Blowup of Jang's equation at outermost marginally trapped surfaces

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Abstract. The aim of this paper is to collect some facts about the blowup of Jang's equation. First, we discuss how to construct solutions that blow up at an outermost MOTS. Second, we exclude the possibility that there are extra blowup surfaces in data sets with non-positive mean curvature. Then we investigate the rate of convergence of the blowup to a cylinder near a strictly stable MOTS and show exponential convergence near a strictly stable MOTS.

1 Introduction

In the paper [AM07], inspired by an idea of Schoen [Sch04], we constructed marginally outer trapped surfaces (MOTS) in the presence of barrier surfaces by inducing a blow-up of Jang's equation.

In this note, we take a slightly different perspective. Let (M, g, K) be a data set with non-empty outer boundary $\partial^+ M$ and assume that we are given an outermost MOTS Σ in (M, g, K). Outermost means that there is no other MOTS on the outside of Σ. From [AM07] it follows that (M, g, K) always contains a unique such surface, or does not contain outer trapped surfaces at all, provided ∂M is outer untrapped. As stated in theorem 3.1, we show that there exists a solution f to Jang's equation that blows up at Σ, assuming that ∂M is inner and outer untrapped. By blow-up we mean, that outside from Σ the function f is such that graph f is a smooth submanifold of $M \times \mathbf{R}$ with a cylindrical end converging to $\Sigma \times \mathbf{R}$. There is however a catch, as

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f may blow up at other surfaces, too. These surfaces are marginally inner trapped. In theorem 3.4 we show that the other blow-up surfaces can not occur if the data set has non-positive mean curvature.

Before turning to these results, we introduce some notation in section 2. Section 3 proceeds with the construction of the the blow-up. We will not go into details here, but emphasize the general idea and point to the results needed from the paper [AM07]. In section 4, we show that near strictly stable MOTS, the blow-up is at most logarithmic. Turning the picture sideways, and writing the blow-up solution as the graph of a function u on the cylinder, this logarithmic growth turns into an exponential decay of u and its derivatives.

2 Preliminaries

Let (M, g, K) be an initial data set for the Einstein equations. That is M is a 3-dimensional manifold, g a Riemannian metric on M and K a symmetric 2-tensor. We do not require any energy condition to hold.

Assume that ∂M is the disjoint union $\partial M = \partial^- M \cup \partial^+ M$, where $\partial^\pm M$ are smooth surfaces without boundary. We refer to $\partial^- M$ as the inner boundary and endow it with the normal vector ν pointing into M. The outer boundary $\partial^+ M$ is endowed with the normal ν pointing out of M. We denote by $H[\partial M]$ the mean curvature of ∂M with respect to the normal vector filed ν, and by $P[\partial M] = \mathrm{tr}_{\partial M} K$ the trace of the tensor K restricted to the 2-dimensional surface ∂M. Then the inward and outward expansions of ∂M are defined by

$$\theta^\pm[\partial M] = P[\partial M] \pm H[\partial M].$$

Assume that $\theta^+[\partial^- M] = 0$, and that $\theta^+[\partial^+ M] > 0$ and $\theta^-[\partial^+ M] < 0$.

If $\Sigma \subset M$ is a smooth, embedded surface homologous to $\partial^+ M$, then Σ bounds a region Ω together with $\partial^+ M$. In this case, we define $\theta^\pm[\Sigma]$ as above, where H is computed with respect to the normal vector field pointing into Ω (that is in direction of $\partial^+ M$). Σ is called marginally outer trapped surface (MOTS), if $\theta^+[\Sigma] = 0$. We say that ∂M is an outermost MOTS, if there is no other MOTS in M, which is homologous to $\partial^+ M$. In [AM07] it is proved that for any initial data set (M, g, K) which contains a MOTS, there is also an outermost MOTS surrounding it.

