_I and let

$$F^\tau_{f,T} = \sum_{\{n_I\}=0}^{\infty} \prod_I \frac{(\tau_I - T_I)^{n_I}}{n_I!} \prod_I X_I^{n_I} \cdot f \tag{4.15}$$

where X_I denotes the Hamiltonian vector field of C'_I. One can show that the X_I are weakly commuting, hence the sequence of the application of the X_I in (4.15) is irrelevant on the constraint surface [8]. One can show that (4.15) is a weak Dirac observable, i.e. it Poisson commutes with all the C_I on the constraint surface. One can also show [9] that the evolution in τ_I has a Hamiltonian generator $H_I(\{\tau_J\})$ which is defined via $\{H_I(\{\tau_J\}), F^\tau_{f,T}\} := dF^\tau_{f,T}/d\tau_I$ for those f which have vanishing Poisson brackets with the T_I and their conjugate momenta. However, these Hamiltonians are not granted to be either positive or independent of the τ_J [9]. The physical meaning of $F^\tau_{f,T}$ is that it is the value of f in the gauge[15] when T_I assumes the value τ_I. See [8] for more details.

In General Relativity the label set of the I's takes countably infinite cardinality and there are open issues with the convergence of (4.15). In particular, the fact that (4.15) is only a weak Dirac observable and $H_I(\{\tau_J\})$ a weak Hamiltonian is mathematically rather inconvenient. Moreover, the inversion of the matrix A which is required in order to compute $F^\tau_{f,T}$ at least order by order is practically difficult for general choices of the T_I which is why it is important to supply physical input towards choosing the "clocks" T_I. Notice also that in General Relativity the C_I will involve both the spatial diffeomorphism and the Hamiltonian constraint even if f is spatially diffeomorphism invariant because the Hamiltonian constraints do not close among themselves, they are proportional to a diffeomorphism constraint. This can be circumvented when the T_I themselves are also spatially diffeomorphism invariant [8], however, it is difficult to choose an algebraically independent set of such functions which also satisfy the requirement on A and which can be considered as canonical configuration variables.

These comments reveal that (4.15) is practically difficult to handle unless one manages to simplify it drastically, in particular the matrix A should be simple. We claim that this is precisely what we managed to do in this paper: Let b_I be an orthonormal basis of $L_2(\sigma, d^3x)$ such that b_I also has finite $L_1(\mathbb{R}, d^3x)$ norm[16]. Let $x \mapsto \tau(x)$ be an arbitrary function and define $\tau_I := < b_I, \tau >$, $T_I := < b_I, \phi >$

[15] Notice that $T_I = \tau_I$ can be considered as a gauge fixing condition.

[16] For $\sigma = \mathbb{R}^3$ these could be Hermite functions.

, $\pi_I =< b_I, \pi >$, $C'_I :=< b_I, C'^{\rm tot} >$ where we assume that $\tau, \phi, \pi, C'^{\rm tot}$ have at least finite sup norm on σ. Then we find for our H_τ that $H_\tau = \sum_I(\tau_I - T_I)(C_I - \pi_I)$. Notice that $A_{IJ} = \{C'_I, T_J\} = \delta_{IJ}$ is the unit matrix so that $C_I = C'_I$. Therefore, our $O_f(\tau)$ coincides with $F^\tau_{f,T}$ up to the fact that we use $\tau = const.$ rather than arbitrary $\tau(x)$. That $F^\tau_{f,T}$ Poisson commutes (even strongly) with the C'_I is now a consequence of the fact that our C'_I do close among themselves, namely they form an Abelean subalgebra of the constraint algebra.

What is unclear however from the general relational framework is that this $F^\tau_{f,T}$ should Poisson commute with the spatial diffeomorphism constraint because formula (4.1) does not involve the spatial diffeomorphism constraint at all and the T_I are not at all spatially diffeomorphism invariant. Since $F^\tau_{f,T}$ coincides with $O_f(\tau)$ with H_τ replaced by $\int_\sigma d^3x(\tau(x) - \phi(x))H(x)$ we see that this is automatically the case *if and only if* $\tau = const.$. This explains why, for gauge invariance reasons, $\tau = const.$ is the only reasonable choice for a function and why[17] $\tau_I = \tau \int d^3x b_I$ is only a one parameter family of time evolutions rather than a "bubble time evolution". The generator of this τ evolution is then $\sum_I < 1, b_I >< b_I, H >= H(1) = H$ which now has the advantage not to have an explicit τ dependence and which is positive by our construction.

Thus we see that, by mere coincidence, the Brown – Kuchař mechanism helps to **drastically simplify the relational framework**. Rather than computing the infinite number of series (4.15) and inverting complicated infinite dimensional matrices there is only one series (4.1) and no matrix to invert. There is a distinguished notion of time generated by an invariant, positive and time independent physical Hamiltonian and the observables that we compute are strong observables, they have vanishing Poisson brackets with all constraints everywhere in the phase space, not only the constraint surface. Since there is only one series, convergence issues and, in quantum theory, operator ordering issues are much easier to settle.

Remarks:

1. There is an issue which we have not considered so far: The functions $H(M), C'^{\rm tot}(M), \pi(M)$ may not converge for $M = \tau = const.$ if σ is not bounded and/or may not be functionally differentiable if σ has a boundary. In fact, from the case of a massless Klein Gordon field which corresponds to the choice $L(I/2) = I/2$ and in case that the geometry is asymptotically flat one would assume that π and therefore necessarily H decay only as $1/r^2$ with respect to an asymptotic radial coordinate and thus H_τ would blow up. We will assume that this problem is absent by a judicious choice of phantom model with corresponding fall – off conditions for both π and H. For instance we could use compact σ without boundary and these issues would be absent. This is not sufficient for asymptically flat boundary conditions or in $k = 0, 1$ cosmology. In the model we derive in the next section this problem will be avoided automatically because the Hamiltonian turns out to have compact support, at least when the scalar field is close to homogeneous and the spatial diffeomorphism constraint holds.

2. One may worry that due to square roots which enter into the construction of $C'^{\rm tot}(M)$ its Hamiltonian vector field is singular or zero on the constraint surface. Indeed, that would be the case if one would drop the phantom field altogether and consider $H(x)$ as an alternative choice of the Hamiltonian constraint for GR and the remaining matter fields which together with the spatial diffeomorphism constraints $D_a(x)$ actually form a Lie algebra without structure functions [5]. However, we do not drop the phantom field and due to the term $\pi(x) = C^{\rm tot}(x) -$

[17] This explains why we need $||b_I||_1 < \infty$.

$H(x)$ the Hamiltonian vector field does not vanish on the constraint surface and it is not singular because $H(x)$ is not constrained to vanish, rather $\pi(x) = -H(x)$.

3. The observables that we have constructed here are strong Dirac observables with respect to the new Hamiltonian constraints $C'^{\rm tot}(x)$ but since we have used the constraints in order to transform between this constraint and the original one, the observables will be only weak Dirac observables with respect to the original Hamiltonian constraints $C^{\rm tot}(x)$.

4. We use the phantom field values $\phi(x)$ as physical clocks. Under the gauge transformations generated by the new Hamiltonian constraint $C'^{\rm tot}$ it transforms as $\delta\phi(x) = \{C'^{\rm tot}(M), \phi(x)\} = M(x)$. Since the lapse function M is required to be everywhere positive, it follows that under this unphysical time evolution $\phi(x)$ evolves *strictly monotonously* for all x. Therefore $\phi(x)$ is classically a perfect (i.e globally on phase space) clock for all x.

5 Physical Scalar Field Models: Dirac – Born – Infeld Phantom k – Essence Lagrangeans

In the previous sections we have outlined why particular scalar field models in general allow to deparametrise GR. What is left to do is to exhibit a (set of) model(s) which leads to a Hamiltonian that reduces to that of the standard model when the geometry is flat. This is the task of the present section. The analysis displayed here is far from complete and we will satisfy ourselves by finding one suitable model. More general or improved models are left for future research.

5.1 Selection of the Model

The class of models we will look at is already constrained by the necessity to be able to solve (3.4)

$$K = [L'(I/2)]^2 \; (I+V) \tag{5.1}$$

algebraically for $I = J(K,V)$. Hence (5.1) should lead to an algebraic equation at most of fourth order in I. We will restrict attention to models which only lead to quadratic equations in order to avoid the algebraic complications associated with Cardano's and Ferrari's formulas for cubic and quartic equations resepctively. This restricts us to functions L that satisfy

$$[L'(I/2)]^2 = \frac{a+bI}{\delta+\xi I+\zeta I^2} \tag{5.2}$$

where a, b, δ, ξ, ζ are real constants such that the square roots that enter the integral L of (5.2), when evaluated at the solution $I = J$, of (5.1) have positive arguments.

