Spherically symmetric gravitating shell as a reparametrization-invariant system

P. Hájíček
Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland
(Received 6 August 1997; published 31 December 1997)

The subject of this paper is spherically symmetric thin shells made of a baryotropic ideal fluid which moves under the influence of its own gravitational field as well as that of a central black hole; the cosmological constant is assumed to be zero. The general super-Hamiltonian derived in a previous paper is rewritten for this spherically symmetric special case. The dependence of the resulting action on the gravitational variables is trivialized by a transformation due to Kuchař. The resulting variational principle depends only on shell variables, is reparametrization invariant, and includes both first- and second-class constraints. Several equivalent forms of the constrained system are written down. The exclusion of the second-class constraints leads to a super-Hamiltonian which appears to overlap with that by Ansoldi *et al.* in a quarter of the phase space. As the Kuchař variables are singular at the horizons of both Schwarzschild spacetimes inside and outside the shell, the dynamics is first well defined only inside of 16 disjoint sectors. The 16 sectors are, however, shown to be contained in a single, connected symplectic manifold and the constraints are extended to this manifold by continuity. Poisson brackets between no two independent spacetime coordinates of the shell vanish at any intersection of two horizons. [S0556-2821(98)00502-5]

PACS number(s): 04.60.Ds, 04.20.Fy

I. INTRODUCTION

Spherically symmetric thin shells are popular models used extensively in the study of a number of phenomena: properties of classical gravitational collapse [1], properties of classical black holes [2], quantum gravitational collapse [3], the dynamics of domain walls in the early Universe [4,5], the back reaction in the Hawking effect [6], entropy on black holes [7], and the quantum theory of black holes [8], to mention just a few examples.

Attempts to derive a Hamiltonian formalism for such shells are for example Refs. [9,6,10,11,5,12]. The Hamiltonian (or super-Hamiltonian) is either guessed directly from equations of motion (Refs. [9,11]), or derived from a variational principle guessed for the spherically symmetric system consisting of dust shells and gravity (Refs. [6,10]), or it is derived from the Lagrangian formalism based on the sum of the Einstein-Hilbert action and an action for an ideal fluid either after reducing the action by spherical symmetry [5] or without any assumption about symmetry [12].

In Ref. [12], both the super-Hamiltonian and the symplectic structure are derived from the Einstein-Hilbert-ideal-fluid variational principle. In this sense, the symplectic structure is unique; it contains a boundary term at the hypersurface of the shell and it turns out that the momentum conjugate to the surface area of the shell is the (hyperbolic) angle between the shell and the foliation hypersurface (''Kijowski momentum'' [13]; see also Ref. [14]). This momentum will play an important role in our calculations.

In the present paper, we shall derive a super-Hamiltonian and a symplectic form for the spherically symmetric ideal fluid shells, starting from the general formula of Ref. [12]. For the sake of simplicity, we shall also assume that the cosmological constant and all fields different from gravity are zero. Our leading principle is the reparametrization invariance. Thus, the result must be a super-Hamiltonian rather than a Hamiltonian. One problem is then how the variables describing the gravitational field around the shell can be made to disappear from the action so that the final formalism contains the shell variables only. As most of these gravitational variables just describe a gauge, one possible method is to choose a gauge and to reduce the system, as for instance in Refs. [10,15]; then, however, the reparametrization invariance is lost. We find a suitable tool in a transformation due to Kuchař [16]. This transformation trivializes the gravitational part of the equations of motion to such an extent that they do not contain any more information about the motion of the shell. The boundary terms that result from the Kuchař transformation contribute to the shell part of the symplectic form. They not only modify the Kijowski momentum but provide additional terms so that this part itself becomes nondegenerate; thus, the symplectic structure of the shell emerges. Several equivalent forms of the variational principle can be written down.

For example, one of the resulting phase spaces is locally described by four pairs of conjugate quantities, namely, (E_+,T_+), (E_-,T_-), (P_+,R_+), and (P_-,R_-), where $T_\pm$ and $R_\pm$ are the Schwarzschild coordinates, $E_\pm$ is the Schwarzschild mass, and $P_\pm$ is the modified Kijowski momenta; the sign + refers to the outside and − to the inside Schwarzchild spacetimes. There are then three constraints: (1) the super-Hamiltonian constraint $C_s=0$ is (roughly) the time-time component of Israel's matching condition at the shell and it is a primary constraint, (2) the continuity condition $R_+-R_-=0$ is another primary, and (3) the Poisson bracket $\chi:=\{C_s,R_+-R_-\}$ fails to vanish, so $\chi=0$ is a secondary constraint. The two constraint functions χ and R_+-R_- form a second-class pair. The second-class constraints can be solved for $[R]$ and $\overline{P}$,[1] and the solution can be sub-

[1]We adhere to the usual notation in the theory of thin shells: for any, possibly discontinuous, function X at the shell $\overline{X}:=(X_++X_-)/2$ and $[X]:=X_+-X_-$, where $X_\pm$ are the limits of X at the shell, X_+ from right, and X_- from left.

stituted back into the action; in this manner, a partially reduced system with three pairs of conjugate variables (E_+,T_+), (E_-,T_-), and $([P],\overline{R})$ and just one constraint $C_s^r=0$ is obtained. In four sectors of the phase space, C_s^r has a similar form as the super-Hamiltonian of Ref. [5], which has been derived in a completely different way. The origin of the second-class constraints is in the additional conditions by which the general Einstein-Hilbert-ideal-fluid action must be supplemented in order that the system with a shell be well defined: the so-called continuity conditions (see Ref. [12], and the next section).

The Schwarzschild coordinates $(T_\pm,R_\pm)$ of the shell are singular at the horizons of the spacetimes inside and outside the shell. Each of these two spacetimes is separated by the horizons into four quadrants. As a consequence of this, the phase space of the system is split up into 16 disjoint sectors. The dynamical trajectories that result from the action can be smoothly matched through the horizons, because the shells are regular there. This suggests that there are dynamical variables which are regular at the horizons; we try the Eddington-Finkelstein and Kruskal transformations. The first one leads to an atlas of 16 overlapping Darboux charts covering a single, connected extension of the old phase space; this extension is not maximal, however, because the points at the intersections of the horizons are not covered. The second transformation leads to one single chart covering the maximal extension of the old phase space. The constraints have smooth extensions to new phase space in both cases. The Poisson brackets between the Kruskal coordinates u and v of the shell does not vanish and we show that this must be true for *any* spacetime coordinates that are regular at the intersection of two horizons.

Our super-Hamiltonians are reparametrization invariant, but rather complicated: they depend on momenta through exponentials and square roots. Some problems arise immediately. For example, the problem of quantizing such complicated super-Hamiltonians or the problem of relation between the super-Hamiltonians of the present paper and that of Ref. [11], which is not only reparametrization invariant but also quadratic in momenta. These problems will not be addressed here.

The plan of the paper is as follows. In Sec. II, we introduce the general formula for the super-Hamiltonian from Ref. [12]. In this way, the paper becomes self-contained. In Sec. III, the assumption of spherical symmetry is formulated, the dynamical variables are adapted to the symmetry, and the action is expressed as a functional of these variables. In Sec. IV, the Kuchař transformation is performed and an effective shell super-Hamiltonian is derived. Section V is devoted to the study of the shell action obtained in Sec. IV. We check that correct equations of motion result from it, investigate the structure of the constrained system defined by the action, and remove the second-class constraint by a partial reduction. Finally, Sec. VI addresses the problem of the singularity at the horizons. We use the units such that $c=G=1$ (c is the velocity of light in vacuum and G is Newton constant).

II. THE SPACETIME AND THE SHELL

In this section, we describe the spacetime with the shell, introduce the basic ideas and quantities, and collect the equations from Ref. [12] that will be needed as a starting point of our investigation. Let $(\mathcal{M},g)$ be an assymptotically flat globally hyperbolic spacetime and let a thin shell of ideal fluid move along a timelike hypersurface Σ in $\mathcal{M}$; Σ divides the spacetime into two parts, $\mathcal{M}_+$ and $\mathcal{M}_-$, so that $\mathcal{M}_+$ is adjacent to the infinity where the observers are. Let $x^\mu_{(\pm)}$ be some coordinates in $\mathcal{M}_\pm$ and ξ^α be some in Σ. No relation between the coordinates $x^\mu_{(-)}$ and $x^\mu_{(+)}$ is assumed. Let $x^\mu_{(\pm)}(\xi)$ be the embedding functions of Σ in $\mathcal{M}_\pm$. We assume that (see Ref. [12])

$$\gamma_{\alpha\beta}(\xi)=\left(g_{(-)\mu\nu}\frac{\partial x^\mu_{(-)}}{\partial\xi^\alpha}\frac{\partial x^\nu_{(-)}}{\partial\xi^\beta}(x_{(-)}(\xi))\right)_-=\left(g_{(+)\mu\nu}\frac{\partial x^\mu_{(+)}}{\partial\xi^\alpha}\frac{\partial x^\nu_{(+)}}{\partial\xi^\beta}(x_{(+)}(\xi))\right)_+,\qquad(1)$$

where the symbols $(\,)_\pm$ denote the limits from the four-volumes $\mathcal{M}_\pm$ towards Σ, $g_{(\pm)\mu\nu}$ is the metric in $\mathcal{M}_\pm$ with respect to the coordinates $x^\mu_{(\pm)}$, and $\gamma_{\alpha\beta}(\xi)$ is the metric in Σ with respect to ξ^α. Equations (1) are called *continuity relations*.

Let $\{S_t\}$ be a foliation of $\mathcal{M}$ by Cauchy hupersurfaces S_t, where t runs through some real interval and let $S_{(\pm)t}:=S_t\cap\mathcal{M}_\pm$. We assume that S_t are (continuous) hypersurfaces in M and that $S_{(\pm)t}$ are smooth hypersurfaces in $\mathcal{M}_\pm$, for all t. The Arnowitt-Deser-Misner- (ADM-)like formalism described in Ref. [12] is based on a choice of coordinates $x^\mu_{(\pm)}$ that are adapted to the foliation $\{S_t\}$ on one hand and to Σ on the other. Such coordinates satisfy the following requirements. First,

$$x^0_{(\pm)}=t,\qquad \xi^0=t;$$

then, $x^k_{(\pm)}$, $k=1,2,3$, can be considered as coordinates on S_t and ξ^K, $K=1,2$, as coordinates om $\Sigma\cap S_t$. Second, the embedding functions $x^\mu_{(\pm)}(\xi)$ defining Σ satisfy

$$x^0_{(\pm)}(\xi^0,\xi^1,\xi^2)=\xi^0,$$

$$\frac{d}{d\xi^0}x^k_{(\pm)}(\xi^0,\xi^1,\xi^2)=0$$

for all $(\xi^0,\xi^1,\xi^2)\in\Sigma$ and $k=1,2,3$. Thus, the vector $\partial/\partial t$ is tangential to Σ. The functions

$$y^k_{(\pm)}(\xi^1,\xi^2):=x^k_{(\pm)}(\xi^0,\xi^1,\xi^2)$$

can be considered as embedding functions of the surface $\Sigma\cap S_t$ in the hypersurfaces $S_{(\pm)t}$; they are independent of the time coordinate t. Thus, the dynamics of the shell is completely determined by the time dependence of the metric and of the matter fields along Σ. This leads to a great simplification of the formalism and of the variational procedure, but to no restriction of generality, see Ref. [12] for a discussion of this point. We shall often leave out the index t in the sequel.