Let $\Sigma \subset M$ be a MOTS and consider a normal variation of Σ in M, that is a map $F : \Sigma \times (-\varepsilon, \varepsilon) \to M$ such that $F(\cdot, 0) = \mathrm{id}_\Sigma$ and $\frac{\partial}{\partial s}\big|_{s=0} F(p, s) = f\nu$, where f is a function on Σ and ν is the normal of Σ. Then the change of θ^+ is given by

$$\left.\frac{\partial\theta^+[F(\Sigma,s)]}{\partial ds}\right|_{s=0} = L_M f,$$

where L_M is a quasi-linear elliptic operator of second order along Σ. We will not need the precise formula for L_M here, we instead refer to [AMS05] and [AMS07] for details. The only fact we will need here is that L_M has a principal eigenvalue λ, which is real. If λ is non-negative Σ is called stable, and if λ is positive, Σ is called strictly stable. In particular, if Σ is strictly stable as a MOTS, there exists an outward deformation strictly increasing θ^+.

In $\bar{M} = M \times \mathbf{R}$, we consider Jang's equation [Jan78, SY81] for the graph of a function $f : M \to \mathbf{R}$. Let $N := \operatorname{graph} f = \{(x,z) : z = f(x)\}$. The mean curvature $\mathcal{H}[f]$ of N with respect to the downward normal is given by

$$\mathcal{H}[f] = \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)$$

define $\bar{K}$ on $\bar{M}$ by $\bar{K}_{(x,z)}(X,Y) = K_x(\pi X, \pi Y)$, where $\pi : T\bar{M} \to TM$ denotes the orthogonal projection onto the horizontal tangent vectors. Let

$$\mathcal{P}[f] = \operatorname{tr}_N \bar{K}.$$

Then Jang's equation becomes

$$\mathcal{J}[f] = \mathcal{H}[f] - \mathcal{P}[f] = 0. \tag{2.1}$$

3 The blowup

The main result of this paper is that we can construct a solution to Jang's equation which blows up at the outermost MOTS in (M,g,K) and has zero Dirichlet boundary data at $\partial^+ M$. In fact, we chose the assumptions on the outer boundary $\partial^+ M$ so that we can prescribe arbitrary Dirichlet data there. The focus here lies on the blow-up in the interior, so that we do not investigate the optimal conditions for $\partial^+ M$.

Theorem 3.1. *If (M,g,K) be an initial data set with $\partial M = \partial^- M \cup \partial^+ M$ such that $\partial^- M$ is an outermost MOTS, $\theta^+[\partial^+ M] > 0$ and $\theta^-[\partial^+ M] < 0$, then there exists an open set $\Omega_0 \subset M$ and a function $f : \Omega_0 \to \mathbf{R}$ such that*

1. *$M \setminus \Omega_0$ does not intersect ∂M,*

2. *$\theta^-[\partial\Omega_0] = 0$ with respect to the normal vector pointing into Ω_0,*

3. *$\mathcal{J}[f] = 0$,*

4. *$N = \operatorname{graph} f$ is asymptotic to the cylinder $\partial^- M \times \mathbf{R}^+$,*

5. *$N = \operatorname{graph} f$ is asymptotic to the cylinder $\partial\Omega_0 \times \mathbf{R}^-$, and*

6. *$f|_{\partial^+ M} = 0$.*

For data sets (M, g, K) which do not contain surfaces with $\theta^- = 0$, the above theorem implies the following result.

Corollary 3.2. *If (M, g, K) is as in theorem 3.1, and in addition there are not subsets $\Omega \subset M$ with $\theta^-[\partial\Omega] = 0$ with respect to the normal pointing out of Ω, then there exists a function $f : M \to \mathbf{R}$ such that*

1. *$\mathcal{J}[f] = 0$,*

2. *$N = \operatorname{graph} f$ is asymptotic to the cylinder $\partial^- M \times \mathbf{R}^+$,*

3. *$f|_{\partial^+ M} = 0$.*

Remark 3.3. Analogous results hold if (M, g, K) is asymptotically flat with appropriate decay of g and K instead of having an outer boundary $\partial^+ M$. Then the assertion $f|_{\partial^+ M} = 0$ in theorem 3.1 has to be replaced by $f(x) \to 0$ as $x \to \infty$.

The proof of theorem 3.1 is largely based on the tools developed in [SY81] and [AM07]. Thus we will not include all details here, but provide a summary, which facts will have to be used.

Proof. We will assume that (M, g, K) is embedded into (M', g', K') which extends M beyond the boundary $\partial^- M$ such that $\partial^- M$ lies in the interior of M'.