The integral of (5.2) can be carried out for all values of a, b, δ, ξ, ζ but in general involves complicated inverse trigonometric functions and logarithms. This would be bad because we also need to solve the Hamiltonian constraint for K later on and such functions would lead to transcendental equations which cannot be solved algebraically. In order to avoid transcendental equations we have to specialise (5.2) to one of the following cases:

$$\text{i.} \quad [L'(I/2)]^2 = \frac{b^2}{4[a+bI/2]} \tag{5.3}$$

$$\text{ii.} \quad [L'(I/2)]^2 = a^2$$

$$\text{iii.} \quad [L'(I/2)]^2 = (\frac{3}{2}b)^2[a+bI/2] \tag{5.4}$$

which are readily integrated to

$$\begin{array}{lll} \text{i.} & L(I/2)=-(\beta+\epsilon\sqrt{a+bI/2}) & (5.5) \\ \text{ii.} & L(I/2)=-\beta+aI/2 & \\ \text{iii.} & L(I/2)=-(\beta+\epsilon\sqrt{a+bI/2}^3) & (5.6) \end{array}$$

where a, b, β are real constants on which we will impose some restrictions in what follows and $\epsilon=\pm1$. Remarkably, the Lagrangean i. in (5.5) is precisely of the form of a Dirac – Born – Infeld type of phantom field with constant potential if we set $\epsilon=-1$, see e.g. the first reference in [6] and references therein.

Let us first discuss case i. which turns out to be the right choice. Equation (5.1) now leads to

$$K=\frac{b^2(I+V)}{4(a+bI/2)} \quad\Rightarrow\quad J(K,V)=\frac{4aK-b^2V}{b^2-2Kb} \tag{5.7}$$

It follows that

$$a+bJ/2=\frac{a-bV/2}{1-2K/b} \tag{5.8}$$

which should be positive independently of the range of both K, V. Since K, V are manifestly positive, in order that the square root in (5.5) is well defined (real valued) and in order that both sides of equation (5.3) are positive we are forced to choose $a>0,\ b<0$. It is remarkable that this is possible although J can take either sign, in fact J is not bounded from below, however, it is bounded from above by $|2a/|b|$!

It follows that

$$\begin{aligned} \frac{C^{\text{phantom}}}{\sqrt{\det(q)}} &= \frac{K}{L'(J/2)}-L(J/2) \\ &= \epsilon\sqrt{a+Jb/2}[1-2K/b]+\beta \\ &= \epsilon\sqrt{[a+Jb/2][1-2K/b]^2}+\beta \\ &= \epsilon\sqrt{[a-bV/2][1-2K/b]}+\beta \\ &= -\frac{C}{\sqrt{\det(q)}}=-c \end{aligned} \tag{5.9}$$

where the last equality holds when the Hamiltonian constraint is satisfied.

Using the Brown – Kuchař key identity $V=d/K$ which holds when the spatial diffeomorphism constraint holds and taking the square of (5.9) we find

$$(c+\beta)^2=a+d-\frac{bd}{2K}-\frac{2aK}{b} \tag{5.10}$$

This leads to a quadratic equation for K which we can write as

$$(K-A)^2=A^2-B,\ \ A:=-\frac{b}{4a}[(c+\beta)^2-d-a],\ B:=d\frac{b^2}{4a} \tag{5.11}$$

Since the left hand side of (5.11) is positive, the right hand side of (5.11) is constrained to be positive as well. In order to to extend (5.11) off the constraint surface we write the two roots of (5.11) as[18]

$$K=A\pm\sqrt{\frac{1}{2}(A^2-B+|A^2-B|)} \tag{5.12}$$

[18]Lemma: For real numbers x, y the equation $x^2=y$ implies $x^2=(y+|y|)/2$.
Proof: If $x^2=y$ holds then $y\geq0$ so the second identity follows. The second identity implies the first for $y\geq0$, however it implies $x=0$ for $y<0$.

Since K does not automatically vanish when $B = 0$, only the positive sign in (5.12) is meaningful. Moreover, since K is manifestly positive, also the right hand side is constrained to be positive. We extend (5.12) off the constraint surface by

$$K = \frac{1}{2}\left[A + \sqrt{\frac{1}{2}(A^2 - B + |A^2 - B|)} + \left|A + \sqrt{\frac{1}{2}(A^2 - B + |A^2 - B|)}\right|\right] \tag{5.13}$$

which is now manifestly positive. If we want to avoid absolute values then we can write the solution explicitly in the form

$$K = G(c, d) := -\frac{b}{4a}[(c + \beta)^2 - a - d] + \sqrt{\{-\frac{b}{4a}[(c + \beta)^2 - a - d]\}^2 - \frac{d}{a}(b/2)^2} \tag{5.14}$$

This means in particular that we get the conditions $A^2 \geq B, (c+\beta)^2 \geq a+d$ (recall that $a > 0,\ b < 0$). Hence also $(c + \beta)^2 - (a + d) \geq 2\sqrt{a}\sqrt{d}$, i.e.

$$|c + \beta| \geq \sqrt{d} + \sqrt{a} \tag{5.15}$$

This condition, if imposed at all $x \in \sigma$, is gauge invariant and invariant under the physical evolution. To see this, notice that all quantities in (5.15) are spatial scalars, therefore (5.15) at x is mapped to $\varphi(x)$ where it also holds by assumption. Next, all $H(x),\ x \in \sigma$ Poisson commute with $H(M)$ for all M and with $H = H(M)_{M=1}$ in particular. Thus, if $H(x)^2 = [\det(q)G(c, D)](x)$ is positive and meaningful for all $x \in \sigma$ then it will be in every gauge and under the physical time evolution. Thus, the extension off the constraint surface (5.13) which displays H in manifestly positive form is only required fo the purpose of quantisation. For classical purposes, (5.14) is completely sufficient since anyway we are only interested in the portion of phase space where (5.15) holds.

Notice that for spatially homogeneous phantom fields ϕ the function d is constrained to vanish. Hence, for $d = 0$ (5.14) reduces to

$$K = \frac{|b|}{4a}[(c + \beta)^2 - a] + |(c + \beta)^2 - a|] \tag{5.16}$$

which is manifestly non negative and now there are no conditions on c, d, β, a. We want this expression to coincide with c^2 for large c in order that the Hamiltonian density equals $H(x) = C(x) = (\sqrt{\det(q)}c)(x)$ in this limit. This enforces $b \approx -2a$ and $|\beta|, a$ small. Specifically we could set $\beta^2 = a$ in oder to remove the constant term in (5.16) and $b = -2a$ in order that the coefficient of c^2 equals unity. We may choose a as small as we want in order to suppress the term linear in c in (5.16). For those values the Lagrangean becomes simply $L = -\sqrt{a}[\epsilon\sqrt{1 - I} + \delta]$ where $\delta = \pm 1$. In case that σ is not compact it makes sense to choose $\beta^2 < a$ (e.g. $\beta = 0$) because then (5.16) vanishes in regions where $(c + \beta)^2 \leq a$. Hence, whatever the fall – off conditions of the fields are at infinity, as long as they fall off at all, the support of (5.16) will be compact, at least when the diffeomorphism constraint holds and when ϕ is homogeneous. Therefore, our model can be used, under the assumptions made, also if σ is not compact and/or has a boundary. For other values of β, a and in particular if there is a cosmological constant present, then the action converges anyway only if σ is compact and we then also require $\partial\sigma = \emptyset$ in order to avoid boundary terms.

Once we have chosen $b = -2a, \beta = 0$, (5.14) reduces to

$$K = \frac{1}{2}[c^2 - a - d] \pm \sqrt{\{\frac{1}{2}[c^2 - a - d]\}^2 - ad} \tag{5.17}$$

It is then tempting to perform the limit $a \to 0$ so that (5.17) becomes

$$K = \frac{1}{2}([c^2 - d] \pm |c^2 - d|) \tag{5.18}$$

which has the advantage to be manifestly positive even for $d \neq 0$. Since for $b = -2a = \beta = 0$ the Lagrangean from which (5.14) was derived vanishes identically, (5.18) must come from a different Lagrangean. Indeed, it comes from the constrained, incompressible dust Lagrangean introduced by Brown and Kuchař in their original work [1]. Unfortunately, it is not of the form $L(I/2)$, rather $L = -\frac{1}{2}\lambda(1-I)$ where λ is a Lagrange multiplier field[19]. The canonical formulation of this action leads to the contribution to the Hamiltonian constraint given by $C^{\text{dust}} = \frac{1}{2}[\pi^2/(\lambda\sqrt{\det(q)})+(1+V)\sqrt{\det(q)}\lambda]$ and elimination of λ by its equation of motion results in $C^{\text{dust}} = \sqrt{\pi^2(1+V)}$. One can show explicitly by solving (5.18) for c in order to obtain C^{phantom} that (5.18) cannot come from a Lagrangean of the form $L(I/2)$ by applying the procedure for the inverse problem mentioned at the end of section 3.2. We therefore prefer scalar matter (5.14) over the dust matter (5.18) which we feel to be awkward due to the Lagrange multiplier λ. What is awkward about this action is that the Euler – Lagrange equations for λ require $I = 1$ and do not allow to solve for λ while after Legendre transformation the constraint equation for λ only can be solved by eliminating λ[20]. and to forget about $I = 1$.