The 3+1 decomposition of the metric $g_{(\pm)\mu\nu}$ can be described as follows (see Ref. [17]):

$$g^{00}_{(\pm)}=-N^{-2}_{(\pm)}, \quad g_{(\pm)0k}=N_{(\pm)k},$$

$$g_{(\pm)kl}=q_{(\pm)kl}, g_{\pm}=-N^2_{\pm}q_{\pm},$$

where $N_{\pm}$ is the lapse and $N_{(\pm)k}$ the shift in $S_{\pm}$, $q_{(\pm)kl}$ is the metric induced in $S_{\pm}$ by $g_{(\pm)\mu\nu}$, $g_{\pm}$ is the determinant of $g_{(\pm)\mu\nu}$, and $q_{\pm}$ that of $q_{(\pm)kl}$. (We work with adapted coordinates.) The 2+1 decomposition of the metric $\gamma_{\alpha\beta}$ is analogous:

$$\gamma^{00}=-\nu^{-2}, \quad \gamma_{0K}=\nu_K,$$

$$\lambda_{KL}=\gamma_{KL}, \quad \gamma=-\lambda\nu^2,$$

where λ_{KL} is the metric of the surface $\Sigma\cap S$ with respect to the coordinates ξ^K and λ its determinant. The 2+1 decomposition of the continuity relations (1) is

$$\nu=\sqrt{N^2_{\pm}-(N^{\perp}_{(\pm)})^2}, \tag{2}$$

$$\nu_K=N_{(\pm)k}\frac{\partial y^k_{(\pm)}}{\partial\xi^K}, \tag{3}$$

$$\lambda_{KL}=q_{(\pm)kl}\frac{\partial y^k_{(\pm)}}{\partial\xi^K}\frac{\partial y^l_{(\pm)}}{\partial\xi^L}, \tag{4}$$

where

$$N^{\perp}_{(\pm)}=N_{(\pm)k}m^k_{(\pm)},$$

and $m^k_{(\pm)}$ is the unit normal vector to $\Sigma\cap S$ tangent to $S_{(\pm)}$ and oriented from $S_{(-)}$ to $S_{(+)}$ (towards the observers). This orientation will be often used, so we call it *right* orientation.

An important role is played by the (hyperbolic) angle $\alpha_{\pm}$ of the two hypersurfaces Σ and $S_{(\pm)}$ which is defined by

$$\sinh\alpha_{\pm}:=-g_{(\pm)\mu\nu}n^{\mu}_{(\pm)}\tilde{m}^{\nu}_{(\pm)}, \tag{5}$$

where $n^{\mu}_{(\pm)}$ is the future-oriented unit normal to $S_{(\pm)}$ and $\tilde{m}^{\nu}_{(\pm)}$ is the right-oriented unit normal to Σ in $\mathcal{M}_{\pm}$. One easily proves that

$$N_{\pm}=\nu\cosh\alpha_{\pm}, \quad N^{\perp}_{(\pm)}=\nu\sinh\alpha_{\pm}.$$

Another important quantity is the second fundamental form l_{KL} of the surface $\Sigma\cap S$ in $S_{(\pm)}$, which is defined by

$$l_{(\pm)KL}:=m_{(\pm)k|l}\frac{\partial y^k_{(\pm)}}{\partial\xi^K}\frac{\partial y^l_{(\pm)}}{\partial\xi^L};$$

here the bar denotes the covariant derivative associated with the metric $q_{(\pm)kl}$ in $S_{(\pm)}$. We reserve semicolons for the covariant derivative defined by $g_{\mu\nu}$ in $\mathcal{M}$ and colons for that by $\gamma_{\alpha\beta}$ in Σ. The trace $l^{KL}\gamma_{KL}$ of l_{KL} will be denoted by l.

The matter of the shell is assumed to be relativistic barotropic perfect fluid. Its description follows the pattern given in Refs. [18,12]; let us collect the relevant formulas.

The mass points of the fluid fill the so-called matter space Z which is a two-dimensional manifold for a shell. The coordinates z^A, $A=1,2$, in Z can be thought of as Lagrangian coordinates of the fluid. The state of the fluid is described by the "fields" $z^A(\xi)$. The matter space carries a scalar density $h(z)$, which determines the mole or particle density of the fluid in the matter space. The mole (particle) current j^{α} in Σ is given by

$$j^{\alpha}=h\epsilon^{\alpha\beta\gamma}z^1_{\beta}z^2_{\gamma},$$

where we use the abbreviation

$$z^A_{\alpha}:=\frac{\partial z^A}{\partial\xi^{\alpha}}.$$

The current j^{α} is identically conserved, $j^{\alpha}_{:\alpha}=0$. j^{α} defines the three-velocity $u^{\alpha}(\xi)$ and the rest mole (particle) density $n(\xi)$ in Σ by

$$j^{\alpha}=\sqrt{|\gamma|}nu^{\alpha},$$

where $\gamma_{\alpha\beta}u^{\alpha}u^{\beta}=-1$.

The information about the consecutive relations of the fluid is encoded in the quantity $e(n)$ that gives the energy per mole in the rest frame of the fluid as a function of the mole density n. Then, the surface tension $-p$ of the fluid is determined by (see Ref. [18])

$$p=n^2\frac{de}{dn}.$$

The dynamics of the fluid in the fixed background three-spacetime Σ can be derived from the Lagrangian

$$L_m=-\sqrt{|\gamma|}\rho(n),$$

where $\rho:=ne(n)$ denotes the rest mass density of the fluid. The stress-energy tensor density

$$T^{\alpha\beta}=\sqrt{|\gamma|}((\rho+p)u^{\alpha}u^{\beta}+p\gamma^{\alpha\beta}) \tag{6}$$

satisfies the relation

$$T^{\alpha\beta}(x)=2\frac{\delta I_m}{\delta\gamma_{\alpha\beta}(x)}, \tag{7}$$

where I_m is the action of the fluid,

$$I_m=\int_{\Sigma}d^3\xi L_m.$$

It also satisfies (the Noether identity)

$$T^{\alpha}_{\beta}=L_m\delta^{\alpha}_{\beta}-\frac{\partial L_m}{\partial z^A_{\alpha}}z^A_{\beta}. \tag{8}$$

The momenta p_A of the fluid are defined by

$$p_A := \frac{\partial L_m}{\partial z_0^A}.$$

The negative component $-T_0^0$ of the stress-energy tensor in the adapted coordinates ξ^α is the Hamiltonian of the fluid [12]. In Ref. [12], the following important formulas have been derived:

$$T_0^0 = -\nu\sqrt{\lambda}\widetilde{T}^{\perp\perp} - \nu^K\sqrt{\lambda}\widetilde{T}_K^{\perp} \tag{9}$$

and

$$\frac{\partial T_0^0}{\partial \lambda_{KL}} = \frac{1}{2}T^{KL}, \tag{10}$$

where

$$\widetilde{T}^{\perp\perp} = \frac{n}{\rho'(j^0)^2}\lambda^{KL}z_K^A z_L^B p_A p_B + \rho, \tag{11}$$

$$\widetilde{T}_K^{\perp} = z_K^A p_A. \tag{12}$$

We have introduced the symbols

$$\widetilde{T}^{\perp\perp} = \frac{1}{\sqrt{|\gamma|}}T^{\alpha\beta}\widetilde{n}_\alpha\widetilde{n}_\beta, \quad \widetilde{T}_K^{\perp} = \frac{1}{\sqrt{|\gamma|}}T_K^\alpha\widetilde{n}_\alpha,$$

where $\widetilde{n}^\alpha$ is the future-oriented unit normal to the surface $\Sigma \cap S$ in Σ.

Finally, the master formula, the Hamiltonian of the whole system consisting of the shell and gravity (the gravitational field being also dynamical) reads (for a derivation, see Ref. [12])

$$\check{\mathcal{H}} = \int_{S_+} d^3x(N_+C + N_{(+)}^k C_k) + \int_{S_-} d^3x(N_-C + N_{(-)}^k C_k) + \int_{S\cap\Sigma} d^2\xi(\nu C_s + \nu^K C_{sK}) + \frac{1}{8\pi}\int_{S\cap\Sigma^+} d^2\xi L_0^0, \tag{13}$$

where C is the ADM super-Hamiltonian and C_k is the ADM supermomentum,

$$C = \frac{1}{16\pi}\left(\frac{2\pi^{kl}\pi_{kl} - \pi^2}{2\sqrt{q}} - \sqrt{q}R^{(3)}\right),$$

$$C_k = -\frac{1}{8\pi}\pi_{k|l}^l,$$

π^{kl} is the ADM momentum for gravity, and $R^{(3)}$ the curvature scalar of the metric q_{kl}. The surface super-Hamiltonian C_s and the surface supermomentum C_{sK} at the shell are given by

$$C_s = -\frac{1}{8\pi}[\widetilde{\pi}^{\perp\perp}\sinh\alpha - l\cosh\alpha] + \widetilde{T}_s^{\perp\perp},$$

$$C_{sK} = -\frac{1}{8\pi}[\widetilde{\pi}_K^{\perp} + \alpha_{,K}] + \widetilde{T}_{sK}^{\perp},$$

where

$$\widetilde{\pi}^{\perp\perp} = \frac{\pi^{kl}}{\sqrt{q}}m_k m_l, \quad \widetilde{\pi}_K^{\perp} = \frac{\pi^{kl}}{\sqrt{q}}q_{lr}m_k\frac{\partial y^r}{\partial \xi^K},$$

$$\widetilde{\pi}_{KL} = \frac{\pi^{kl}}{\sqrt{q}}\frac{\partial y^k}{\partial \xi^K}\frac{\partial y^l}{\partial \xi^L}.$$

The symplectic form is

$$\Omega(\delta\mathbf{X},\dot{\mathbf{X}}) = \frac{1}{16\pi}\int_S d^3x\,(\delta\pi^{kl}\dot{q}_{kl} - \delta q_{kl}\dot{\pi}^{kl}) + \frac{1}{16\pi}\int_{S\cap\Sigma} d^2\xi(\delta[\alpha]\dot{\sqrt{\lambda}} - \delta\sqrt{\lambda}[\dot{\alpha}]) + \int_{S\cap\Sigma} d^2\xi(\delta p_A\dot{z}^A - \delta z^A\dot{p}_A) - \frac{1}{16\pi}\int_{S\cap\Sigma^+} d^2\xi\,(\delta\alpha^+\dot{\sqrt{\lambda^+}} - \delta\sqrt{\lambda^+}\dot{\alpha}^+), \tag{14}$$

where the quantities with the superscript plus sign concern the hypersurface Σ^+ and

$$\delta\mathbf{X} = (\delta\pi^{kl}(x), \delta q_{kl}(x), \delta[\alpha(\xi)], \times\delta\sqrt{\lambda}(\xi), \delta p_a(\xi), \delta z^a(\xi), \delta\sqrt{\lambda^+(\xi)}, \delta\alpha^+(\xi)),$$

$$\dot{\mathbf{X}} = (\dot{\pi}^{kl}(x), \dot{q}_{kl}(x), [\alpha(\dot{\xi})], \dot{\sqrt{\lambda(\xi)}}, \times\dot{p}_a(\xi), \dot{z}^a(\xi), \dot{\sqrt{\lambda^+(\xi)}}, \dot{\alpha}^+(\xi))$$

are two vectors tangential to the symplectic manifold of the system.