Let $\partial^- M = \cup_{i=1}^N \Sigma_i$ where the Σ_i are the connected components of ∂M. As ∂M is an outermost MOTS, each of the Σ_i is stable [AM07, Corollary 5.3].

Following the proof of [AM07, Theorem 5.1], we deform $\partial^- M$ to a surface Σ_s by pushing the components Σ_i out of M, into the extension M'. To this end, let $\phi_i > 0$ be the principal eigenfunction of the stability operator of Σ_i and extend the vector field $X_i = -\phi_i \nu_i$ to a neighborhood of Σ_i in M'. Flowing Σ_i by X_i yields a family of surfaces Σ_i^s, $s \in [0, \varepsilon)$ so that the Σ_i^s form a smooth foliation for small enough ε with $\Sigma_i^s \in M' \setminus M$. If Σ_i is strictly stable then

$$\frac{\partial}{\partial s}\bigg|_{s=0} \theta^+[\Sigma_s] = -\lambda\phi < 0,$$

where λ is the principal eigenvalue of Σ_i. Thus, for small enough ε, we have $\theta^+[\Sigma_i^s] < 0$ for all $s \in (0, \varepsilon)$.

If Σ_i has principal eigenvalue $\lambda = 0$, then the Σ_i^s satisfy

$$\frac{\partial}{\partial s}\bigg|_{s=0} \theta^+[\Sigma_s] = 0.$$

In this case it is possible to change the data K' on Σ_i^s as follows

$$\tilde{K} = K' - \tfrac{1}{2}\phi(s)\gamma_s. \tag{3.1}$$

where γ_s is the metric on Σ_s and ϕ is a smooth function $\phi : [0, \varepsilon] \to \mathbf{R}$. The operator $\tilde{\theta}^+$, which means θ^+ computed with respect to the data $\tilde{K}$ instead of K', satisfies

$$\tilde{\theta}^+[\Sigma_i^s] = \theta^+[\Sigma_i^s] - \phi(s).$$

It is clear from equation (3.1) that ϕ can be chosen such that $\phi(0) = \phi'(0) = 0$ and $\tilde{\theta}^+[\Sigma_i^s] < 0$ for all $s \in (0, \varepsilon)$ provided ε is small enough. Then $\tilde{K}$ is $C^{1,1}$ when extended by K to the rest of M.

Replace each original boundary component Σ_i of M by a surface Σ_i^ε as constructed above, and replace K' with $\tilde{K}$, such that the following properties are satisfied. Let $\tilde{M}$ denote the manifold with boundary components Σ_i^ε resulting from this procedure. Thus we construct from (M, g, K) a data set $(\tilde{M}, g', \tilde{K})$ with the following properties:

1. $M \subset \tilde{M}$ with $g'|_M = g$, $\tilde{K}|_M = K$, and $\partial^+ M = \partial^+ \tilde{M}$,
2. $\theta^+[\partial^- \tilde{M}] < 0$, and
3. the region $\tilde{M} \setminus M$ is foliated by surfaces Σ_s with $\theta^+(\Sigma_s) < 0$.

The method developed in section 3.2 in [AM07] now allows the modification of the data $(\tilde{M}, g', \tilde{K})$ to a new data set, which we also denote by $(\tilde{M}, \tilde{g}, \tilde{K})$, although $\tilde{K}$ changes in this step. This data set has the following properties

1. $M \subset \tilde{M}$ with $g'|_M = g$, $\tilde{K}|_M = K$, and $\partial^+ M = \partial^+ \tilde{M}$,

2. $\theta^+[\partial^-\tilde{M}] < 0$,

3. $H[\partial^-\tilde{M}] > 0$ where H is the mean curvature of $\partial^- M$ with respect to the normal pointing out of $\partial^-\tilde{M}$,

4. the region $\tilde{M} \setminus M$ is foliated by surfaces Σ_s with $\theta^+(\Sigma_s) < 0$.