Let us now consider the other cases. First of all, while a can be as small as we want in order to suppress unwanted terms in (5.14), the case $a = 0$ is singular. The case $a = 0$, that is, $L(I/2) = \sqrt{I/2}$ or $L(I/2) = \sqrt{I/2}^3$ has to be treated independently. For $L(I/2) = \sqrt{I/2}$ we obtain an expression for the physical Hamiltonian which contains negative powers of c, d which do not resemble the usual Hamiltonian. For $L(I/2) = \sqrt{I/2}^{3/2}$ as well as for case iii. in (5.3) we are driven to an algebraic equation of ninth order for K which can no longer be solved by quadratures, hence this model is ruled out for purely mathematical reasons. Finally, the same analysis carried out for case ii. in (5.3) leads to $K = -c \pm \sqrt{c^2 - d}$ which should be positive, hence c is constrained to be negative. Thus $C^{\text{tot}} = \pi + \sqrt{\det(q)}\sqrt{-c + \sqrt{c^2 - d}}$ and the Hamiltonian would be $H(x) = \sqrt{\det(q)}\sqrt{-c + \sqrt{c^2 - d}}(x)$ which for small d becomes $\sqrt{2\det(q)|c|}$. Thus, the Klein Gordon Lagrangen produces the square root of c which is also not what we want.

Hence the only suitable model corresponds to case i. and we now proceed to explore its properties.

5.2 Physical Properties of the Model

Let us first check that $h_1(c,d) := \sqrt{G(c,d)}$ of (5.14) satisfies the PDE (3.18). Since the calculation is not entirely trivial we display here some intermediate steps for the convenience of the reader. To simplify the computation we notice that $\sqrt{G}$ satisfies (3.18) with $n = 1$ if and only if G satisfies

[19] In their work [1] they actually used $U_\mu = T_{,\mu} + W_a X^a_{,\mu}$ instead of $\Phi_{,\mu}$ where W_a, X^a are additional six scalar fields and defined $I = -g^{\mu\nu}U_\mu U_\nu$. Then $\Phi := T$ is a local clock field and $X^a,\ a = 1,2,3$ are local position fields and together they are assumed to provide a local coordinate system of M. The functions W_a are also Lagrange multipliers. We simply set $X^a, W_a = 0$ here.

[20] If we compare this to the string, then this would be as if we would go from the covariant Polyakov string action to the Nabu – Goto string action by eliminating the worldsheet metric (the analog of λ here) by its equation of motion. However, both string actions are worlsheet covariant while this is not the case here.

(3.18) with $n = 2$. We find

$$\frac{\partial G}{\partial d} = \frac{b}{4aR}(G - b/2) \quad (5.19)$$
$$\frac{\partial G}{\partial c} = -\frac{b}{2aR}G(c+\beta) \quad (5.20)$$

where $R := \sqrt{A^2 - B}$, $A := b(a + d - (c+\beta)^2)/4a$, $B = (b/2)^2 d/a$. Then (3.18) with $n = 2$ and $h_2 := G$ becomes

$$\begin{aligned}
& d[\frac{\partial G}{\partial d}]^2 - \frac{1}{4}[\frac{\partial G}{\partial c}]^2 - G[\frac{\partial G}{\partial d}] && (5.21)\\
= & (\frac{b}{4aR})^2[(d(G-b/2) - 4aRG/b)(G-b/2) - G^2(c+\beta)^2] \\
= & (\frac{b}{4aR})^2[\frac{4a}{b}G^2(\frac{b}{4a}[d-(c+\beta)^2] - R) + G(2aR - bd) + (b/2)^2 d] \\
= & (\frac{b}{4aR})^2[\frac{4a}{b}G(A+R)(A-R) - aG^2 + G(2aR - bd) + (b/2)^2 d] \\
= & (\frac{b}{4aR})^2[\frac{4a}{b}GB - aG^2 + G(2aR - bd) + (b/2)^2 d] \\
= & (\frac{b}{4aR})^2[-aG^2 + 2aRG + (b/2)^2 d] \\
= & a(\frac{b}{4aR})^2[-G^2 + 2RG + B] \\
= & a(\frac{b}{4aR})^2[G(R + [R-G]) + B] \\
= & a(\frac{b}{4aR})^2[(R+A)(R-A) + B] \\
= & 0 && (5.22)
\end{aligned}$$

which is what we wanted to show.

Next, in order to understand the meaning of ϵ and β or δ, we compute the equation of state of the model. The energy momentum tensor with our signature convention is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{|\det(g)|}} \frac{\partial \sqrt{|\det(g)|}L(I/2)}{\partial g^{\mu\nu}} = -[g_{\mu\nu}(\beta + \epsilon\sqrt{a + Ib/2}) + \frac{\epsilon b}{2}\frac{\Phi_{,\mu}\Phi_{,\nu}}{\sqrt{a + bI/2}}] \quad (5.23)$$

Energy density and pressure become in the perfect fluid approximation $T_{\mu\nu} = \rho n_\mu n_\nu + p(g_{\mu\nu} + n_\mu n_\nu)$ (with respect to our unit timelike vector field n normal to the foliation introduced in section 3)

$$\rho = T_{\mu\nu}n^\mu n^\nu = \beta + \epsilon\sqrt{(a - bV/2)(1 - 2K/b)} \quad (5.24)$$
$$p = \frac{1}{3}(\rho + g^{\mu\nu}T_{\mu\nu}) = -\frac{1}{3}(3(\beta + \epsilon) + \frac{\epsilon b}{2}\frac{V}{\sqrt{\frac{a - bV/2}{1 - 2K/b}}}) \quad (5.25)$$

The equation of state "field" is defined by $w := p/\rho$ and becomes for spatially homogeneous ϕ for which $V = 0$

$$w(y) = -\frac{\beta + \epsilon\sqrt{a}/y}{\beta + \epsilon\sqrt{a}y} \quad (5.26)$$

where $y = \sqrt{1 - 2K/b} \geq 1$. For $\beta = 0$ we get $w = -1/y^2$ i.e $-1 \leq w \leq 0$ independent of ϵ, a. For $\beta \cdot \epsilon > 0$ we also get $-1 \leq w \leq 0$. For $\beta \cdot \epsilon < 0$ let $e = |\beta|/\sqrt{a}$. If $e > 1$ then again $-1 \leq w \leq 0$. For $e < 1$, $w(y)$ has a maximum at $y_e = 1/e + \sqrt{1/e^2 - 1} > 1$ given by $w(y_e) = (1/y_e)^2 < 1$ and $w(1) = -1,\ w(\infty) = 0$ hence $-1 \leq w \leq 1$. Finally for $e = 1$ we get $0 \leq w(y) = 1/y \leq 1$.

Next, the speed of sound, for spatially homogeneous ϕ is given by

$$c_s^2 := \frac{\partial p(\rho)}{\partial \rho} = \frac{\partial p(y)/\partial y}{\partial \rho(y)/\partial y} = +\frac{1}{y^2} > 0 \tag{5.27}$$

independently of $\beta, a\epsilon$

Now if ϕ would be the only observable scalar field then we would need $\rho \geq 0,\ c_s^2 > 0$ for stability reasons, hence $\epsilon = +1$ in order that there are no restrictions on the range of K, V and $\beta \geq -\sqrt{a}$ i.e. $e \leq 1$. Then in order to get inflation and the recent accelarated expansion ($w < -1/3$) of the universe (dark energy) we must choose either $\beta \geq 0$ or at least $-\sqrt{a} < \beta$. Since, however, the phantom field is pure gauge and has other purposes, it will not be associated with the physical inflaton and/or dark energy. Thus we keep the ranges of ϵ, a, b, β unrestricted up to the requirement that $a > 0, b \approx -2a; \beta^2$ and a small. We will see however, that the physical evolution equations of the next section still impose further restrictions.

Let us set $b = -2a$ for definiteness. Then we conclude that we obtain the two parameter set of Dirac – Born – Infeld type Lagrangeans

$$L = \sqrt{|\det(g)|}(-\beta + \alpha\sqrt{1 + g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}}) = \alpha\sqrt{|\det(g)|}(-1 + \sqrt{1 + g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}}) + (\alpha - \beta)\sqrt{|\det(g)|} \tag{5.28}$$

where $\alpha := -\epsilon\sqrt{a} \neq 0$ is a real number with $|\alpha|$ small and β is any real number with $|\beta|$ small. For small I the first term in (5.28) becomes to linear order $\alpha\sqrt{|\det(g)|}g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}/2$ which up to the constant α is just the usual Lagrangean for a massless Klein Gordon field. Thus, to this order the Lagrangean has the correct sign in front of the kinetic term for $\alpha < 0$ and it becomes a phantom in the cosmological sense for $\alpha > 0$. The second term represents a contribution by $\alpha - \beta$ to the cosmological constant which is a positive contribution for $\alpha - \beta$ negative. Since the cosmological term can always be absorbed into the contribution C to the Hamiltonian constraint by gravity and the remaining (physical) matter, a natural choice would be $\beta = \alpha$ which would then imply $e = 1$ hence $0 \leq w \leq 1$.