The equations of motion follow from the variation formula (cf. Ref. [12])

$$\delta\check{\mathcal{H}} = \Omega(\delta\mathbf{X},\dot{\mathbf{X}}) + \frac{1}{16\pi}\int_{S\cap\Sigma^+} d^2\eta\ \gamma_{\alpha\beta}\delta Q^{\alpha\beta}, \tag{15}$$

where $Q^{\alpha\beta} := L\gamma^{\alpha\beta} - L^{\alpha\beta}$ and $L := L^{\alpha\beta}\gamma_{\alpha\beta}$. This formula plays a double role. By deriving it from the Lagrange formalism carefully considering all boundary terms, we find what is the symplectic form of the system. By comparing the right-hand side (RHS) and the left-hand side (LHS) coefficients at the variations of the same variable, we obtain the equations of motion.

The last terms in Eqs. (13) and (15) determine the so-called *control mode* (see Ref. [19]). In fact, there must be one such term for each infinity, see the next section. Σ^+ is a timelike surface that forms a boundary of S and it will be pushed to infinity eventually, $L_{\alpha\beta}$ is the second fundamental form of Σ^+ defined by

$$L_{\alpha\beta} := \widetilde{m}_{\mu;\nu}\frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial x^\nu}{\partial \xi^\beta},$$

$\widetilde{m}^\mu$ being the external (with respect to the volume closed by Σ^+) unit normal to Σ^+, and $x^\mu(\xi)$ are the embedding functions defining Σ^+. The usual canonical equations hold only if the last term in Eq. (15) vanishes. This means that the field $Q^{\alpha\beta}$ must be kept fixed at Σ^+. In Ref. [13], a more natural control mode is described; we obtain it if we perform a Legendre transformation from $\check{\mathcal{H}}$ to $\mathcal{H}$ by

$$\mathcal{H}=\check{\mathcal{H}}-\frac{1}{16\pi}\int_{\Sigma^+\cap S}d^2\xi\ \gamma_{KL}Q^{KL} \tag{16}$$

so that the boundary term in Eq. (15) becomes

$$\frac{1}{16\pi}\int_{\Sigma^+\cap S}d^2\xi\ (\gamma_{00}\delta Q^{00}+2\gamma_{0K}\delta Q^{0K}-Q^{KL}\delta\gamma_{KL}). \tag{17}$$

If the surface Σ^+ is shifted to infinity and if the usual falloff conditions on q_{kl}, π^{kl}, N, and N_k are met, then the on-shell value of $\mathcal{H}$ is the ADM mass and the expression (17) vanishes (see Ref. [13]). We will pass to this description directly in the spherically symmetric case.

III. SPHERICAL SYMMETRY

In this section, we substitute the spherically symetric values of the physical fields and foliation into the Hamiltonian (13) and the symplectic form (14). We start with the transformation of the volume terms following closely the notation by Kuchař [16].

There are coordinates t, r, ϑ, and φ such that the spacetime metric has the form

$$ds^2=-(N^2-N_r^2\Lambda^{-2})dt^2+2N_r dt dr+\Lambda^2 dr^2+R^2 d\vartheta^2 +R^2\sin^2\vartheta d\varphi^2$$

with the square root of the determinant

$$\sqrt{-g}=N\Lambda R^2\sin\vartheta,$$

where $N(t,r)$, $N^r(t,r)$, $\Lambda(t,r)$, and $R(t,r)$ are some functions of t and r. We assume that $r\in(-\infty,\infty)$, that $r=\pm\infty$ are spacelike infinities and that the equation $r=0$ defines the shell. We further assume that the coordinates are continuous across the shell. We shall leave out the indices $\pm$, but we will keep in mind that some components of the metric (N, N_r, Λ, etc.) are discontinuous across the shell.

The folii t = const carry the metric q_{kl}:

$$ds^2=\Lambda^2 dr^2+R^2 d\vartheta^2+R^2\sin^2\vartheta d\varphi^2$$

with the square root of the determinant

$$\sqrt{q}=\Lambda R^2\sin\vartheta.$$

The shell hypersurface Σ can be described by the coordinates t, ϑ, and φ and the metric $\gamma_{\alpha\beta}$ satisfying the continuity relations (1) is

$$ds^2=-(N^2-N_r^2\Lambda^{-2})dt^2+R^2 d\vartheta^2+R^2\sin^2\vartheta d\varphi^2.$$

The components of the unit future-oriented vector n normal to S are $n_\mu=-N\delta^0_\mu$. The corresponding second fundamental form K_{kl} can easily be calculated; its two independent components are

$$K_{rr}=-\frac{\Lambda}{N}(\dot{\Lambda}-(\Lambda N^r)'),$$

$$K_{\vartheta\vartheta}=-\frac{R}{N}(\dot{R}-N^r R'),$$

where the prime denotes the derivative with respect to r and the dot that with respect to t. Then, the ADM momentum π^{kl} is determined by

$$\pi^{rr}=-\frac{2R\sin\vartheta}{\Lambda N}(\dot{R}-N^r R'),$$

$$\pi^{\vartheta\vartheta}=-\frac{\Lambda R^2\sin\vartheta}{N}\left(\frac{1}{\Lambda R^2}(\dot{\Lambda}-(N^r\Lambda)')+\frac{1}{R^3}(\dot{R}-N^r R')\right).$$

We obtain for the Liouville form

$$\theta=\int dr d\vartheta d\varphi\ \pi^{kl}dq_{kl}$$
$$=-16\pi\int dr\left\{\frac{R}{N}(\dot{R}-N^r R')d\Lambda+\left(\frac{R}{N}(\dot{\Lambda}-(N^r\Lambda)') +\frac{\Lambda}{N}(\dot{R}-N^r R')\right)dR\right\}.$$

Let us set, with Kuchař,

$$P_\Lambda=-\frac{R}{N}(\dot{R}-N^r R'),$$

$$P_R=-\frac{R}{N}(\dot{\Lambda}-(N^r\Lambda)')-\frac{\Lambda}{N}(\dot{R}-N^r R').$$

Hence,

$$\pi^{rr}=\frac{2P_\Lambda}{\Lambda}\sin\vartheta,\qquad \pi^{\vartheta\vartheta}=\frac{P_R}{R}\sin\vartheta,$$

and

$$\theta=16\pi\int dr(P_\Lambda d\Lambda+P_R dR). \tag{18}$$

The volume terms in the super-Hamiltonian become

$$\frac{1}{16\pi}\int_{S_\pm} d^3x\ (NC+N^kC_k)$$

$$=\int_{S_\pm} dr\Bigg\{N^r(-\Lambda P'_\Lambda+R'P_R)+N\left(\frac{\Lambda}{2R^2}P_\Lambda^2-\frac{1}{R}P_\Lambda P_R-\frac{\Lambda R^2}{4}R^{(3)}\right)\Bigg\}, \qquad (19)$$

where

$$R^{(3)}=-4\frac{R''}{\Lambda^2R}+4\frac{\Lambda'R'}{\Lambda^3R}-2\frac{R'^2}{\Lambda^2R^2}+\frac{2}{R^2} \qquad (20)$$

is the curvature scalar of the metric q_{kl} and the hypersufaces $S_\pm$ are defined by $\pm r>0$.

The surface terms containing only the geometrical quantities are our next task. The shell surface Σ is defined by $r=0$. Thus, for the normal m^k to $\Sigma\cap S$ in S, we have

$$m_k=\Lambda\,\delta_k^r,$$

and the normal $\tilde{m}^\mu$ to Σ in M is

$$\tilde{m}_\mu=\frac{1}{\sqrt{g^{rr}}}\delta_\mu^r=\frac{\Lambda N}{\sqrt{N^2-N_r^2\Lambda^{-2}}}\delta_\mu^r.$$

Then

$$N^\perp=N^r\Lambda=\frac{N_r}{\Lambda},\qquad \nu=\sqrt{N^2-N_r^2\Lambda^{-2}}$$

and

$$\sinh\alpha=\frac{N_r}{\Lambda\nu}.$$

The definitions of λ_{KL}, l_{KL}, $\tilde{\pi}^{\perp\perp}$, and $\tilde{\pi}_K^\perp$ yield

$$\lambda_{KL}=\begin{pmatrix}R^2, & 0\\ 0, & R^2\sin^2\vartheta\end{pmatrix},\qquad \sqrt{\lambda}=R^2\sin\vartheta,$$

$$l_{KL}=\frac{R'}{R\Lambda}\lambda_{KL},\qquad l=\frac{2R'}{R\Lambda},\qquad \tilde{\pi}^{\perp\perp}=\frac{2}{R^2}P_\Lambda,\qquad \tilde{\pi}_K^\perp=0.$$

Hence, the surface term in the Hamiltonian (13) becomes

$$\int_{\Sigma\cap S} d^2\xi\ (\nu C_s+v^KC_{sK})$$

$$=\nu\left(\left[-P_\Lambda\sinh\alpha+\frac{RR'}{\Lambda}\cosh\alpha\right]+M(R)\right)_{r=0} \qquad (21)$$

and an analogous term in the symplectic form (14) is

$$\frac{1}{16\pi}\int_{\Sigma\cap S} d^2\xi\ \sqrt{\lambda}\left(\frac{\dot{\lambda}}{\lambda}\delta[\alpha]-[\dot{\alpha}]\frac{\delta\lambda}{\lambda}\right)$$

$$=(\delta[\alpha]R\dot{R}-R\,\delta R[\dot{\alpha}])_{r=0}. \qquad (22)$$

The matter space Z will carry the coordinates $z^1=\Theta$, $z^2=\Phi$, and the mole density $h=\sin\Theta$ [in fact, any scalar factor in front of h can be swallowed by $e(n)$]; the matter fields $z^A(\xi)$ will simply be

$$\Theta(t,\vartheta,\varphi)\equiv\vartheta,\qquad \Phi(t,\vartheta,\varphi)\equiv\varphi.$$

Thus, $z_K^A=\delta_K^A$, $\dot{z}^A=0$, and we obtain

$$j^\alpha=(h,0,0),\qquad p_A=0,\qquad n=\frac{\sin\vartheta}{\sqrt{\lambda}}=\frac{1}{R^2}.$$

The fluid Hamiltonian is

$$-T_0^0=\nu R^2\sin\vartheta\,\rho=\nu e\sin\vartheta$$

or

$$-\int_{\Sigma\cap S} d\vartheta\, d\varphi\ T_0^0=4\pi\nu e. \qquad (23)$$

We introduce the so-called *mass function* $M(R):=4\pi e(R^{-2})$; the meaning of it is the total rest mass of the shell of radius R (see Ref. [11]). As the momentum p_A is identically zero, there is no contribution to the symplectic form by the matter.