By section 3.3 in [AM07] this enables us to solve the boundary value problem

$$\begin{cases} \mathcal{J}[f_\tau] = \tau f_\tau & \text{in } \tilde{M} \\ f_\tau = \frac{\delta}{2\tau} & \text{on } \partial^-\tilde{M} \\ f_\tau = 0 & \text{on} \partial^+\tilde{M} \end{cases} \tag{3.2}$$

where δ is a lower bound for H on $\partial^- M$. The solvability of this equation follows, provided an estimate for the gradient at the boundary can be found. The barrier construction at $\partial^-\tilde{M}$ was carried out in detail in [AM07], whereas the barrier construction at $\partial^+\tilde{M}$ is standard due to the stronger requirement that $\theta^+[\partial^+ M] > 0$ and $\theta^-[\partial^+ M] < 0$.

The solution f_τ to equation (3.2) satisfies an estimate of the form

$$\sup_{\tilde{M}} |f_\tau| + \sup_{\tilde{M}} |\nabla f_\tau| \leq \frac{C}{\tau}, \tag{3.3}$$

where C is a constant depending only on the data $(\tilde{M}, \tilde{g}, \tilde{K})$ but not on τ.

The gradient estimate implies in particular that there exists an $\varepsilon > 0$ independent of τ such that

$$f_\tau(x) \geq \tfrac{\delta}{4\tau} \quad \forall x \text{ with } \operatorname{dist}(x, \partial^-\tilde{M})$$

The graphs N_τ have uniformly bounded curvature in $\tilde{M} \times \mathbf{R}$ away from the boundary. This allows the extraction of a sequence $\tau_i \to 0$ such that the N_{τ_i} converge to a manifold N, cf. [AM07, Proposition 3.8], [SY81, Section 4]. This convergence determines three subsets of $\tilde{M}$:

$$\Omega_- := \{x \in M : f_{\tau_i}(x) \to -\infty \text{ as } i \to \infty\},$$

$$\Omega_0 := \{x \in M : \limsup_{i\to\infty} |f_{\tau_i}(x)| < \infty\},$$

$$\Omega_+ := \{x \in M : f_{\tau_i}(x) \to \infty \text{ as } i \to \infty\}.$$

From the fact that the f_τ blow up near $\partial^-\tilde{M}$, we have that $\Omega_+ \neq \emptyset$ and Ω_+ contains a neighborhood of $\partial^-\tilde{\Omega}$. As already noted in [SY81] $\partial\Omega_+ \setminus \partial\tilde{M}$ consists of MOTS. As the region $\tilde{M} \setminus M$ is foliated by surfaces with $\theta^+ < 0$, we must have that $\Omega_+ \supset (\tilde{M} \setminus M)$ and hence $\partial\Omega_+$ is a MOTS in M. As $\partial^- M$ was assumed to be an outermost MOTS in M, we conclude that the closure of Ω_+ is $\tilde{M} \setminus M$.

The barriers near $\partial^+ M$ are so that they imply that the f_τ are uniformly bounded near $\partial^+ M$. Thus Ω_0 contains a neighborhood of $\partial^+ M$ and $\Omega_0 \subset M$.

The limit manifold N over Ω_0 is a graph satisfying $\mathcal{J}[f_\tau] = 0$, and has the desired asymptotics. □

We will now discuss another possibility to assert that the resulting graph is nonsingular on M that is that $M = \Omega_0$ in theorem 3.1.

Theorem 3.4. *Let (M, g, K) be as in theorem 3.1 with $\operatorname{tr} K \leq 0$. Then in the assertion of theorem 3.1 we have that $\Omega_0 = M$, that is f is defined on M and has no other blow-up than near $\partial^- M$.*

Proof. This follows from a simple argument using the maximum principle. Let f_τ be a solution to the regularized problem

$$\mathcal{H}[f_\tau] - \mathcal{P}[f_\tau] - \tau f_\tau = 0 \tag{3.4}$$

in $\tilde{M}$, as in the proof of theorem 3.1. We claim that f_τ can not have a negative minimum in the region where the data is unmodified. Assume that $x \in M$ is such a minimum. There we have $\mathcal{H}[f_\tau] \geq 0$, and since $\operatorname{graph} f$ is horizontal at x we have that

$$\mathcal{P}[f_\tau] = \operatorname{tr} K \leq 0.$$

thus the right hand side of (3.4) is non-negative, whereas τf_τ is assumed to be negative, a contradiction.