Summarising, we have found a simple scalar field model which in the physical situation of interest, that is, a roughly homogeneous phantom field in order that $\phi(x) \approx \tau = const.$ is a good approximation for a physical clock, gives rise to a satisfactory physical Hamiltonian. It decays sufficiently fast at spatial infinity (in fact has compact support) when simultaneously the spatial diffeomorphism constraint holds. Decay requirements can be avoided if there is a non – vanishing cosmological constant so that σ is compact and we impose also $\partial\sigma = \emptyset$ in this case to avoid boundary terms.

It would be interesting to improve the model in order that it is applicable also in situations where σ is bounded and has a boundary. Also, for mathematical reasons in particular in view of quantisation (see the next section) it would be more convenient to have a manifestly positive Hamiltonian. A possible starting point is the manifestly positive extension (5.13) off the positivity constraint surface. Finally it might be possible to find a model such that H approximates $C(N = 1)$ even when V is not close to zero. One way to investigate this would be to solve the inverse problem mentioned at the end or section 3.2. We leave this to future research.

6 Consequences for Cosmology

The Hamiltonian constraint for our model is given by
$C'^{\rm tot}(x) = \pi(x) + \sqrt{\det(q)G(c,d)}(x) =: \pi(x) + H(x)$ and we see that the Hamiltonian becomes for small V and large $c(x)^2 \gg \alpha^2, \beta^2$ approximately $H = \int_\sigma d^3x C(x)$ which is just the integrated contribution, of the gravitational and non − phantom like matter degrees of freedom, to the original Hamiltonian constraint. It would result from the canonical Hamiltonian by choosing the lapse to equal unity, the shift to equal zero and by dropping the phantom field contribution from $C^{\rm tot}$, $D_a^{\rm tot}$. This explains why in the presence of our particular phantom field model chosen, evolution with unit lapse and zero shift with respect to the canonical, original "Hamiltonian" $H^{\rm canon}(N,\vec{N}) = C^{\rm tot}(N) + D^{\rm tot}(\vec{N})$ of functions on phase space not involving ϕ *approximately* equals the physical evolution of the non − phantom like degrees of freedom.

We will now illustrate the meaning of "approximately" in the context of isotropic and homogeneous minisuperspace models, that is, FRW cosmology. The presence of the phantom field, while not directly observable, will still have an important impact on the conceptual (interpretation) and technical (matter content) aspects of the FRW equations as well as on their validity. Namely, we will see that the FRW equations are only an approximation to the actual physical time evolution of observable quantities generated by the physical Hamiltonian. This also serves to explain the formalism in a simple context without the field theoretic complications[21].

We consider the experimentally almost confirmed case of a spatially flat ($k=0$) model for which the usual FRW line element is given by $ds^2 = -dt^2 + a(t)^2\delta_{ab}dx^a dx^b$ where a is called the scale factor. Comparing with the general ADM line element $ds^2 = -[N^2 + q_{ab}N^aN^b]dt^2 + 2q_{ab}N^b dx^a + q_{ab}dx^a dx^b$ we read off $N=1,\ N^a=0,\ q_{ab} = a(t)^2\delta_{ab} =: Q\delta_{ab}$. Here we work with dimensionless spatial coordinates x^a so that a has dimension cm while the unphysical time (or foliation parameter) t has dimension cm. We also take our scalar field to have dimension cm so that $I = (d\phi/dt)^2$ is dimensionless.

We begin by specialising the canonical formulation of GR to isotropic and homogeneous situations. The extrinsic curvature $K_{ab} = (\dot{q}_{ab} - \mathcal{L}_{\vec{N}}q_{ab})/(2N)$ where $\mathcal{L}$ denotes the Lie derivative reduces to $K_{ab} = a\ da/dt\delta_{ab}$, hence the momentum $P^{ab} = \frac{1}{2}\sqrt{\det(q)}[q^{ac}q^{bd} - q^{ab}q^{cd}]K_{cd}$ conjugate[22] to q_{ab} becomes $P^{ab} = -\dot{a}\delta^{ab} =: P_Q\delta^{ab}/3$. Here Q, P_Q are canonically conjugate. The canonical transformation from $(Q = a^2, P_Q)$ to $(a, P = 2aP_Q)$ reveals $P = -6a\ da/dt$. The spatial diffeomorphism constraint vanishes identically if spatial homogeneity is assumed and the contribution to the Hamiltonian constraint of the gravitational degrees of freedom and the cosmological term becomes

$$C^{\rm grav} = \frac{1}{2}(\sqrt{\det(q)}[(q^{ac}q^{bd} - q^{ab}q^{cd})K_{ab}K_{cd} - R^{(3)}] + 2\Lambda\sqrt{\det(q)}) = -\frac{P^2}{12a} + \Lambda a^3 \tag{6.1}$$

where $R^{(3)}$ is the curvature scalar of q_{ab} which vanishes identically.

For ordinary matter we will make as usual a perfect fluid Ansatz for the energy momentum tensor $T_{\mu\nu} = \rho_m n_\mu n_\nu + p_m(g_{\mu\nu} + n_\mu n_\nu)$ with $n^\mu = \delta^t_\mu$ where energy density ρ_m and pressure are related by $-3a^2 p_m = d(\rho_m a^3)/da$. For instance for a Klein - Gordon field $\rho_m = (\pi_m^2/a^6 + U)/2$ where U is its potential and thus $p_m = (\pi_m^2/a^6 - U)$. Since in general $c^{\rm grav} = -2[G_{\mu\nu} + \frac{1}{2}\Lambda g_{\mu\nu}]n^\mu n^\nu$ we find that the contribution to the Hamiltonian constraint of gravity and non − phantom matter reads in general

[21] Notice that as usual we quotient all equations by the infinite coordinate volume $\int d^3x$ in the $k = 0, -1$ models.

[22] Recall that the gravitational and cosmological action are multiplied by $1/(16\pi G) = 1/2$ which explains the factor 1/2 in front here.

$C = C^{\rm grav} + a^3\rho_m$. If there is a phantom present, as we advertised in the present paper then the total Hamiltonian constraint reads $C^{\rm tot} = C + a^3\rho^{\rm phantom} = C + C^{\rm phantom}$.

As we showed in section (5.1) the phantom field contribution to the Hamiltonian constraint is given by (remember $b = -2a,\ V = 0$)

$$\rho^{\rm phantom} = C^{\rm phantom}/a^3 = \beta - \epsilon\sqrt{K+\alpha^2} = \beta - \epsilon\sqrt{\pi^2_{\rm phantom}/a^6 + \alpha^2} =: \beta - \epsilon\alpha\sqrt{1+x} \tag{6.2}$$

where we now take $\alpha > 0,\ \epsilon = \pm 1$ and have introduced the "deviation parameter"

$$x := \frac{E^2}{\alpha^2 a^6} \tag{6.3}$$

which will be crucial for what follows. We noticed that the gauge invariant quantity $\pi_{\rm phantom}$ is also a constant of the physical motion because the physical Hamiltonian $H = -\pi_{\rm phantom}$ does not involve ϕ (it is cyclic) and therefore we have denoted the energy squared constant of motion by $E^2 := \pi^2_{\rm phantom}$. Since (6.2) has the same dimension as the cosmological constant Λ we see that α, β have dimension cm^{-2}, π, E have dimension cm and x is dimensionless. The phantom pressure is given by

$$p_{\rm phantom} = -\frac{1}{3a^2} d(a^3\rho_{\rm phantom})/da = -\beta + \epsilon\alpha^2 a^3/\sqrt{E^2 + \alpha^2 a^6} = -\beta + \epsilon\frac{\alpha}{\sqrt{1+x}} \tag{6.4}$$

This gives an equation of state

$$w_{\rm phantom} = \frac{p_{\rm phantom}}{\rho_{\rm phantom}} = -\frac{\beta - \epsilon\alpha/y}{\beta + \epsilon\alpha y} \tag{6.5}$$

with $y = \sqrt{1+x}$. The phantom speed of sound is given by

$$c^2_{\rm phantom} = \frac{dp_{\rm phantom}/dy}{d\rho_{\rm phantom}/dy} = \frac{1}{y^2} \tag{6.6}$$

which is always positive independently of β, α, ϵ.

The canonical "Hamiltonian" of the theory without the phantom is $H^{\rm canon} = C$. Without the phantom, however, it is constrained to vanish and therefore should not be interpreted as a Hamiltonian, rather as a Hamiltonian constraint which generates gauge transformations and not any observable evolution. With the phantom, C is not constrained to vanish. The *physical Hamiltonian*, with the phantom present then follows from (5.16) with $b = -2a$ (remember $d = 0$ identically in exactly homogeneous cosmology)

$$H = a^3\sqrt{\frac{1}{2}[(c+\beta)^2 - \alpha^2] + |(c+\beta)^2 - \alpha^2|]} \tag{6.7}$$

which approaches $|C| = a^3|c|$ for $|c| \gg |\beta|, \alpha$. It vanishes identically when $|c + \beta| \le \alpha$ (this never happens on the constraint surface). Hence, as long as $c > 0$ and $c \gg |\beta|, \alpha$ the physical Hamiltonian H, in presence of the phantom, is in good agreement with the canonical Hamiltonian constraint, in absence of the phantom.