Collecting the results (19), (18), (21), (22), and (23), we obtain the Hamiltonian for the spherically symmetric system:

$$\begin{aligned}\mathcal{H}=&\int_{r<0} dr\ \Bigg\{N^r(-\Lambda P'_\Lambda+R'P_R)+N\left(\frac{\Lambda}{2R^2}P_\Lambda^2-\frac{1}{R}P_\Lambda P_R-\frac{\Lambda R^2}{4}R^{(3)}\right)\Bigg\}\\&+\int_{r>0} dr\ \Bigg\{N^r(-\Lambda P'_\Lambda+R'P_R)+N\left(\frac{\Lambda}{2R^2}P_\Lambda^2-\frac{1}{R}P_\Lambda P_R-\frac{\Lambda R^2}{4}R^{(3)}\right)\Bigg\}\\&+\nu\left(\left[-P_\Lambda\sinh\alpha+\frac{RR'}{\Lambda}\cosh\alpha\right]+M(R)\right)_{r=0}\\&+E(\infty)+E(-\infty),\end{aligned} \qquad (24)$$

where $E(\pm\infty)$ is the ADM energy at $r=\pm\infty$ and $R^{(3)}$ is given by Eq. (20). The symplectic form reads

$$\Omega(\delta\mathbf{X},\dot{\mathbf{X}})=\int_0^\infty dr\ (\delta P_\Lambda\dot{\Lambda}-\delta\Lambda\dot{P}_\Lambda+\delta P_R\dot{R}-\delta R\dot{P}_R)$$
$$+\int_{-\infty}^0 dr\ (\delta P_\Lambda\dot{\Lambda}-\delta\Lambda\dot{P}_\Lambda+\delta P_R\dot{R}-\delta R\dot{P}_R)$$
$$+(\delta[\alpha]R\dot{R}-R\delta R[\dot{\alpha}])_{r=0}, \tag{25}$$

where

$$\delta\mathbf{X}$$
$$=(\delta P_\Lambda(r),\delta\Lambda(r),\delta P_R(r),\delta R(r),\delta[\alpha]_{r=0},\delta(R^2/2)_{r=0}),$$
$$\dot{\mathbf{X}}=(\dot{P}_\Lambda(r),\dot{\Lambda}(r),\dot{P}_R(r),\dot{R}(r),[\dot{\alpha}_{r=0}],(R^2/2)^{\cdot}_{r=0}).$$

The equation of motion can be obtained from the variation formula

$$\delta\mathcal{H}=\Omega(\delta\mathbf{X},\dot{\mathbf{X}}).$$

The same equations of motion can be obtained from a Hamiltonian action I, if we employ the corresponding Liouville form instead of the symplectic one:

$$I=\int dt\Bigg(\int_{-\infty}^0 dr\ (P_\Lambda\dot{\Lambda}+P_R\dot{R})+\int_0^\infty dr\ (P_\Lambda\dot{\Lambda}+P_R\dot{R})$$
$$+([\alpha]R\dot{R})_{r=0}-\mathcal{H}\Bigg). \tag{26}$$

We have assumed that the fields N, N^r, Λ, R, P_Λ, and P_R satisfy the usual falloff conditions as described by Ref. [16] in detail.

IV. THE KUCHAŘ TRANSFORMATION

The Kuchař transformation is a canonical transformation of the gravitational volume variables so that the new variables can be neatly separated into the true degrees of freedom and the variables that indicate a point in the solution spacetime. An example is given in Ref. [16] where the spherically symmetric gravity is studied. The transformation leads to a pair of physical variables (one degree of freedom) and to the remaining variables being the Schwarzschild time $T(r)$, the curvature radius $R(r)$, and the conjugate momenta. The foliation of each spacetime solution remains completely arbitrary. As a byproduct, the equations of motion for gravity become trivial. This will help us to express the action (26) through shell variables alone without restricting the reparametrization invariance.

A. Transformation to E and P_E

In Ref. [16], the transformation is performed in two steps. This subsection goes the first one transforming the variables $(P_\Lambda,\Lambda,P_R,R)$ to (P_E,E,P_R,R). The transformation can be written as follows:

$$E=\frac{R}{2}(1-F_1F_2),$$
$$P_E=\frac{\Lambda P_\Lambda}{RF_1F_2},$$
$$\mathrm{P}_R=P_R-\frac{(F_1F_2+1)}{F_1F_2}\frac{\Lambda P_\Lambda}{2R}-\frac{R}{2}\left(\ln\left|\frac{F_1}{F_2}\right|\right)',$$

where the useful abbreviations F_1 and F_2 are

$$F_1=\frac{R'}{\Lambda}+\frac{P_\Lambda}{R},\quad F_2=\frac{R'}{\Lambda}-\frac{P_\Lambda}{R}.$$

The inverse transformation is

$$\Lambda=\sqrt{-FP_E^2+F^{-1}R'^2},$$
$$P_\Lambda=\frac{RFP_E}{\Lambda}, \tag{27}$$
$$P_R=\mathrm{P}_R+\frac{F+1}{2}P_E+\frac{R}{F}\left(\frac{FP_E}{R'}\right)'\frac{R'^2}{\Lambda^2},$$

where

$$F:=\frac{R-2E}{R}, \tag{28}$$

and the Λ's on the RHSs must be expressed with the help of the first equation.

The following important relations hold [16]

$$F=F_1F_2,\quad P_E=-T'. \tag{29}$$

The transformation of the volume part of the Liouville form has the form [16]

$$P_\Lambda d\Lambda+P_RdR=P_EdE+\mathrm{P}_RdR+\left(\frac{RdR}{2}\ln\left|\frac{F_1}{F_2}\right|\right)'+\cdots, \tag{30}$$

where the dots denote a differential of some function on the phase space, which can be discarded. In Ref. [16], the r-derivative term on the RHS of Eq. (30) could also be thrown away because the asymptotic values of the differentiated function vanished. In our case, however, this term gives a nontrivial contribution to the shell part of the Liouville form:

$$\int_{-\infty}^0 dr(R_\Lambda d\Lambda+P_RdR)+\int_0^\infty dr(P_\Lambda d\Lambda+P_RdR)$$
$$=\int_{-\infty}^0 dr(R_EdE+\mathrm{P}_RdR)+\int_0^\infty dr(R_EdE+\mathrm{P}_RdR)$$
$$+\left(\left[\ln\sqrt{\left|\frac{F_2}{F_1}\right|}\right]RdR\right)_{r=0}. \tag{31}$$

Let us study the geometrical meaning of the last term. The meaning of any quantity in the canonical formalism is given by the role it plays in the classical solutions. We can, therefore, assume that the canonical equations are satisfied. The only canonical equation we need is

$$P_\Lambda = -\frac{R}{N}(\dot{R} - R'N^r);$$

it implies that

$$F_{1,2} = \left(\frac{1}{\Lambda}\frac{\partial}{\partial r} \mp \frac{1}{N}\left(\frac{\partial}{\partial t} - N^r\frac{\partial}{\partial r}\right)\right)R. \qquad (32)$$

We also have

$$\frac{\partial}{\partial t} = Nn + N^r\Lambda m,$$

$$\frac{\partial}{\partial r} = \Lambda m,$$

where m is the right-oriented unit vector normal to $S\cap\Sigma$ and tangential to S, and n is the future-oriented unit vector normal to S at $S\cap\Sigma$ (these vectors carry, of course, the indices $\pm$ that we are leaving out provisionally); we call (n,m) *the foliation frame*. It follows that

$$F_{1,2} = (m^\mu \mp n^\mu)\frac{\partial R}{\partial x^\mu}.$$

Clearly, $m^\mu \mp n^\mu$ are radial null vectors; F_1 vanishes at the left-going (past) and F_2 at the right-going (future) horizon (see Ref. [16]). The meaning of the logarithm in Eq. (31) will be evident if we introduce the so-called *Schwarzschild frame* (n_S, m_S) defined by the conditions that the frame (n_S, m_S) is orthonormal, future- and right-oriented, and such that at least one of its vectors (as a differential operator) annihilates the function R. The horizons divide the Kruskal manifold into four quadrants $Q_{\rm I}$–$Q_{\rm IV}$. We identify them as follows: $Q_{\rm I}$ is adjacent to the right infinity, $Q_{\rm II}$ to the left one, $Q_{\rm III}$ to the future singularity, and $Q_{\rm IV}$ to the past one. The Schwarzschild frame is well defined only *inside* the four quadrants, and its components with respect to the Schwarzschild coordinates T and R there are given by the Table I. Let us define the angle β as the hyperbolic rotation angle from the Schwarzschild to the foliation frame:

$$n = n_S\cosh\beta + m_S\sinh\beta,$$

$$m = n_S\sinh\beta + m_S\cosh\beta.$$

Then,

$$m \mp n = e^{\mp\beta}(m_S \mp n_S)$$

and

$$(m^\mu \mp n^\mu)\partial_\mu R = e^{\mp\beta}(m_S^\mu \mp n_S^\mu)\partial_\mu R.$$

Working with Table I, we obtain from it that

$$\left|\frac{F_2}{F_1}\right| = e^{2\beta}$$

in all quadrants. The final result is, therefore, simply

TABLE I. Components of the Schwarzschild frame.

	n_S	m_S
$Q_{\rm I}$	$\left(\frac{1}{\sqrt{\lvert F\rvert}},0\right)$	$(0,\sqrt{\lvert F\rvert})$
$Q_{\rm II}$	$\left(-\frac{1}{\sqrt{\lvert F\rvert}},0\right)$	$(0,-\sqrt{\lvert F\rvert})$
$Q_{\rm III}$	$(0,-\sqrt{\lvert F\rvert})$	$\left(\frac{1}{\sqrt{\lvert F\rvert}},0\right)$
$Q_{\rm IV}$	$(0,\sqrt{\lvert F\rvert})$	$\left(-\frac{1}{\sqrt{\lvert F\rvert}},0\right)$

$$\left[\ln\sqrt{\left|\frac{F_2}{F_1}\right|}\right]RdR = [\beta]RdR, \qquad (33)$$

and the first step of the Kuchař transformation changes the shell part of the Liouville form as follows:

$$([\alpha]RdR)_{r=0} \to ([\alpha+\beta]RdR)_{r=0}.$$

The definition (5) implies that α is the angle of the hyperbolic rotation from the foliation frame to the *shell frame* $(\tilde{n},\tilde{m})$. Here, the vector $\tilde{n}$ is future oriented, orthogonal to $S\cap\Sigma$, and tangential to Σ, $\tilde{m}$ is right-oriented and orthogonal to Σ at $S\cap\Sigma$. We have, from Eq. (5),

$$\tilde{n} = n\cosh\alpha + m\sinh\alpha,$$

$$\tilde{m} = n\sinh\alpha + m\cosh\alpha.$$

Thus, $\alpha+\beta$ is the angle of the hyperbolic rotation from the Schwarzschild to the shell frame. Let us define

$$P = (\alpha+\beta)R;$$

P is independent of the foliation and *singular* at the horizons.

The constraints C and C_r are written down in terms of the new variables in Ref. [16]. More interesting for us is that these constraints can be replaced by an equivalent pair C_1 and C_2 that is much simpler [16]:

$$C_1 = E'(r), \quad C_2 = \mathrm{P}_R(r).$$

The shell constraint contains the expression

$$\tilde{C} := -P_\Lambda\sinh\alpha + \frac{RR'}{\Lambda}\cosh\alpha$$

that reads in the new variables as follows [cf. Eqs. (27) and (29)]:

$$\tilde{C} = \frac{RFT'}{\Lambda}\sinh\alpha + \frac{RR'}{\Lambda}\cosh\alpha.$$

TABLE II. $\sinh\beta$ and $\cosh\beta$ by means of the canonical variables.