Since we know that in the limit $\tau \to 0$, the functions f_τ must blow-up in the modified region which lies in Ω_+, we infer a lower bound for f_τ from the above argument. Thus $\Omega_0 = M$ as claimed. □

4 Asymptotic behavior

Here, we shall discuss a refinement of [SY81, Corollary 2], which says that $N = \operatorname{graph} f$ converges uniformly in C^2 to the cylinder $\partial^- M \times \mathbf{R}$ for large values of f. A barrier construction allows us to determine the asymptotics of this convergence. Before we present our result, recall the statement of [SY81, Corollary 2]:

Theorem 4.1. *Let $N = \operatorname{graph} f$ the manifold constructed in the proof of theorem 3.1 and let Σ be a connected component of $\partial^- M$. Let U be a neighborhood of Σ with positive distance to $\partial^- M \setminus \Sigma$.*

Then for all $\varepsilon > 0$ there exists a $\bar{z} = \bar{z}(\varepsilon)$, depending also on the geometry of (M, g, K), such that $N \cap U \times [\bar{z}, \infty)$ can be written as the graph of a function u over $\Sigma \times [\bar{z}, \infty)$, so that

$$|u(p,z)| + |\,{}^C\nabla u(p,z)| + |\,{}^C\nabla^2 u(p,z)| < \varepsilon.$$

for all $(p, z) \in \Sigma \times [\bar{z}, \infty)$. Here, ${}^C\nabla$ denotes covariant differentiation along $\Sigma \times \mathbf{R}$.

If Σ is strictly stable, we can in fact say more about u.

Theorem 4.2. *In the situation of theorem 4.1, where in addition Σ is strictly stable, we have that*

$$|u(p,z)| + |\,{}^C\nabla u(p,z)| + |\,{}^C\nabla^2 u(p,z)| \leq C \exp(-\delta z).$$

Here C and $\delta > 0$ are constants depending only on the data (M, g, K).

Proof. Let $\phi > 0$ be the principal eigenfunction on Σ and denote by ν the normal vector field of Σ pointing into M. Extend the vector field $X = \phi\nu$ to a neighborhood of Σ and consider the flow of Σ along X. That is let $\Psi : \Sigma \times (-\varepsilon, \varepsilon) \to M$ the map solving

$$\frac{\partial \Psi}{\partial s}(x, s) = X(\Psi(x, s)). \tag{4.1}$$

This generates a family of surfaces $\Sigma_s = \Psi(\Sigma, s)$ which form a local foliation near Σ with lapse β such that for s small enough, say $s \in [0, \bar{s}]$,

$$\theta^+[\Sigma_s] \geq \kappa s$$

for some positive $\kappa > 0$. Denote the region swiped out by these Σ_s by U. Note that $\partial U = \Sigma \cup \Sigma_{\bar{s}}$ and $\operatorname{dist}(\Sigma_{\bar{s}}, \Sigma) > 0$. We can assume that $\operatorname{dist}(\Sigma_{\bar{s}}, \partial M) > 0$. On U we consider functions w which are constant on the Σ_s. Denote this constant by $\phi(s)$. For such functions Jang's operator can be computed as follows

$$\mathcal{J}[w] = \frac{\phi'}{\beta\sigma}\theta^+ - \left(1 + \frac{\phi'}{\beta\sigma}\right) P - \sigma^{-2} K(\nu, \nu) + \frac{\phi''}{\beta^2\sigma^3} - \frac{\phi'}{\beta^3\sigma^3}\frac{\partial \beta}{\partial s},$$

where $\sigma^2 = 1 + \beta^{-2}(\phi')^2$, and ϕ' denotes the derivative of ϕ with respect to s. As already mentioned β is the lapse of the foliation of the Σ_s. The quantities θ^+, $K(\nu, nu)$ and P are computed on the respective Σ_s.

Assuming a lower bound on ϕ', an upper bound on β, β^{-1} and $|\frac{\partial \beta}{\partial s}|$ we can estimate $\mathcal{J}[w]$ as follows, using $\theta^+[\Sigma_s] \geq \kappa s$,

$$\mathcal{J}[w] \leq -c_1 \kappa s + \frac{c_2}{|\phi'|^2} + c_3 \frac{\phi''}{|\phi'|^3}, \tag{4.2}$$

with $c_1 > 0$.