We will now derive the FRW equations from the canonical formalism and compare them with the evolution equations of physical observables. We begin with the standard FRW equations which consist of a set of two equations. The first one is just the constraint equation $C^{\rm tot} = 0$ with P eliminated by the equation of motion for a which gives a condition on $(da/dt)^2$. The second one involves d^2a/dt^2 and is obtained by eliminating dp/dt by its equation of motion. Notice that by "equation of motion" we mean actually gauge transformations generated by $C^{\rm tot}$. On the other hand, we can compute evolution equations generated by the physical Hamiltonian H.

1. *Gauge Transformations generated by C^{tot}: Standard FRW Equations*
   The gauge transformation for a is

$$\frac{da}{dt} := \{C^{\text{tot}}, a\} = \frac{\partial C}{\partial p} = -\frac{P}{6a} \tag{6.8}$$

   Inserting (6.8) into the constraint equation $C^{\text{tot}} = 0$ we therefore find the first one of the FRW equations

$$3(\frac{da}{dt})^2/a^2 = \Lambda + \rho_m + \rho_{\text{phantom}} \tag{6.9}$$

   The second equation is obtained by solving (6.9) for da/dt, taking the second derivative and using the conservation law $d\rho/dt + 3(\rho + p)da/dt = 0$ which gives

$$3\frac{d^2a}{dt^2}/a = \Lambda - \frac{1}{2}(\rho_m + 3p_m + \rho_{\text{phantom}} + 3p_{\text{phantom}}) \tag{6.10}$$

   This is the second FRW equation for spatial curvature $k = 0$ and using units for which $8\pi G = 1$.

2. *Physical Time Evolution generated by H: Modified FRW equations*
   We can now compute the time evolution of physical observables. The general formula (4.2) specialises to

$$O_f(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!}\{H_\tau, f\}_{(n)} = \sum_{n=0}^{\infty} \frac{(\tau - \phi)^n}{n!}\{H, f\}_{(n)}, \quad H_\tau = (\tau - \phi)H \tag{6.11}$$

   with

$$H = \sqrt{(C + \beta a^3)^2 - \alpha^2 a^6} \tag{6.12}$$

   and f can be any function on the cosmological minisuperspace phase space independent of π, ϕ because the spatial diffeomorphism constraint vanishes identically. Obviously, just as in the full theory $dO_f(\tau) = \{H, O_f(\tau)\} = O_{\{H,f\}}(\tau)$. We now see that

$$O_f(\tau) \approx f(\tau - \phi) \equiv f(t), \quad t := \tau - \phi \tag{6.13}$$

   where $t \mapsto f(t),\ f(0) = f$ is the solution of the evolution equation for f without the phantom when treating the Hamiltonian constraint C as a Hamiltonian. The unphysical time parameter t of that unphysical time evolution is now interpreted as $t = \tau - \phi$. Hence all the cosmological evolution equations remain approximately intact (under the restrictions on c made above), however, we now have justified why that evolution corresponds to observation and we have interpreted the time parameter t as composed of the pure gauge phantom time ϕ and the physical time parameter τ.

   Thus, the phantom has nicely reconciled the mathematical framework (gauge theory) with observation (FRW equations). It is pure gauge and one would be tempted to conclude that its presence does not have any observational consequences beyond modifying the Hamiltonian constraint. This is of course wrong because, at least in our model, we cannot have $C \equiv H$ just $C \approx H$ and we now proceed to compute the associated modifications. Hence, what we need to do is to repeat the steps (6.8) – (6.10) where the first FRW equation $C^{\text{tot}} = C + a^3\rho_{\text{phantom}} = 0$ must be expressed in terms of observable quantities and instead of the gauge transformation

$d/dt(.) = \{C^{\text{tot}}, .\}$ we now have actual physical evolution $d/d\tau(.) = \{H, .\}$. The physical evolution equation for the observable O_a corresponding to a is[23]

$$\frac{dO_a}{d\tau} = \{H, O_a\} = O_{\{H,a\}} = O_{\frac{\partial H}{\partial C}\{C,a\}} = -O_{\frac{\partial H}{\partial C}} O_P/(6O_a) \tag{6.14}$$

Here

$$\frac{\partial H}{\partial C} = \frac{C + \beta a^3}{H} = \epsilon \frac{\sqrt{E^2 + \alpha^2 a^6}}{E} = \epsilon\sqrt{1 + 1/x} \tag{6.15}$$

Using (6.14) in $O_{C^{\text{tot}}} = 0$ we find the first modified FRW equation[24]

$$\frac{3}{O_a^2}(\frac{dO_a}{d\tau})^2 = (1 + \frac{1}{O_x})[\Lambda + O_{\rho_m} + O_{\rho_{\text{phantom}}}] \tag{6.16}$$

The second again follows with the definition of pressure by taking the second physical time derivative of O_a

$$\frac{3}{O_a}\frac{d^2O_a}{d\tau^2} = (1 + \frac{4}{O_x})\Lambda - \frac{1}{2}\{[O_{\rho_m} + O_{\rho_{\text{phantom}}}](1 - \frac{5}{O_x}) + 3[O_{p_m} + O_{p_{\text{phantom}}}](1 + \frac{1}{O_x}))\} \tag{6.17}$$

Notice that $O_x = E^2/(\alpha^2 O_a^6)$. We now interpret (6.16) and (6.17): For $O_x \to \infty$ they look exactly like standard FRW equations with an additional phantom matter component. In fact, at $O_x = \infty$ these two equations are identical with (6.9) and (6.10) under the substitution $(t, a(t)) \to (\tau, O_a(\tau))$. That additional matter component behaves at early times ($O_a \to 0$) of the universe ($O_x \to \infty$) as

$$\rho_{\text{phantom}} \to -\epsilon E/a^3, \;\; p_{\text{phantom}} \to -\beta \tag{6.18}$$

Hence it behaves like dust matter for $\epsilon = -1$ as $O_a \to 0$ provided we set $\beta = 0$. It therefore could serve as a dark matter candidate then. At later times, when $O_a \to \infty$ or $O_x \to 0$ we get

$$\rho_{\text{phantom}} \to \beta - \epsilon\alpha, \; p_{\text{phantom}} \to -\beta + \alpha\epsilon \tag{6.19}$$

so it behaves like a positive/negative cosmological constant for $\epsilon = -1$ or $\epsilon = +1$ respectively. In order to get a positive contribution to the cosmological constant and to retain these interpretations, we thus should choose $\beta = 0, \epsilon = -1$ which is exactly of the form used in the model [4].

However, our purposes are different here: The phantom field is there in order to deparametrise the theory and to provide a positive, physical Hamiltonian which approximates C since C is used in the standard model with flat space. Since $C^{\text{tot}} = C + \rho_{\text{phantom}} = 0$ we are forced to take $\rho_{\text{phantom}} < 0$ in order to have $C > 0$. This excludes the $\epsilon = -1$ case[25] and we must use $\epsilon = +1$. Hence, the phantom has negative energy, positive pressure and behaves like

[23] The convenient observation here is that the map $f \mapsto O_f$ is an automorphism. In particular if $f = F(a, P, \phi, \pi)$ for some function F then $O_f = F(O_a, O_P, \tau, \pi)$.

[24] $O_{C^{\text{tot}}} = C^{\text{tot}} = 0$ follows because C^{tot} is already gauge invariant. Here we need to use the generalisation of the map $f \mapsto O_f$ mentioned in section 4.

[25] More precisely we could take $C'^{\text{tot}} = \pi \pm \sqrt{C^2 - \alpha^2 a^6}$, i.e. $H = \mp\sqrt{C^2 - \alpha^2 a^6} \approx \pm|C|$ for small a and α. In order that this approximates C we shoulld have $C < 0$ or $C > 0$ respectively which requires $\rho_{\text{phantom}} > 0$ or $\rho_{\text{phantom}} < 0$ respectively. However, we should have $\rho_{\text{phantom}} < 0$ in order to avoid a big rip singularity as we will show below. Also $C > 0$ in flat space.

negative energy dust in the early universe while it apporaches a negative cosmological constant in the late universe. Since the gravitational contribution to C is negative, we must have $\rho_m + \rho_{\text{phantom}} > 0$ in order that $C^{\text{tot}} = 0$. Hence the total energy density of matter is positive which is sufficient in order to obtain a stable theory, that is, the usual energy conditions are satisfied [12]. In fact the phantom can be compensated for by a k – essence field with positive energy [4] or by ordinary positive energy dust matter and a positive cosmological constant term.