	Q_{I}	Q_{II}	Q_{III}	Q_{IV}
$\sinh\beta$	$\frac{T'\sqrt{\lvert F\rvert}}{\Lambda}$	$-\frac{T'\sqrt{\lvert F\rvert}}{\Lambda}$	$-\frac{R'}{\Lambda\sqrt{\lvert F\rvert}}$	$\frac{R'}{\Lambda\sqrt{\lvert F\rvert}}$
$\cosh\beta$	$\frac{R'}{\Lambda\sqrt{\lvert F\rvert}}$	$-\frac{R'}{\Lambda\sqrt{\lvert F\rvert}}$	$\frac{T'\sqrt{\lvert F\rvert}}{\Lambda}$	$-\frac{T'\sqrt{\lvert F\rvert}}{\Lambda}$

This can be expressed by means of the angle $\alpha+\beta=P/R$. The foliation frame has the following components with respect to the Schwarzschild coordinates:

$$n=\left(\frac{R'}{F\Lambda},\frac{FT'}{\Lambda}\right),\quad m=\left(\frac{T'}{\Lambda},\frac{R'}{\Lambda}\right),$$

and this holds in all quadrants. It follows that $\sinh\beta$ and $\cosh\beta$ is related to T' and R' as given in Table II. The following notation will enable us to write formulas valid in all quadrants simultaneously: let

$$\mathrm{sh}_+ x := \cosh x,\quad \mathrm{sh}_- x := \sinh x,$$

and let a and b be signs defined by Table III. Then,

$$\tilde{C}=bR\sqrt{|F|}\ \mathrm{sh}_a\frac{P}{R}.$$

To summarize, the Hamiltonian action I of the system reads

$$I=\int dr\biggl\{\int_{-\infty}^{0}dr(P_E\dot{E}+\mathrm{P}_R\dot{R}-N_1C_1-N_2C_2)$$
$$+\int_0^{\infty}dr(P_E\dot{E}+\mathrm{P}_R\dot{R}-N_1C_1-N_2C_2)+([P]\dot{R})_{r=0}$$
$$+\nu\left(\left[bR\sqrt{|F|}\ \mathrm{sh}_a\frac{P}{R}\right]+M(R)\right)_{r=0}-E(\infty)-E(-\infty)\biggr\},\tag{34}$$

where $E(\infty)$ and $E(-\infty)$ are the ADM masses at each of the spacelike infinities.

B. Transformation to T and P_T

The second step of the Kuchař transformation concerns the variables E and P_E and the boundary terms at the infinities. We shall use a slightly modified version of the Kuchař procedure in this section.

TABLE III. The signs a and b.

	Q_{I}	Q_{II}	Q_{III}	Q_{IV}
a	+	+	−	−
b	+	−	−	+

Each given boundary term at the infinities assume some particular boundary condition; in our case, the lapse function $N(\pm\infty)$ must be kept equal to 1. We need more freedom, however. Such a freedom is achieved in Ref. [16] by parametrizing the system at the infinities. This can be done by introducing the coordinates $T(\pm\infty)$ of the hypersurface S_t at $r=\pm\infty$. In Ref. [16], it is shown that

$$N(\pm\infty)=\pm\dot{T}(\pm\infty)$$

and the term $E(\infty)+E(-\infty)$ in the Hamiltonian (24) or in the action (34) is to be replaced by $E(\infty)\dot{T}(\infty)-E(-\infty)\dot{T}(-\infty)$. Then, all variations can be performed, including arbitrary variation of N at both infinities, and the result are valid equations [16].

The term $E(\infty)\dot{T}(\infty)-E(-\infty)\dot{T}(-\infty)$ in the action can of course be considered as a part of the Liouville form; thus, the parametrized action contains the Liouville term:

$$\dot{\theta}=\int_{-\infty}^{0}dr(P_E\dot{E}+\mathrm{P}_R\dot{R})+\int_0^{\infty}dr(P_E\dot{E}+\mathrm{P}_R\dot{R})+([\psi]R\dot{R})_{r=0}$$
$$-E(\infty)\dot{T}(\infty)+E(-\infty)\dot{T}(-\infty).\tag{35}$$

The next step is to introduce the new variable $T(r)$ that satisfies the relation $P_E=-T'$ [see Eq. (29)] and to find the corresponding conjugate momentum. This can be done by a transformation that concerns only the variables E, P_E, $E(\pm\infty)$, and $T(\pm\infty)$. The relevant parts of the Liouville form are

$$\dot{\theta}_+=\int_0^{\infty}dr\ P_E\dot{E}-E(\infty)\dot{T}(\infty)$$

and

$$\dot{\theta}_-=\int_{-\infty}^{0}dr\ P_E\dot{E}+E(-\infty)\dot{T}(-\infty).$$

Let us substitute $-T'$ for P_E in $\dot{\theta}_+$ and transfer the primes and overdots as follows:

$$\dot{\theta}_+=-\int_0^{\infty}dr\ E'\dot{T}-(E\dot{T})_{r=0}+\left(-\int_0^{\infty}dr\ ET'\right)^{\cdot}.$$

Similarly one obtains, for $\dot{\theta}_-$

$$\dot{\theta}_-=-\int_{-\infty}^{0}dr\ E'\dot{T}+(E\dot{T})_{r=0}+\left(-\int_{-\infty}^{0}dr\ ET'\right)^{\cdot}.$$

Hence, $P_T=-E'$, and the constraints simplify even further:

$$C_1=-P_T,\quad C_2=\mathrm{P}_R.\tag{36}$$

If we introduce the notation

$$\lim_{r=0\pm}T(r)=T_\pm,\quad \lim_{r=0\pm}E(r)=E_\pm,$$

then the action (34) in the new variables reads

$$I=\int dt\Bigg\{\int_{-\infty}^{0}dr(P_T\dot{T}+\mathrm{P}_R\dot{R})+\int_{0}^{\infty}dr(P_T\dot{T}+\mathrm{P}_R\dot{R})+([P]\dot{R}-E_+\dot{T}_++E_-\dot{T}_-)_{r=0}-\int_{-\infty}^{0}dr(N_1C_1+N_2C_2)-\int_{0}^{\infty}dr(N_1C_1+N_2C_2)-\nu\left(\left[bR\sqrt{|F|}\ \mathrm{sh}_a\left(\frac{P}{R}\right)\right]+M(R)\right)_{r=0}\Bigg\}, \quad (37)$$

where C_1 and C_2 are given by Eq. (36) and a and b by Table III.

The dynamical equations for the variables T, R, P_T and P_R describing the gravitational field around the shell that result from the action (37) are

$$\dot{T}=-N_1,\quad \dot{R}=N_2 \quad (38)$$

and

$$P_T=0,\quad \mathrm{P}_R=0. \quad (39)$$

The first pair (38) does not impose any limitations on $\dot{T}$ and $\dot{R}$ because the Lagrange multipliers N_1 and N_2 are arbitrary. The second pair (39) implies that $E(r)$ is constant along each slice, $E(t,r)=E_+(t)$ and $E(t,r)=E_-(t)$. This together with $\mathrm{P}_R=0$ does not even imply that the spacetime outside the shell is Schwarzschild one.

The nontrivial part of the dynamics is completely contained in the shell equations. The shell Hamiltonian depends on the variables ν R, $E_\pm$, $P_\pm$ and on the discrete variables $a_\pm$ and $b_\pm$. It does not depend on $T_\pm$. It follows immediately that $\dot{E}_\pm=0$. This, together with the volume equations (38) and (39) is equivalent to Schwarzschild solution being the spacetime outside the shell.

We can, therefore, replace the action (37) by an effective shell action of the form

$$I_s=\int dt(P_+\dot{R}-E_+\dot{T}_+-P_-\dot{R}+E_-\dot{T}_--\nu C_s), \quad (40)$$

where

$$C_s=b_+R\sqrt{|F_+|}\ \mathrm{sh}_{a_+}\frac{P_+}{R}-b_-R\sqrt{|F_-|}\ \mathrm{sh}_{a_-}\frac{P_-}{R}+M(R), \quad (41)$$

is the super-Hamiltonian of the shell and $F_\pm$ are given by Eq. (28). We interpret the solutions $E_\pm$, $T_\pm(t)$, $P_\pm(t)$ and $R(t)$ as embedding formulas in two Schwarzschild spacetimes with energies $E_\pm$ and coordinates (T,R).

The discrete variables $a_\pm$ and $b_\pm$ describe the different sectors of the extended phase space. If the shell crosses a horizon in the spacetime to its left or right, some of the signs will change. There are 16 sectors; some of these, however, will have empty intersection with the constraint surface. Observe that $a_\pm$ is not an independent variable, but a function of R and $E_\pm$:

$$a_\pm=\mathrm{sgn}F_\pm. \quad (42)$$

The action (40) describes the motion inside the sectors and it becomes singular at sector boundaries. The variables $P_\pm$ and $T\pm$ diverge and $a_\pm$ and $b_\pm$ are not defined at the boundaries.

V. PROPERTIES OF THE SHELL ACTION

In this section, we study the properties of the action (40). We derive the equations of motion, clarify the structure of constraints and reveal a geometrical meaning of the super-Hamiltonian.

A. The equations of motion

Let us vary the action (40). The variation of ν gives the Hamiltonian constraint

$$b_+R\sqrt{|F_+|}\ \mathrm{sh}_{a_+}\frac{P_+}{R}-b_-R\sqrt{|F_-|}\ \mathrm{sh}_{a_-}\frac{P_-}{R}+M(R)=0, \quad (43)$$

and the variations of $P_\pm$ result in

$$\frac{\dot{R}}{\nu}=b_+\sqrt{|F_+|}\ \mathrm{sh}_{-a_+}\frac{P_+}{R}, \quad (44)$$

$$\frac{\dot{R}}{\nu}=b_-\sqrt{|F_-|}\ \mathrm{sh}_{-a_-}\frac{P_-}{R}, \quad (45)$$

which implies another constraint,

$$b_+\sqrt{|F_+|}\ \mathrm{sh}_{-a_+}\frac{P_+}{R}-b_-\sqrt{|F_-|}\ \mathrm{sh}_{-a_-}\frac{P_-}{R}=0. \quad (46)$$

The variation with respect to $E_\pm$ yields

$$-\dot{T}_\pm-\nu b_\pm\frac{R}{2\sqrt{|F_\pm|}}\mathrm{sh}_{a_\pm}\frac{P_\pm}{R}\frac{\partial|F_\pm|}{\partial E_\pm}.$$

For the calculation of the derivative of $|F_\pm|$, we take Eq. (42) into account:

$$\frac{\partial|F_\pm|}{\partial E_\pm}=-\frac{2a_\pm}{R}.$$

Then, the following equation results:

$$\frac{\dot{T}_\pm}{\nu}=\frac{a_\pm b_\pm}{\sqrt{|F_\pm|}}\mathrm{sh}_{a_\pm}\frac{P_\pm}{R}. \quad (47)$$

The variation of $T_\pm$ leads to

$$\dot{E}_\pm=0. \quad (48)$$

Finally, varying R, we obtain

$$-\dot{P}_++\dot{P}_--\nu\frac{\partial C_s}{\partial R}. \quad (49)$$

Equations (43)–(49) form the complete set of dynamical equations for the shell. Some discussion of these equations is in order.

First, we show that Eq. (49) is a consequence of Eqs. (43)–(45), (48), and of $\dot{R}\neq 0$ (the last relation is generically satisfied along each trajectory). Indeed, the time derivative of the super-Hamiltonian C_s must vanish as a consequence of Eq. (43):

$$\frac{\partial C_s}{\partial R}\dot{R}+\frac{\partial C_s}{\partial E_+}\dot{E}_+ +\frac{\partial C_s}{\partial E_-}\dot{E}_- +\frac{\partial C_s}{\partial P_+}\dot{P}_+ +\frac{\partial C_s}{\partial P_-}\dot{P}_- =0.$$

The second and third terms on the LHS vanish because of Eq. (48). For the last two terms, we obtain from Eqs. (44) and (45):

$$\frac{\partial C_s}{\partial P_+}=\frac{\dot{R}}{\nu},\quad \frac{\partial C_s}{\partial P_-}=-\frac{\dot{R}}{\nu}.$$

Hence,

$$\left(\frac{\partial C_s}{\partial R}+\frac{\dot{P}_+}{\nu}-\frac{\dot{P}_-}{\nu}\right)\dot{R}=0,$$

and this shows the claim.