Choosing $\phi(s) = a \log s$ for some constant a, we first find that

$$\phi'(s) = \frac{a}{s} \qquad \phi''(s) = -\frac{a}{s^2}$$

so that

$$\frac{1}{|\phi'|^2} = \frac{s^2}{a^2} \qquad \frac{\phi''}{|\phi'|^3} = \frac{s}{a^2}.$$

For small s and large enough a, depending only on c_1, c_2, c_3 and κ, the first term on the right hand side of (4.2) dominates the other terms, so that $\mathcal{J}[w] \leq 0$ for this choice of ϕ. Hence, we obtain a super-solution w with

$$\mathcal{J}_\tau w \leq 0$$

at least where $w \geq 0$, that is near Σ.

As w blows up near the horizon, and the f_τ are bounded uniformly in τ on $\partial^+ U$, we can translate w vertically to $\bar{w} = w + b$ with a suitable $b > 0$ so that

$$f_\tau|_{\partial^+ U} \leq \bar{w}|_{\partial^+ U}$$

for all $\tau > 0$. Then the maximum principle implies that $f_\tau \leq \bar{w}$ for all $\tau > 0$ in U and consequently the function f constructed in theorem 3.1 also satisfies $f \leq \bar{w}$.

Near Σ, the graph of $\bar{w}$ can be written as the graph of a function $\bar{v}$ over $\Sigma \times (\bar{z}, \infty)$ where v decays exponentially in z. This is due to the fact that by the assumptions β, the parameter s is comparable to the distance to Σ. By the above construction $u \leq v$, where u is the function from theorem 4.1. Thus we find the claimed estimate for u.

Getting the desired estimates for the derivatives of u is then a standard procedure, but as it is a little work to set the stage, we briefly indicate how to proceed.

We choose coordinates of a neighborhood $\Sigma \times \mathbf{R}$ as follows. Let $\Psi : \Sigma \times (-\varepsilon, \varepsilon) \to M$ be the map from equation (4.1). Then extend Ψ to $\bar{\Psi}$ as follows

$$\bar{\Psi} : \Sigma \times (-\varepsilon, \varepsilon) \times \mathbf{R} \to M \times \mathbf{R} : (x, s, z) \mapsto (\Psi(x, s), z).$$

For a function h on $\Sigma \times \mathbf{R}$ we let $\operatorname{graph}_{\bar{\Psi}} h$ be the set

$$\operatorname{graph}_{\bar{\Psi}} h = \{\bar{\Psi}(x, h(x), z) : (x, z) \in \Sigma \times \mathbf{R}\}.$$

From theorem 4.1, it is clear that for large enough $\bar{z}$ the set $N \cap M \times [\bar{z}, \infty)$ can be written as $\operatorname{graph}_{\bar{\Psi}} h$, where h decays exponentially by the above reasoning. We can compute the value of Jang's operator for h as follows

$$\bar{H} - \bar{P} = \mathcal{L}h$$

where $\mathcal{L}$ is a quasi-linear elliptic operator of mean curvature type. To be more precise, $\mathcal{L}h$ has the form

$$\mathcal{L}h = a^{ij}\partial_i\partial_j h + B(\partial h) + \theta^+[\Sigma_{h(x,z)}]$$

where a^{ij} depends on ∂h and B is quadratic in ∂h. By freezing coefficients, we therefore conclude that h satisfies a linear, uniformly elliptic equation of the form

$$a^{ij}\partial_i\partial_j h + b^i\partial_i h + \theta^+[\Sigma_{h(x,z)}] = 0.$$

By construction we have that $|\theta^+[\Sigma_s]| \leq \kappa s$ for some fixed κ. Thus $\theta^+[\Sigma_{h(x,z)}]$ decays exponentially in z.

Now we are in the position to use standard interior estimates for linear elliptic equations to conclude the decay of higher derivatives of h. This decay translates back into the decay of the first and second derivatives of u as the coordinate transformation is smooth and controlled by the geometry of (M, g, K). □

Remark 4.3. If Σ is not strictly stable, but has positive k-th variation, we find that the foliation near Σ satisfies $\theta^+(\Sigma_s) \geq \kappa s^k$. Then a function of the form $\phi(s) = as^{-p}$ with large a and $p = \frac{k-1}{2}$ yields a super-solution. This super-solution can be used to prove that $|u| \leq Cz^{2/(1-k)}$ as above.

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