The reasoning here would then be as follows:
1. Something like a phantom is needed in order to deparametrise the theory and to keep validity of standard model physics and the FRW equations etc.
2. Since the phantom energy is negative we must compensate for it by positive energy matter. The simplest way to do this is to add a k – essence field with energy density $\rho_k = \sqrt{\pi_k^2/a^6 + \gamma^2}$ which just corresponds to an action of the DBI type with α replaced by γ and $\epsilon = -1$. The k – essence momentum π_k is also a constant of the motion.
Thus, *both* the phantom and k – essence fields are called for by the mathematical formalism.

Notice that the transition between the regimes where the usual FRW equations retain their interpretation as evolution equations of observable quantities is controlled by the deviation parameter $O_x = E^2/(\alpha^2 O_a^6)$. The transition occurs at at $O_x = 1$ so $O_a = \sqrt[3]{E/\alpha}$. By choosing α sufficiently small and/or E sufficiently large we can achieve that the transition scale is as large as we need. Notice that α is a kinematical parameter of the Lagrangean while E is a dynamical constant of motion. However, whatever the value of α, equations (6.16) differ drastically from the standard FRW equations beyond $O_x = 1$. That is, the universe evolves completely differently beyond $O_x = 1$. The largest corrections at late O_x are of the order of $1/O_x \propto O_a^6$ which are terms normally not considered in the FRW equations. In fact they will completely dominate then.

Let us see what will qualitatively happen at very late times: We will use realistic matter composed of dust (baryons), radiation and k – essence (for simplicity without potential term) and set

$$\rho_m = B/a^3 + R/a^4 + \sqrt{\pi_k^2/a^6 + \gamma^2} \tag{6.20}$$

where $B, R > 0$ are integration constants and $\gamma > 0$. Notice that also π_k is a constant of motion. The physical evolution equations become (we replace O_a by a etc. for the purpose of this discussion)

$$3(da/d\tau)^2/a^2 = (1 + 1/x)[\Lambda + \rho_m + \rho_{\text{phantom}}] \tag{6.21}$$

and we clearly need that the right hand side is positive for the entire evolution. During radiation domination, $R > 0$ is sufficient. During baryon domination we must have $B + |\pi_k| - |E| > 0$. For $x \to \infty$ we may want to require $\Lambda + \gamma - \alpha \geq 0$. However, we notice one problem at late times: Suppose that the right hand side of (6.21) never vanishes. Then $da/d\tau > 0$ for all τ due to continuity. Expanding (6.21) around $x = 0$ we find

$$3(da/d\tau)^2/a^2 = (1+1/x)\{[\Lambda+\gamma-\alpha] + \frac{1}{2}(\gamma\frac{\pi_k^2\alpha^2}{E^2\gamma^2} - \alpha)x + B\frac{\alpha}{E}\sqrt{x} + R(\frac{\alpha}{E}\sqrt{x})^{4/3} + O(x^2)\} \tag{6.22}$$

Even if $\Lambda + \gamma - \alpha = 0$ the right hand side diverges at $x \to 0$ due to the radiation and baryonic terms present. The leading contribution is then given by

$$(da/d\tau)^2 = B\frac{\alpha^2}{3E^2}a^5 =: \kappa^2 a^5 \tag{6.23}$$

which can be solved by

$$a(\tau)=\frac{1}{(\delta\mp\frac{3}{2}\kappa\tau)^{2/3}} \tag{6.24}$$

with $\kappa>0$, δ is a constant of integration and the upper/lower sign corresponds to $da/d\tau>0/<0$ respectively. Hence, if $da/d\tau>0$ for all τ then we must take the upper sign and $\delta>0$ and would conclude that the universe reaches *infinite* size after the *finite* amount of time $2\delta/(3\kappa)$. The evolution would in fact stop there and is called a big rip singularity. This is clearly undesirable and the only way to avoid this is to tune the parameters in such a way that ρ can vanish while always being non - negative during the evolution. This is only possible if the phantom contribution to the energy budget is negative. The universe would then be able to reach maximum size and then would recollapse. This is granted to happen if the right hand side of (6.21) kinematically can become negative beyond some critical x_c (dynamically it can never happen because $(da/d\tau)^2\geq 0$). In fact it is not difficult to show that we must have $\Lambda+\gamma-\alpha<0$ for this to happen.

One can tune $B,R,\Lambda,E,\pi_k,\gamma,\alpha$ such that there is a radiation, dust and positive vacuum energy era while still $x\gg 1$ during which the FRW equations hold. The radiation era holds for $0\leq a\leq\frac{R}{B+|\pi_k|-E}=:a_r$ during which $\sqrt{x}\geq\frac{E}{\alpha a_r^3}\gg 1$ must hold. The dust era is $a_r\leq a\leq\sqrt[3]{\frac{B+|\pi_k|-E}{\Lambda}}=:a_d$ during which $\sqrt{x}\geq\frac{E}{\alpha a_d^3}\gg 1$ must hold. Since $\Lambda/\alpha<1$ and $a_r<a_d$, both conditions are easily satisfied with the observed values for R,B for $|\pi_k|-E$ of the order of B, $E/B\gg 1$ and E/α sufficiently small. Finally, the vacumm era lasts for $a_d\leq a\leq a_c$ where a_c is the value at which (6.21) vanishes. In order to see that x_c is smaller than $x=1$ at which the FRW equations anyway no longer take their standard form we notice that at $x=1$ the right hand side of (6.22) is still larger than $\Lambda+\sqrt{\gamma^2+\alpha^2\frac{\pi_k^2}{E^2}}-\sqrt{2}\alpha$ which is positive independently of the value of Λ,γ as long as $\pi_k^2/E^2\geq 2$. However, if we associate B with baryonic matter and $|\pi_k|-E$ with dark matter then $(|\pi_k|-E)/B\approx 10$. Since $B/E\ll 1$ we get $|\pi_k|/E\approx 1$ so we should have $\Lambda+\sqrt{\gamma^2+\alpha^2}-\sqrt{2}\alpha>0$ for $\Lambda,\gamma,\alpha>0$ subject to $\Lambda+\gamma<\alpha$. Fix some $\epsilon>0$ and set $y:=\gamma/\alpha$ and $\Lambda/\alpha:=(1-\epsilon-y)$. Then we need to find $0\leq y\leq 1-\epsilon$ such that $1-\epsilon-y+\sqrt{1+y^2}-\sqrt{2}\geq 0$. Set $\kappa:=\sqrt{2}-1+\epsilon$ which is positive and satisfies $\kappa<1$ for $\epsilon<2-\sqrt{2}$. Then $0\leq y=(1-\kappa^2)/(2\kappa)\leq 1-\epsilon$ which implies $1+(1-\epsilon)^2\leq 2$ which is identically satisfied for $0<\epsilon\leq 2$. Hence we may choose any $0<\epsilon\leq 2-\sqrt{2}$ and then $\gamma=\alpha(1-\kappa^2)/2\kappa,\ \Lambda=\alpha(1-\epsilon)-\gamma$ where $\kappa=\sqrt{2}-1+\epsilon$. It is appealing that the cosmological constant during the vacuum era Λ is of the order of α which should be small, thus explaining the smallness of the cosmological constant[26]

Let us summarise once again the observable effect of the phantom:
1. The physical evolution equations of observable quantities have a standard FRW form for large x (small a).
2. The phantom adds additional matter terms with an equation of state $-1\leq w\leq 0$ which evolves from $0\to -1$ as a evolves from $0\to\infty$. It therefore acts like in k – essence, just with negative energy. However, an additional k – essence field is natural in order to compensate the negative phantom energy.
3. In particular, it is wrong that the physical evolution equations have just the FRW form without

[26]For completeness we also mention a scenario without k – essence matter, that is, $\pi_k=\gamma=0$. In this case the same analysis yields the condition $0<E/(B-E)\gg 1$. Then at $x=1$ the energy density becomes $\rho\geq\Lambda+\alpha(B/E-\sqrt{2})$ which is still positive for $\sqrt{2}-1-\epsilon\leq\Lambda/\alpha<1$ where $B=(1+\epsilon)E$. Notice that now $B-E$ corresponds to observed baryonic matter.

phantom matter contribution. It is easy to see that this is due to the fact that $C^{\rm tot} \neq C^{\rm ideal} := C + \pi$ in which case we would have $H = C$ exactly. Rather we have $C^{\rm tot} = C - \sqrt{\pi^2 + \alpha^2 a^6}$ so that $C^{\rm tot} - C^{\rm ideal} = |\pi|(\sqrt{1 + 1/x} - 1)$. It is not difficult to see that there is no Lagrangean of the form $L(I/2)$ which can produce $H = C$ exactly.
4. As x becomes small, the actual evolution equations differ drastically from the FRW form. The transition is roughly at $a_t = \sqrt[3]{E/\alpha}$ where E is the nergy of the universe and α is a parameter of the model which can be tuned to be so small that a_t is way beyond today's value a_0. In order that the universe has infinite observable life time, the parameters can and must be tuned such that the universe in fact recollapses rather than expanding forever.