Second, manipulating Eqs. (44), (45), and (47), we arrive at

$$-|F_\pm|\left(\frac{\dot{T}_\pm}{\nu}\right)^2+\frac{1}{|F_\pm|}\left(\frac{\dot{R}}{\nu}\right)^2=-\ \mathrm{sh}^2_{a_\pm}\frac{P_\pm}{R}+\mathrm{sh}^2_{-a_\pm}\frac{P_\pm}{R}. \tag{50}$$

A useful identity is

$$\mathrm{sh}_a(x+y)=\cosh x\ \mathrm{sh}_a y+\mathrm{sh}x\ \mathrm{sh}_{-a}y, \tag{51}$$

which can easily be derived from the definition

$$\mathrm{sh}_a x=\frac{e^x+a e^{-x}}{2}, \tag{52}$$

and which implies that $\mathrm{sh}^2_a x-\mathrm{sh}^2_{-a}x=a$, independently of x. Thus, the RHS of Eq. (50) is $-a_\pm$. Multiplying the equation by $a_\pm$ and using Eq. (42) yield

$$-F_\pm\left(\frac{\dot{T}_\pm}{\nu}\right)^2+\frac{1}{F_\pm}\left(\frac{\dot{R}}{\nu}\right)^2=-1. \tag{53}$$

This is the ''time equation'' (see Ref. [11]) saying that

$$\left(\left(\frac{\dot{T}_\pm}{\nu}\right),\left(\frac{\dot{R}}{\nu}\right)\right)$$

is a unit timelike vector. It also implies that

$$F_\pm\frac{\dot{T}_\pm}{\nu}=\tau_\pm\sqrt{F_\pm+\left(\frac{\dot{R}}{\nu}\right)^2}, \tag{54}$$

where

$$\tau_\pm:=\mathrm{sgn}\left(F_\pm\frac{\dot{T}_\pm}{\nu}\right). \tag{55}$$

Finally, we obtain, from Eqs. (43) and (47),

$$F_+\frac{\dot{T}_+}{\nu}-F_-\frac{\dot{T}_-}{\nu}=-\frac{M(R)}{R};$$

substituting into this equation from (54) yields the ''radial equation''

$$-\tau_+\sqrt{F_+ +\left(\frac{\dot{R}}{\nu}\right)^2}+\tau_-\sqrt{F_- +\left(\frac{\dot{R}}{\nu}\right)^2}=\frac{M(R)}{R}. \tag{56}$$

This is the Israel equation for spherically symmetric shells written in a way that is valid for all sectors in the case of future-oriented shell motion (see Ref. [11]). Thus, the dynamical equations implied by the action (40) are as they should be.

B. Structure of the constraints

Two constraint functions have been obtained directly from the action (40) by varying it: the super-Hamiltonian C_s and the LHS of Eq. (46), which we denote by χ. The Lagrange multiplier that gives C_s is ν, that for χ is $\overline{P}$, defined by

$$\overline{P}:=\frac{P_+ +P_-}{2}.$$

The Poisson bracket between C_s and χ requires a longer calculation; we quote just the result

$$\begin{aligned}\{\chi,C_s\}\approx &-\frac{E_+-E_-}{2R^2}\left(1+\frac{a_+b_+b_-}{\sqrt{|F_+F_-|}}\ \mathrm{sh}_{a_+a_-}\frac{[P]}{R}\right.\\ &\left.-\frac{2\,\mathrm{sgn}B}{\sqrt{B^2-A^2}}(M'-M/R)\right)\\ &-\frac{a_-b_+b_-\mathrm{sgn}B}{R^2\sqrt{B^2-A^2}}\ \mathrm{arctanh}\frac{A}{B}M(R)\\ &\times\sqrt{|F_+F_-|}\mathrm{sh}_{-a_+a_-}\frac{[P]}{R},\end{aligned}$$

where

$$A:=b_+\sqrt{|F_+|}\ \mathrm{sh}_{a_+}\frac{[P]}{2R}-a_-b_-\sqrt{|F_-|}\ \mathrm{sh}_{a_-}\frac{[P]}{2R}, \tag{57}$$

$$B:=b_+\sqrt{|F_+|}\ \mathrm{sh}_{-a_+}\frac{[P]}{2R}-a_-b_-\sqrt{|F_-|}\ \mathrm{sh}_{-a_-}\frac{[P]}{2R}, \tag{58}$$

and $[P]=P_+-P_-$ is the momentum conjugate to R. We have used the constraints in calculating the bracket, so the equality is only weak ($\approx$). The Poisson bracket is nonzero at the constraint surface, so our system cannot be purely first class and the value of some Lagrange multipliers will be determined by the equations of motion (see, e.g., [20]). Clearly, it is $\overline{P}$ which is determined, for χ depends on it and can, therefore be used to calculate it:

$$\chi = A\cosh\frac{\overline{P}}{R} + B\sinh\frac{\overline{P}}{R}$$

or

$$\overline{P} = -R\ \mathrm{arctanh}\frac{A}{B}. \tag{59}$$

The Lagrange multiplier ν is not restricted by the equations of motion. This means that the system is mixed, containing both first- and second-class constraints. To prove that, we extend the phase space by another conjugate pair, $(\overline{P},\pi)$ and constraint the momentum π to be zero,

$$\pi = 0.$$

This constraint must be enforced by another Lagrange multiplier $\tilde{\nu}$, say, and the corresponding additional term in the action is $-\tilde{\nu}\pi$. The new system is clearly equivalent to the old one, but it has three constraints C_s, χ, and π. We obtain easily

$$\{\pi, C_s\} = \frac{\partial C_s}{\partial \overline{P}} = \chi$$

and

$$\{\chi,\pi\} = \frac{\partial\chi}{\partial\overline{P}} = -\frac{M(R)}{R^2} + \frac{1}{R^2}C_s.$$

Thus, the pair (χ,π) represents the second-class part of the constraint system, and a modification $\tilde{C}_s$ of C_2 defined by

$$\tilde{C}_s := C_s + \frac{\{\chi, C_s\}R^2}{M(R)}\ \pi$$

has weakly vanishing Poisson brackets with both χ and π. The equations $C_s = 0$ and $\pi = 0$ are primary constraints and $\chi = 0$ is a secondary constraint.

Let us write down the action of the extended system

$$I_s^e = \int dt([P]\dot{R} - E_+\dot{T}_+ + E_-\dot{T}_- + \pi\dot{\overline{P}} - \tilde{\nu}\pi - \nu C_s), \tag{60}$$

where we have to substitute $\overline{P} \pm [P]/2$ for $P_\pm$. The method of the Dirac brackets can be applied to I_s^e. An (equivalent) alternative is to get rid of $\overline{P}$ by solving the constraint $\chi = 0$ for it and inserting the solution back into the action (40).

C. Partial reduction

In this subsection, we reduce the system partially by substituting Eq. (59) for $\overline{P}$ into the action (40). First, we make the dependence of C_s on $\overline{P}$ explicit:

TABLE IV. The vector ξ and the Schwarzschild frame.

	Q_{I}	Q_{II}	Q_{III}	Q_{IV}
ξ	$n_S\sqrt{\lvert F\rvert}$	$-n_S\sqrt{\lvert F\rvert}$	$m_S\sqrt{\lvert F\rvert}$	$-m_S\sqrt{\lvert F\rvert}$

$$C_s = RA\sinh\frac{\overline{P}}{R} + RB\cosh\frac{\overline{P}}{R} + M(R),$$

where A and B are given by Eqs. (57) and (58). Equation (59) can be written in the form

$$\sinh\frac{\overline{P}}{R} = -\mathrm{sgn}B\,\frac{A}{\sqrt{B^2 - A^2}},$$

$$\cosh\frac{\overline{P}}{R} = \mathrm{sgn}B\,\frac{B}{\sqrt{B^2 - A^2}}.$$

Hence, we obtain, for C_s,

$$C_s = \mathrm{sgn}BR\sqrt{B^2 - A^2} + M(R).$$

Clearly, the constraint surface intersects only those sectors, where the following conditions are satisfied:

$$\mathrm{sgn}B = -1, \quad |B| > |A|. \tag{61}$$

The definitions (57) and (58) lead to

$$B^2 - A^2 = F_+ + F_- - 2a_-b_+b_-\sqrt{|F_+F_-|}\ \mathrm{sh}_{a_+a_-}\frac{[P]}{R},$$

where we have used the identity (51); we obtain the partially reduced super-Hamiltonian, which we denote by C_s^r,

$$C_s^r = R\ \mathrm{sgn}B \times \sqrt{F_+ + F_- - 2a_-b_+b_-\sqrt{|F_+F_-|}\ \mathrm{sh}_{a_+a_-}\frac{[P]}{R}} + M(R); \tag{62}$$

the corresponding action, which is independent of $\overline{P}$ and implies only one constraint, reads

$$I_s^r = \int dt\{[P]\dot{R} - E_+\dot{T}_+ + E_-\dot{T}_- - \nu C_s^r\}. \tag{63}$$

The super-Hamiltonian (62) in the four sectors where $a_+ = a_- = 1$ seems to be the same as the zero cosmological constant case of the super-Hamiltonian derived in Ref. [5].

The expression under the square root in Eq. (62) reminds of the cosine theorem, and, indeed, it has a simple geometrical interpretation. Consider the vector ξ generating the Schwarzschild time shift. There is a simple expression for ξ in terms of the Schwarzschild frame, because one leg of this frame is always parallel to ξ; for each quadrant, ξ is given by Table IV. Let us find the components of ξ with respect to the shell frame using the transformation between the shell and the Schwarzschild frame

$$n_S = \widetilde{n}\cosh\frac{P}{R} - \widetilde{m}\sinh\frac{P}{R},$$

$$m_S = -\widetilde{n}\sinh\frac{P}{R} + \widetilde{m}\cosh\frac{P}{R}.$$

The result can be summarized by the formula

$$\xi = \widetilde{n} b\sqrt{|F|}\ \mathrm{sh}_a\frac{P}{R} - \widetilde{m} b\sqrt{|F|}\ \mathrm{sh}_{-a}\frac{P}{R}, \tag{64}$$

which is valid in all quadrants; we have left out the indices $\pm$. Comparing Eq. (64) with the original form of the constraints C_s and χ, we can see immediately that

$$C_s = R(\xi^0_{(+)} - \xi^0_{(-)}) + M(R)$$

and

$$\chi = \xi^1_{(+)} - \xi^1_{(-)},$$

where the shell frame components ξ^0 and ξ^1 of the vector ξ are given by Eq. (64). The geometrical meaning of the constraint $\chi = 0$ is, therefore, that the space component of the ''vector difference'' $\xi_+ - \xi_-$ vanishes, and of $C_s = 0$ that the time component of this vector difference equals $-M(R)/R$.

In the case that $\chi = 0$, we have

$$|\xi_+ - \xi_-| = |\xi^0_{(+)} - \xi^0_{(-)}|,$$

where $|\xi_+ - \xi_-|$ is the ''length'' of the ''vector'' $\xi_+ - \xi_-$, defined by

$$|\xi_+ - \xi_-| = \sqrt{|-(\xi^0_{(+)} - \xi^0_{(-)})^2 + (\xi^1_{(+)} - \xi^1_{(-)})^2|}.$$

It follows that

$$C_s = R\ \mathrm{sgn}(\xi^0_{(+)} - \xi^0_{(-)})|\xi_+ - \xi_-| + M(R).$$

Let us calculate the value of $(\xi_+ - \xi_-)^2$ using Eq. (64). The result is

$$-(\xi^0_{(+)} - \xi^0_{(-)})^2 + (\xi^1_{(+)} - \xi^1_{(-)})^2$$
$$= -F_+ - F_- + 2a_-b_+b_-\sqrt{|F_+F_-|}\ \mathrm{sh}_{a_+a_-}\frac{[P]}{R}.$$

This coincides, up to the sign, with the expression under the square root in Eq. (62). It is also clear that the constraint $\chi = 0$ must imply, first, that

$$\mathrm{sgn}B = \mathrm{sgn}(\xi^0_{(+)} - \xi^0_{(-)})$$

and, second, that $B^2 - A^2 > 0$, if the vector difference $\xi_+ - \xi_-$ is timelike. This finishes the clarification of a geometrical meaning of the constraints.