Notice that we are not doubting the validity of Einstein's equations at all. These are completely encoded in the fundamental constraint $C^{\rm tot}$. There are two of these equations. One is $C^{\rm tot} = 0$. The other results by computing the gauge transformation $da/dt = \{C^{\rm tot}, a\}$, to solve this for P, to compute the gauge transformation $dP/dt = \{C^{\rm tot}, P\}$ and to insert this into d^2a/dt^2. This results exactly in (6.16), (6.17) at $x = \infty$ and with t replaced by τ. However, what we critisise is that these are interpreted as physical evolution equations. They are not, they are gauge transformations of non – gauge invariant, unobservable quantities and not evolution equations with respect to a non – vanishing Hamiltonian of observable quantities. What we have done here is to compute the physical evolution of observable quantities generated by a physical Hamiltonian. The resulting equations are, by a judicious choice of phantom field Lagrangean, in good agreement with the standard equations. However we insist that the standard procedure is fundamentally wrong. In particular, the standard FRW equations are drastically *false* in the late universe if our phantom field is realised in nature and provides the physical Hamiltonian which generates the evolution that we actually observe. These reservations hold of course in full generality in all applications of General Relativity.

There is another way to look at what is going on here: What we have done in order to obtain physical evolution is to use the unphysical gauge transformation $d\phi/dt(t) = \{C^{\rm tot}, \phi\}_{\phi=\phi(t)}$ and to solve the condition $\phi(t) = \tau$ for t. This results in the function $\tau \mapsto t_\tau$ which is a non – trivial function on phase space. Now we insert the value t_τ into the unphysical gauge transformation $t \mapsto a(t)$ where $a(t)$ solves $da/dt(t) = \{C^{\rm tot}, t\}_{a=a(t)}$ and obtain $a(t_\tau)$. We claim that $a(t_\tau) = O_a(\tau)$, at least when $C^{\rm tot} = 0$, where $O_a(\tau)$ solves $dO_a/d\tau(\tau) = \{H, O_a(\tau)\}$. To see this, we compute for any function f with equation of motion $df/dt = \{C^{\rm tot}, f\}$

$$\{C^{\rm tot}, f(t_\tau)\} = \{C^{\rm tot}, f\}_{f=f(t), t=t_\tau} + (df/dt)_{t=t_\tau}\{C^{\rm tot}, t_\tau\} = (df/dt)_{t=t_\tau}[1 + \{C^{\rm tot}, t_\tau\}] \tag{6.25}$$

Now choose $f = \phi$ in (6.25) and use that $\tau = \phi(t_\tau)$ is a constant function. Then use $f = a$ in (6.25) to see that $a(t_\tau)$ is an observable. Now on the constraint surface

$$0 = \{C'^{\rm tot}, f(t_\tau)\} = \{\pi, f\}_{f(t), t=t_\tau} + \{H, f\}_{f(t), t=t_\tau} + (df/dt)_{t=t_\tau}[\{\pi, t_\tau\} + \{H, t_\tau\}] \tag{6.26}$$

Choose $f = \phi$ and use that H does not depend on π, then

$$0 = 1 + (d\phi/dt)_{t=t_\tau}[\{\pi, t_\tau\} + \{H, t_\tau\}] \tag{6.27}$$

Insert (6.27) into (6.26) with $f = a$ to obtain

$$0 = \{H, a\}_{a=a(t), t=t_\tau} - (da/dt)_{t=t_\tau}/[(d\phi/dt)_{t=t_\tau}] = \{H, a\}_{a=a(t_\tau)} - da(t_\tau)/d\tau \tag{6.28}$$

Thus $a(t_\tau)$ and $O_a(\tau)$ differ at most by a constant.

What happens here is that $t \mapsto a(t)$, $t \mapsto \phi(t)$ is the parametrisation of a trajectory in phase space (here restricted to the a, ϕ plane) which is obtained by solving the equation $\phi(t) = \tau$ for t and inserting this into $a(t)$ so that we arrive at $\tau \mapsto a(t_\tau)$. We have *deparametrised* the description and now are dealing with the only physically meaningful object, the trajectory itself and not some random parametrisation thereof. Any reparametrisation $t = t(t')$ with $dt/dt' > 0$, that is, a gauge transformation, changes the unphysical functions $r(t), \phi(t)$ but results in the same physical trajectory.

What consequences does this have for the FRW line element

$$ds^2 = -dt^2 + a(t)^2 dx^a dx^b \delta_{ab} \tag{6.29}$$

which is also expressed in terms of the unphysical quantities $t, a(t)$? Let us express the line element in terms of τ by applying the diffeomorphism $t := t_\tau$. In these coordinates (6.29) becomes

$$ds^2 = -d\tau^2 [\frac{d\phi}{dt}(t)]^2_{t=t_\tau} + a(t_\tau)^2 dx^a dx^b \delta_{ab} = -d\tau^2 (1 + \frac{\alpha^2 a(t_\tau)^6}{E^2}) + a(t_\tau)^2 dx^a dx^b \delta_{ab} \tag{6.30}$$

which is again a FRW line element, now expressed in terms of observable quantities, at least for small a. For large a, (6.30) is no longer of FRW form. Again the deviation parameter x has appeared and shows that for sufficiently small a the metric (6.30) expressed in terms of observable quantities is well approximated by the usual FRW form even today.

7 Conclusions and Outlook

It has been known for a long time that the problem of time can be solved in principle by the relational framework due to Rovelli and others. This has never been appreciated as much as it should have been because, while the conceptual, physical framework was clear, the analytical implementation remained largely undeveloped for a long time. With the appearance of [8], analytical methods became available for the first time. Still the framework, in its full generality, remains discouragingly difficult in particular when applied to General Relativity due to the complexity of the analytical expressions which involve summing an infinite number of infinite series, an inversion of infinite dimensional matrices and the computation of an infinite number of different, iterated Poisson brackets[27].

The first main message of the present paper is that when adding appropiate, albeit hypothetical, matter, the complexity of these formulas is drastically reduced. In contrast to the general case, there is only one series to sum, there are no matrices to invert, there is only one kind of iterated Poisson bracket to compute. Hence the formulas that we obtain are remarkably simple. In fact, the classical time evolution in a background dependent theory, say in QCD on Minkowski space, of some observable O such as a Wilson loop function, would also be given by the series

$$O(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \{H_{\rm QCD}, O\}_{(n)} \tag{7.1}$$

where $H_{\rm QCD}$ is the QCD Hamiltonian. Comparing with (4.2) we see that the complexity of the classical time evolution in both theories is comparable!

[27] Drastic simplifications occur as far as the number of relevant constraints is concerned (four instead of infinitely many) when reformulating GR in terms of coordinates that are spacetime scalars [14]. While the resulting observables are relatively simple (although still inversions of non − trivial matrices take place) and physically intuitive, the field variables that one uses are complicated compounds of the the canonical fields and the observables involve polynomials of those evaluated at one specific spatial point. In quantum theory, these observables are therefore presumably too singular because they involve the product of operator valued distributions evaluated at the same point.

The second main message is that, in contrast to the general case, the physical observables we obtain are strong observables and there is just one natural, physical Hamiltonian which does not depend on the physical time parameter. That Hamiltonian is (constrained to be) positive and at least in the physically interesting region in phase space, that Hamiltonian reduces to the canonical Hamiltonian that one usually uses in cosmology and the standard model when the metric is flat. In fact, we manage to completely deparametrise the system irrespective of the other matter present.

The third main message is that the scalar type of matter that we considered here, from the mathematical (to be able to solve algebraic equations) and physical (to obtain a physical Hamiltonian which is close to that of the standard model) perspective naturally leads to Dirac – Born – Infeld (DBI) negative energy phantom fields with constant potential. This negative energy phantom must be compensated for by positive energy matter, most naturally by a k – essence field. Such matter was discussed independently in the cosmology literature in order to provide a candidate for inflation and dark energy. Hence the scalar matter we consider here might actually really exist!

The fourth main message is that, at least for the scalar model we have used here and for which we gave strong motivations, the usual interpretation of the cosmological framework, although fundamentally wrong because gauge transformations of gauge dependent objects are interpreted as actual physical evolution equations of observables, remains valid when analysed in the correct way, that is, by computing the physical evolution of gauge invariant observables. The domain of validity of these equations can be tuned to be arbitrarily long, however, it is manifestly finite when using the physical time parameter corresponding to the physical Hamiltonian. The actual evolution at late times apparently leads to a recollapse.