VI. MATCHING THE SECTORS

The actions (40), (60), and (63) are singular at each horizon $R = 2E_\pm$, because the coordinate $T_\pm$ and the momentum $P_\pm$ diverge. Thus, the actions can be used only inside the 16 sectors; they do not say, at least directly, what happens at the boundary.

The form of the singularity in $P_\pm$ can be inferred from Eq. (64): both the vector ξ and the shell frame $(\widetilde{n},\widetilde{m})$ are smooth objects, so the components are to be smooth, too. It follows that

$$\mathrm{sh}_{a_\pm}\frac{P_\pm}{R} \sim \frac{1}{\sqrt{|F_\pm|}}, \quad \mathrm{sh}_{-a_\pm}\frac{P_\pm}{R} \sim \frac{1}{\sqrt{|F_\pm|}} \tag{65}$$

at the horizons.

This section will be based on a transformation of the extended action (60) that may be interesting for other purposes, too. First, we introduce the variables $R_\pm$ by

$$R = \frac{R_+ + R_-}{2}, \tag{66}$$

$$\pi = -R_+ + R_-. \tag{67}$$

The meaning of the variables R_+ and R_- is simply that they give the values of the function R at the shell from the right and from the left, respectively. Thus, the constraint $\pi = 0$ is nothing but the only remaining continuity condition from Eq. (1). Let us substitute Eqs. (66) and (67) into the Liouville part of the action I^e_s:

$$[P]\dot{\overline{R}} - [R]\dot{\overline{P}} - E_+\dot{T}_+ + E_-\dot{T}_-$$
$$= [P]\dot{\overline{R}} + \overline{P}[\dot{R}] - E_+\dot{T}_+ + E_-\dot{T}_- - (\overline{P}[R])^{\cdot}$$
$$= P_+\dot{R}_+ - E_+\dot{T}_+ - P_-\dot{R}_- + E_-\dot{T}_- - (\overline{P}[R])^{\cdot},$$

where we have used the well-known formula $[XY] = \overline{X}[Y] + \overline{Y}[X]$, valid for any two functions X and Y.

The terms

$$b_\pm\overline{R}\sqrt{\left|1 - \frac{2E_\pm}{\overline{R}}\right|}\ \mathrm{sh}_{a_\pm}\frac{P_\pm}{\overline{R}}$$

that result in the super-Hamiltonian after the substitution (66) can be replaced by

$$b_\pm R_\pm\sqrt{|F_\pm|}\ \mathrm{sh}_{a_\pm}\frac{P_\pm}{R_\pm},$$

where

$$F_\pm = 1 - \frac{2E_\pm}{R_\pm}.$$

Indeed, $R_\pm = \overline{R} \mp \pi/2$, so the replacement amounts to using the constraint $\pi = 0$ in the super-Hamiltonian; such a procedure does not change the equations of motion because it preserves the constraint surface (see Refs. [20] and [21]). Finally, we arrive at the action

$$I^f_s = \int dt(P_+\dot{R}_+ - E_+\dot{T}_+ - P_-\dot{R}_- + E_-\dot{T}_- + \overline{\nu}[R] - \nu C^f_s), \tag{68}$$

where

$$C_s^f = b_+ R_+ \sqrt{|F_+|}\ \mathrm{sh}_{a_+} \frac{P_+}{R_+} - b_- R_- \sqrt{|F_-|}\ \mathrm{sh}_{a_-} \frac{P_-}{R_-} + M(\overline{R}). \tag{69}$$

The Liouville part in the action (68) is split up into two pieces, each being of the form $P\dot{R} - E\dot{T}$, where T and R are coordinates in a spacetime—the Schwarzschild spacetime left or right to the shell—and P and $-E$ are the conjugate momenta. This enables us to generate transformations of the coordinates on the phase space from transformations of coordinates (T,R) on the Schwarzschild spacetime.

We observe first that the transformation from the Schwarzschild coordinates (T,R) to the Eddington-Finkelstein coordinates (U,R) or (V,R) can be completed to a canonical transformation. This is not as trivial as it may seem: the problem is that the transformation of the coordinates contains the momentum $-E$. The dependence on E is harmless for the Eddington-Finkelstein transformation; it is more serious for the transformation to the Kruskal coordinates.

A. Eddington-Finkelstein coordinates

Let us study the Eddington-Finkelstein transformations. As these transformations do not change the coordinate R, it is not necessary to distinguish R_+ from R_- if we are performing it. Thus, we can substitute $\pi = 0$ everywhere into the action (68): $R_+ = R_- = R$ and $\overline{R} = R$. In this way, we return to the action (40). In the following formulas, we shall also suppress the annoying indices $\pm$.

The first Eddington-Finkelstein transformation, in each quadrant and on each side of the shell, is given by

$$R_U = R,$$

$$U = T - R - 2E \ln\left|\frac{R}{2E} - 1\right|;$$

a suitable ansatz for the new momenta P_{UR} and P_U is

$$P_{UR} = P + R \ln\sqrt{|F|}, \tag{70}$$

$$P_U = P_T = -E.$$

A similar ansatz for the second transformation is

$$R_V = R,$$

$$V = T + R + 2E \ln\left|\frac{R}{2E} - 1\right|,$$

$$P_{VR} = P - R \ln\sqrt{|F|}, \tag{71}$$

$$P_V = P_T = -E.$$

To show that the transformations are canonical, we calculate dU and dV:

$$dU = dT - \frac{R}{R-2E} dR - 2\left(\ln\left|\frac{R}{2E} - 1\right| - \frac{R}{R-2E}\right) dE,$$

$$dV = dT + \frac{R}{R-2E} dR + 2\left(\ln\left|\frac{R}{2E} - 1\right| - \frac{R}{R-2E}\right) dE,$$

and substitute this into the Liouville form. We obtain

$$P_{UR} dR + P_U dU - P dR - P_T dT = dG,$$

where

$$G = E^2 \ln\left|\frac{R}{2E} - 1\right| + \frac{RE}{2} + \frac{R^2}{2} \ln\sqrt{|F|}.$$

Similarly,

$$P_{VR} dR + P_V dV - P dR - P_T dT = -dG.$$

The transformation of the super-Hamiltonian C_s depends on the transformation of the term

$$bR\sqrt{|F|}\ \mathrm{sh}_a \frac{P}{R}.$$

We obtain in each of the four quadrants that

$$bR\sqrt{|F|}\ \mathrm{sh}_a \frac{P}{R} = \frac{bR}{2}(\mathrm{e}^{P_U/R} + F \mathrm{e}^{-P_U/R})$$

for the transformation to the U charts (we have left out the indices U and V at R). From the definition of b in Table III, we can see that b is continuous inside each U chart U_{I} and U_{II}. Let us define the sgn b_U by $b_U := b$ so that

$$b_U = +1 \quad \text{in} \quad U_{\mathrm{I}} := \overline{Q_{\mathrm{I}} \cup Q_{\mathrm{IV}} \setminus H^+}$$

and

$$b_U = -1 \quad \text{in} \quad U_{\mathrm{II}} := \overline{Q_{\mathrm{II}} \cup Q_{\mathrm{III}} \setminus H^+}.$$

At the future horizons H^+, where $T = +\infty$, $U \to +\infty$ and $P_U \to -\infty$ in such a way that $F\exp(-P_U/R)$ is smooth.

The transformation to the V charts V_{I} and V_{II} is analogous:

$$bR\sqrt{|F|}\ \mathrm{sh}_a \frac{P}{R} = \frac{abR}{2}(\mathrm{e}^{-P_V/R} + F \mathrm{e}^{P_V/R}).$$

We define $b_V := ab$ so that we have

$$b_V = +1 \quad \text{in} \quad V_{\mathrm{I}} := \overline{Q_{\mathrm{I}} \cup Q_{\mathrm{III}} \setminus H^-}$$

and

$$b_V = -1 \quad \text{in} \quad V_{\mathrm{II}} := \overline{Q_{\mathrm{II}} \cup Q_{\mathrm{IV}} \setminus H^-}.$$

Again, the super-Hamiltonian has continuous extension to each V chart. At the past horizon H^-, where $T = -\infty$, $V \to -\infty$ and $P_V \to +\infty$ in such a way that $F\exp(P_V/R)$ is smooth.

The result of this section can be interpreted as a new, connected phase space that is covered by 16 charts which overlap and that contains all of the 16 disjoint sectors of the old phase space; the super-Hamiltonian has a continuous extension to the new phase space. The origins of the Kruskal manifolds remain excluded, however.

B. Kruskal coordinates

The Kruskal coordinates u and v are regular everywhere inside the Kruskal manifold (but they are ''singular'' at the infinity). Thus, they are suitable to cover the missing points where the horizons intersect.

In each quadrant, the transformation between the Schwarzschild coordinates (T,R) and the Kruskal coordinates (u,v) is given by (see, e.g., Ref. [17])

$$\frac{R}{2E}=K(-uv), \tag{72}$$

$$\frac{T}{2E}=\ln\left|\frac{v}{u}\right|, \tag{73}$$

where the function $K:(-1,\infty)\mapsto(0,\infty)$ is a smooth bijection defined by its inverse

$$K^{-1}(x)=(x-1)\mathrm{e}^{x}, \tag{74}$$

and where the signs of the Kruskal coordinates are defined to be

$$u<0 \quad \text{in } Q_{\mathrm{I}}\cup Q_{\mathrm{IV}}, \quad u>0 \quad \text{in } Q_{\mathrm{II}}\cup Q_{\mathrm{III}},$$

$$v<0 \quad \text{in } Q_{\mathrm{II}}\cup Q_{\mathrm{IV}}, \quad v>0 \quad \text{in } Q_{\mathrm{I}}\cup Q_{\mathrm{III}}.$$

To begin with, we derive some useful relations. Equation (72) implies

$$F=\frac{K-1}{K} \tag{75}$$

(we leave out the argument of K; it will always be $-uv$). Equations (72) and (74) imply

$$-uv=K^{-1}\left(\frac{R}{2E}\right)=\left(\frac{R}{2E}-1\right)\mathrm{e}^{R/2E}=F\frac{R}{2E}\mathrm{e}^{R/2E}$$

or

$$F=-\frac{uv}{K\mathrm{e}^{K}}. \tag{76}$$

Combining Eqs. (75) and (76), we obtain that

$$K-1=-\mathrm{e}^{-K}uv. \tag{77}$$

The next step is to find a smooth ''momentum'' to replace $P_\pm$. We know from the previous subsection that P_{UR} is smooth at the past horizon H^-, where $v=0$, and P_{VR} at the future horizon H^+, where $u=0$. Equations (70) and (76) give

$$P_{UR}=P+\frac{R}{2}\ln|v|+\text{smooth at } H^-,$$

and, analogously,

$$P_{VR}=P-\frac{R}{2}\ln|u|+\text{smooth at } H^+.$$

Accordingly, the function $\tilde{P}$ defined by

$$\tilde{P}:=P+\frac{R}{2}\ln\left|\frac{v}{u}\right|$$

might be smooth everywhere. This suggests that we try the following transformation of momenta:

$$P=\tilde{P}-\tilde{E}K\ln\left|\frac{v}{u}\right|, \tag{78}$$

$$E=\tilde{E}, \tag{79}$$

and check whether or not the symplectic form expressed by means of the variables u, v, $\tilde{P}$, and $\tilde{E}$ is regular everywhere (from now on, we shall leave out the tilde over E). Recall that all equations are written without the indices $\pm$; in fact, Eq. (78) reads, if written properly,

$$P_\pm=\tilde{P}_\pm-\tilde{E}_\pm K_\pm\ln\left|\frac{v_\pm}{u_\pm}\right|,$$

where $K_\pm=K(-u_\pm v_\pm)$, etc.