The future application of this framework lies of course in the quantum theory. The framework presented here, as well as any other application of the relational Ansatz so far, is purely classical. In order to promote the framework to the quantum theory, the functions H should be promoted to positive self – adoint operators and the functions H_τ and $H(M)$ to self –adjoint operators[28]. If we find operator orderings such that $[\hat{H}(M), \hat{H}(M')] = 0$, $[\hat{H}(M), \hat{H}_\tau] = i\hbar\hat{H}(M)$ then the quantum observables are given by

$$\widehat{O_f}(\tau) = \exp(i\hat{H}_\tau/\hbar)\hat{f}\exp(-i\hat{H}_\tau/\hbar) \tag{7.2}$$

where the unitary operators displayed are defined by the spectral theorem. They manifestly commute, under the assumptions made, with $\exp(i\hat{H}(M)/\hbar)$ and the operator ordering problem for the observables would be solved[29]

Given these assumptions, the fact that then a positive, fundamental Hamiltonian is available could enable one to solve the vacuum problem in quantum cosmology: Namely, in usual semiclassical quantum cosmology one neglects quantum gravity and applies the framework of quantum field theory on curved spacetimes [15]. The issue is that in cosmology the background metric is not stationary and therefore the problem becomes, roughly speaking, to choose a point of unphysical time and at that time a definition of annihilation operators (for the free fields) suggested by the energy density function of the matter in question in order to select a vacuum state. This is highly ambiguous and the cosmological evolution does not keep the vacuum intact but rather causes constant particle

[28]These operators are supposed to be spatially diffeomorphism invariant, see [10] for a quantum implementation of the diffeomorphism group within LQG.

[29]If one cannot find a model in which all expressions of which one has to take the square root are manifestly positive, then we may be able to compute the spectrum of the (regulated) operators without the square root and restrict the Hilbert space to the "subspace" on which all of them take positive (generalised) eigenvalues. This is possible because the operators are supposed to commute. The square root would then be well defined on that subspace as has been pointed out in [1]. Alternatively one can try to use the manifestly positive substitute expressions (5.13).

production. On the other hand, if one has a fundamental Hamiltonian at one's disposal, then it is natural to define a vacuum state as a minimal (zero) energy (eigen)state. This would circumvent this problem of initial conditions.

Having a physical Hamiltonian and physical observables at one's disposal one can also hope to develop physical scattering and S – Matrix theory. Namely, while we drastically simplified the relational framework, it will be still very hard to compute $\widehat{O_f}(\tau)$ explicitly to all orders. Here one will use the series in order to perform perturbation theory in the way outlined in [11], say within the framework of LQG: Given approximate physical states which can be obtained by using semiclassical techniques of LQG [16], we can concentrate them on regions in phase space where $\phi(x) \approx \tau$. Then expectation value computations of physical observables can be terminated after a few terms in the power series and only a small number of iterated commutators has to be computed. This should work especially nice in applications to quantum cosmology [11] within LQG.

In summary, there is much left to do in order to make this framework practically applicable and it is worthwhile to explore the space of Lagrangeans which lead to deparametrisation further. However, we feel that conceptually the framework is quite clear, the complexity has been drastically reduced, its validity has been checked in a cosmological setting and the remaining technical tasks to be solved have been identified.

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References

[1] J. Brown and K. Kuchař. Dust as a standard of space and time in canonical quantum gravity. *Phys. Rev.* **D51** (1995), 5600-5629. [gr-qc/9409001]

[2] K. Kuchař and J. Romano. Gravitational constraints which generate a lie algebra. *Phys. Rev.* **D51** (1995), 5579-5582. [gr-qc/9501005]

[3] M. Doran and C. Wetterich. Quintessence and the cosmological constant. *Nucl. Phys. Proc. Suppl.* **124** (2003), 57-62. [astro-ph/0205267]

[4] C. Armendariz-Picon, V. F. Mukhanov and Paul J. Steinhardt. A Dynamical solution to the problem of a small cosmological constant and late time cosmic acceleration. *Phys. Rev. Lett.* **85** (2000), 4438-4441. [astro-ph/0004134]
C. Armendariz-Picon, V. F. Mukhanov and Paul J. Steinhardt. Essentials of k essence. *Phys. Rev.* **D63** (2001) 103510. [astro-ph/0006373]

[5] F. Markopoulou. Gravitational Constraint Combinations Generate a Lie Algebra. *Class. Quant. Grav.* **13** (1996), 2577-2584. [gr-qc/9601038]

[6] R. Barbieri, D. Nanopoulos and D. Wyler. An SU(5) Model With Hierarchical Fermion Masses And A Phantom Axion. *Phys. Lett.* **B106** (1981), 303.
H.Q. Lu. Phantom cosmology with a nonlinear Born-Infeld type scalar field. *Int. J. Mod. Phys.*

D14 (2005), 355-362. [hep-th/0312082]
Wei Fang, H.Q. Lu, Z.G. Huang and K.F. Zhang. Phantom cosmology with the Born-Infeld type scalar field. [hep-th/0409080]
M. Sami and A. Toporensky. Phantom field and the fate of universe. *Mod. Phys. Lett.* **A19** (2004), 1509. [gr-qc/0312009]
P. Singh, M. Sami and N. Dadhich. Cosmological dynamics of phantom field. *Phys. Rev.* **D68** (2003), 023522. [hep-th/0305110]
Zu-Yao Sun and You-Gen Shen. Phantom cosmology with non-minimally coupled real scalar field. *Gen. Rel. Grav.* **37** (2005), 243-251. [gr-qc/0410096]

[7] C. Rovelli. *Quantum Gravity*, (Cambridge University Press, Cambridge, 2004).
C. Rovelli. *Class. Quantum Grav.* **8** (1991), 297.
C. Rovelli. *Class. Quantum Grav.* **8** (1991), 317.
C. Rovelli. *Phys. Rev.* **D43** (1990), 442.
C. Rovelli. *Phys. Rev.* **D43** (1990), 2638.

[8] B. Dittrich. Partial and complete observables for Hamiltonian constrained systems. [gr-qc/0411013]

[9] T. Thiemann. Reduced phase space quantization and Dirac observables. *Class. Quant. Grav.* **23** (2006), 1163-1180. [gr-qc/0411031]

[10] T. Thiemann. *Modern Canonical Quantum General Relativity*, (Cambridge University Press, Cambridge, at press). Draft available as [gr-qc/0110034]
C. Rovelli. Loop quantum gravity, *Living Rev. Rel.* **1** (1998), 1. [gr-qc/9710008]
T. Thiemann. Lectures on loop quantum gravity. *Lect. Notes Phys.* **631** (2003), 41-135. [gr-qc/0210094]
A. Ashtekar and J. Lewandowski. Background independent quantum gravity: a status report. *Class. Quant. Grav.* **21** (2004), R53. [gr-qc/0404018]
L. Smolin. An invitation to loop quantum gravity. [hep-th/0408048]

[11] J. Brunnemann and T. Thiemann. On (cosmological) singularity avoidance in loop quantum gravity. *Class. Quant. Grav.* **23** (2006), 1395-1428. [gr-qc/0505032]

[12] R. M. Wald. *General Relativity*, (The University of Chicago Press, Chicago, 1989).

[13] A. Ashtekar, T. Pawlowski and P. Singh. Quantum nature of the big bang. *Phys. Rev. Lett.* **96** (2006), 141301. [gr-qc/0602086]
A. Ashtekar, T. Pawlowski and P. Singh. Quantum Nature of the Big Bang: An Analytical and Numerical Investigation. I.. [gr-qc/0604013]
A. Ashtekar, T. Pawlowski and P. Singh. Quantum Nature of the Big Bang: Improved dynamics. [gr-qc/0607039]

[14] B. Dittrich. Partial and complete observables for canonical general relativity. [gr-qc/0507106]

[15] R. Wald. *Quantum field theory in curved space-time and black hole thermodynamics*, (The University of Chicago Press, Chicago, 1995).

[16] T. Thiemann. Quantum Spin Dynamics (QSD): VII. Symplectic structures and continuum lattice formulations of gauge field theories. *Class. Quant. Grav.* **18** (2001) 3293-3338. [hep-th/0005232]
T. Thiemann. Gauge Field Theory Coherent States (GCS): I. General properties. *Class. Quant. Grav.* **18** (2001), 2025-2064. [hep-th/0005233]
T. Thiemann and O. Winkler. Gauge Field Theory Coherent States (GCS): II. Peakedness properties. *Class. Quant. Grav.* **18** (2001) 2561-2636. [hep-th/0005237]
T. Thiemann and O. Winkler. Gauge Field Theory Coherent States (GCS): III. Ehrenfest theorems. *Class. Quantum Grav.* **18** (2001), 4629-4681. [hep-th/0005234]

T. Thiemann and O. Winkler. Gauge field theory coherent states (GCS): IV. Infinite tensor product and thermodynamic limit. *Class. Quantum Grav.* **18** (2001), 4997-5033. [hep-th/0005235]
H. Sahlmann, T. Thiemann and O. Winkler. Coherent states for canonical quantum general relativity and the infinite tensor product extension. *Nucl. Phys.* **B606** (2001) 401-440. [gr-qc/0102038]
T. Thiemann. Complexifier coherent states for canonical quantum general relativity. *Class. Quant. Grav.* **23** (2006) 2063-2118. [gr-qc/0206037]