Let us transform the action to the variables $(u_\pm,v_\pm,\tilde{P}_\pm,E_\pm)$. Equations (72) and (73) yield

$$dR=2KdE-2EK'(v\,du+u\,dv),$$

$$dT=2\ln\left|\frac{v}{u}\right|dE+2E\left(-\frac{du}{u}+\frac{dv}{v}\right).$$

This together with Eq. (78) implies

$$PdR-EdT=\tilde{P}dR-2E(K^2+1)\ln\left|\frac{v}{u}\right|dE+E^2\left(2KK'v\ln\left|\frac{v}{u}\right|+\frac{2}{u}\right)du+E^2\left(2KK'u\ln\left|\frac{v}{u}\right|-\frac{2}{v}\right)dv.$$

The first term on the RHS is smooth and the rest is singular. To get rid of it, we observe that

$$d\left(-E^2(K^2+1)\ln\left|\frac{v}{u}\right|\right)=-2E(K^2+1)\ln\left|\frac{v}{u}\right|dE+E^2\left(2KK'v\ln\left|\frac{v}{u}\right|+\frac{K^2+1}{u}\right)du+E^2\left(2KK'u\ln\left|\frac{v}{u}\right|-\frac{K^2+1}{v}\right)dv.$$

This identity implies

$$PdR-EdT=-2EKd\tilde{P}+E^2(K^2-1)\left(-\frac{du}{u}+\frac{dv}{v}\right)+d\left(-E^2(K^2+1)\ln\left|\frac{v}{u}\right|+2EK\tilde{P}\right).$$

The second term on the RHS is regular; indeed, Eq. (77) gives

$$E^2(K^2-1)\left(-\frac{du}{u}+\frac{dv}{v}\right)=E^2(K+1)\mathrm{e}^{-K}(v\,du-u\,dv).$$

Hence, finally, the Liouville form becomes

$$[PdR-EdT]=[-2EKd\tilde{P}+E^2(K+1)\mathrm{e}^{-K}(v\,du-u\,dv)]+d\left[-E^2(K^2+1)\ln\left|\frac{v}{u}\right|+2EK\tilde{P}\right]. \quad (80)$$

The singular part is contained entirely within the last term, which can be discarded without changing the symplectic structure. We postpone the study of the resulting symplectic structure to the next section.

The last nontrivial step in the transformation of the action is to transform the term

$$bR\sqrt{|F|}\ \mathrm{sh}_a\frac{P}{R}$$

in the super-Hamiltonian (69) (the indices $\pm$ are again left out). Using Eqs. (72), (76), and (78), we obtain

$$bR\sqrt{|F|}\ \mathrm{sh}_a\frac{P}{R}=bE\sqrt{K}\mathrm{e}^{-K/2}\sqrt{|uv|}\ \mathrm{sh}_a\left(\frac{\tilde{P}}{2EK}-\ln\sqrt{\left|\frac{v}{u}\right|}\right).$$

Equations (51) and (52) lead then to

$$bR\sqrt{|F|}\ \mathrm{sh}_a\frac{P}{R}=E\sqrt{K}\mathrm{e}^{-K/2}\left(b|u|\exp\left(\frac{\tilde{P}}{2EK}\right)+ab|v|\exp\left(-\frac{\tilde{P}}{2EK}\right)\right).$$

The signs of the Kruskal coordinates as defined at the beginning of this subsection combine with Table III giving that $b|u|=-u$ and $ab|v|=v$ in each quadrant. Thus, we arrive at the expression

$$bR\sqrt{|F|}\ \mathrm{sh}_a\frac{P}{R}=E\sqrt{K}\mathrm{e}^{-K/2}\left(-u\exp\left(\frac{\tilde{P}}{2EK}\right)+v\exp\left(-\frac{\tilde{P}}{2EK}\right)\right). \quad (81)$$

Collecting the results (80) and (81), we obtain the final form of the action

$$I_s^K=\int dt([-2EK\dot{\tilde{P}}+E^2(K+1)\mathrm{e}^{-K}(v\dot{u}-u\dot{v})]+2\tilde{\nu}\,[EK]-\nu C_s^K), \quad (82)$$

where

$$C_s^K=\left[E\sqrt{K}\mathrm{e}^{-K/2}\left(-u\exp\left(\frac{\tilde{P}}{2EK}\right)+v\exp\left(-\frac{\tilde{P}}{2EK}\right)\right)\right]+M(E_+K_++E_-K_-). \quad (83)$$

Let us recall that $[X]=X_+-X_-$ and that $K_\pm=K(-u_\pm v_\pm)$, etc. The action (82) as well as all variables on which it depends, are smooth everywhere in the new phase space. This phase space is covered by the coordinates $u_\pm$, $v_\pm$, $\tilde{P}_\pm$, and $E_\pm$ with ranges $-\infty<u_\pm v_\pm<1$ (Kruskal manifold) $\tilde{P}_\pm\in(-\infty,\infty)$, and $E_\pm\in(0,\infty)$; it is the maximal extension of the old phase space. The super-Hamiltonian (69) as well as the function $[R]=2[EK]$ have continuous extensions to the new phase space.

C. The symplectic form and Poisson brackets

In this subsection, we investigate the properties of the symplectic structure defined by the Liouville form (80). Taking the external derivative of the form (80), we obtain

$$\Omega(\delta\mathbf{X},\dot{\mathbf{X}})=[-2K(\delta E\ \dot{\tilde{P}}-\delta\tilde{P}\ \dot{E})-4E^2\mathrm{e}^{-K}(\delta u\ \dot{v}-\delta v\ \dot{u})+2E(K+1)\mathrm{e}^{-K}(v\,\delta E\ \dot{u}-v\,\delta u\ \dot{E}-u\,\delta E\ \dot{v}+u\,\delta v\ \dot{E})-2EK'(v\,\delta\tilde{P}\ \dot{u}-v\,\delta u\ \dot{\tilde{P}}+u\,\delta\tilde{P}\ \dot{v}-u\,\delta v\ \dot{\tilde{P}})]. \quad (84)$$

In the calculation, we have used Eq. (77) and the identity

$$K'=\frac{1}{K\mathrm{e}^K}, \quad (85)$$

which follows from the definition (74) of K.

The form Ω must be nondegenerate in order to define a symplectic structure. The calculation of the determinant of the corresponding matrix Ω_m can be simplified by writing it in the 2×2 block form

$$\Omega_m=\begin{pmatrix}A & C\\ -C^T & B\end{pmatrix}. \quad (86)$$

Multiplying the second double row by the matrix $-CB^{-1}$ and adding the result to the first double row, one can see immediately that

$$\det\Omega_m=\det(A+CB^{-1}C^T)\det B.$$

After some easy calculation, this leads to

$$\det\Omega_m=\left(\frac{8E^2}{K\mathrm{e}^K}\right)_-^2\left(\frac{8E^2}{K\mathrm{e}^K}\right)_+^2=(8E^2K')_-^2(8E^2K')_+^2.$$

The determinant is nonzero at all points of all Kruskal spacetimes.

The block form (86) helps also to fasten the calculation of the matrix Ω_m^{-1}, which defines the Poisson brackets.[2] We look for Ω_m^{-1} in the form

[2] I shorten the subsequent exposition, because most people today may prefer a MAPLE or MATHEMATICA routine.

$$\Omega_m^{-1} = \begin{pmatrix} U & W \\ -W^T & V \end{pmatrix}$$

and observe that the matrices A, B, U and V all must be proportional to the matrix

$$\epsilon := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The equation $\Omega_m \Omega_m^{-1} = 1$ now decomposes into four equations:

$$AU - CW^T = 1, \quad AW - CV = 0, \tag{87}$$

$$-C^T U - BW^T = 0, \quad -C^T W + BV = 1. \tag{88}$$

From the second equation of Eq. (87) and the well-known identities for ϵ,

$$\epsilon^2 = -1, \quad \epsilon M^T \epsilon = -(\det M)\ M^{-1}, \tag{89}$$

valid for all 2×2 matrices M, we find that W is proportional to C^{T-1}. The rest of Eqs. (87) and (88) contains only two independent linear equations, which determine U and V. A straightforward calculation using Eqs. (85) and (89) then leads to the result

$$\Omega_m^{-1} = \begin{pmatrix} 0 & \frac{K}{2} & -\frac{u}{4E} & \frac{v}{4E} \\ & 0 & \frac{K(K+1)u}{4E} & \frac{K(K+1)v}{4E} \\ & & 0 & \frac{K^2 e^K}{4E^2} \\ & & & 0 \end{pmatrix} \tag{90}$$

(the order of the coordinates is $E, \widetilde{P}, u, v$).

With the help of Eq. (90), we can study Poisson brackets. We observe first that

$$\{u,v\} = \frac{K^2 e^K}{4E^2} \neq 0, \tag{91}$$

and, second, that

$$\{u,E\}_{u=v=0} = 0, \quad \{v,E\}_{u=v=0} = 0. \tag{92}$$

This has interesting consequences. First, there is no Darboux chart such that u and v would be two of the corresponding coordinates. Second, a stronger version of inequality (91) can be proved. Consider an arbitrary pair (x,y) of coordinates in a neighborhood of the horizon crossing $u=v=0$. If x and y are to be independent, they must satisfy

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \neq 0. \tag{93}$$

If they are to be coordinates on the spacetime where the shell moves, they must be independent on $\widetilde{P}$:

$$x = x(u,v,E), \quad y = y(u,v,E).$$

Let us calculate the Poisson bracket of the two coordinates at the horizon crossing. If we take Eqs. (92) into account, we obtain

$$\{x,y\}|_{u=v=0} = \frac{K e^K}{4E^2}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\Bigg|_{u=v=0}.$$

Equations (91) and (93) then imply that this expression is nonzero. We have shown the following theorem.

Theorem 1. Any two independent spacetime coordinates of the shell that are regular at an intersection of two horizons have a nonzero Poisson bracket with each other in a neighborhood of the intersection.

Can this be interpreted as saying that a spacetime manifold is necessarily fuzzy near a horizon in the quantum theory? There are at least two caveats. First, any generic point of any horizon (that is, a point that does not lie at an intersection of two horizons) has a neighborhood, where there *are* commuting coordinates. An example is given in the Sec. VI A. Second, the way from the classical to a quantum theory is longer than it may seem: we had also to define observables, and the observables must satisfy some requirements. For example, their classical counterparts are to have vanishing Poisson brackets with the constraints (for a discussion of this point, see Ref. [21]). The functions u and v as they stand fail to be so. We can safely conclude that some more work is necessary to understand the implications of the theorem.

ACKNOWLEDGMENTS

Useful discussions with H. Friedrich, J. Kijowski, K. V. Kuchař, J. Louko, and B. G. Schmidt are thankfully acknowledged. The author is indebted to the Max-Planck-Institut for Gravitationsphysik, Potsdam and to the Institute for Theoretical Physics of the University of Utah, Salt Lake City, for the nice hospitality and support. Thanks go to National Science Foundation Grant No. PHY9507719 (U.S.A), to the Tomalla Foundation, Zurich and to the Swiss Nationalfonds for a partial support.

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