arXiv:0807.3465v1 [gr-qc] 22 Jul 2008

The Cauchy Problem for Membranes

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Abstract. We show existence and uniqueness for timelike minimal submanifolds (world volume of p-branes) in ambient Lorentz manifolds admitting a time function in a neighborhood of the initial submanifold. The initial value formulation introduced and the conditions imposed on the initial data are given in purely geometric terms. Only an initial direction must be prescribed in order to provide uniqueness for the geometric problem. The result covers non-compact initial submanifolds of any codimension. By considering the angle of the initial direction and vector fields normal to the initial submanifold with the unit normal to the foliation given by the time function we obtain a quantitative description of "distance" to the light cone. This description makes it possible to treat initial data which are arbitrarily close to the light cone. Imposing uniform assumptions give a lower bound for a notion of "time of existence" depending only on geometric quantities involving the length of timelike curves lying in the solution.

AMS classification scheme numbers: 35L15, 35L70, 81T30

Submitted to: *Class. Quantum Grav.*

1. Introduction

Membranes are spacelike submanifolds evolving in a Lorentzian manifold. The equation of motion of a membrane is determined by the condition that the world volume of the membrane, the timelike submanifold swept out during time evolution, is a critical point for the volume functional induced by the ambient manifold. Membranes arise in the context of higher-dimensional extensions of String Theory, where they are called p-branes according to the dimension of the spacelike object. The Euler-Lagrange equation of the volume functional is the vanishing of the mean extrinsic curvature vector of the world volume. This equation will be called the membrane equation in the sequel. Due to the signature of the world volume the membrane equation leads to a system of nonlinear wave equations. Since the membrane equation is invariant under diffeomorphisms of the world volume, it is degenerate.

In this paper we consider the solvability of the initial value problem (IVP) for the membrane equation with initial data only involving geometric quantities. Our focus is on providing short-time existence and uniqueness for the Cauchy problem admitting non-compact initial data, and data which are arbitrarily close to the light cone. The Cauchy problem for closed strings, i. e. a membrane diffeomorphic to the circle line, has been studied in Minkowski space and in globally hyperbolic ambient manifolds by T. Deck [1], and O. Müller [2]. Solutions are obtained as timelike immersions in a neighborhood of the initial data. In these works the question of global-in-time existence was investigated for globally hyperbolic ambient manifolds. Exact solutions of this equation corresponding to pulsating and rotating objects have been studied by H. Nicolai and J. Hoppe in [3].

In the case of the world volume in Minkowski space represented as a graph, global existence for small initial data was shown in [4]. This result was generalized to arbitrary codimensions in [5].

Using techniques previously applied to the Einstein equation by D. Christodoulou and S. Klainerman (cf. [6]), S. Brendle investigated in [7] the stability of a hyperplane satisfying the membrane equation in Minkowski space. In contrast to the work of H. Lindblad (cf. [4]), S. Brendle requires less regularity of the initial data, comparable to that in [6].

Let us state the main problem addressed by this paper in non-technical terms.
Let N^{n+1} be an $(n+1)$-dimensional Lorentzian manifold, and let Σ_0 be an m-dimensional spacelike submanifold of N. Assume ν to be an unit timelike future-directed vector field normal to Σ_0.

Existence

Find an open $(m+1)$-dimensional timelike submanifold Σ of N satisfying

$$H(\Sigma) \equiv 0, \;\; \Sigma_0 \subset \Sigma, \;\; \text{and } \nu \text{ is tangential to } \Sigma. \tag{1}$$

Uniqueness

Show that for Σ_1 and Σ_2 solving the IVP (1), there exists an open set V with $\Sigma_0 \subset V \subset N$ such that

$$V \cap \Sigma_1 = V \cap \Sigma_2.$$

We will refer to N, Σ_0, and ν as ambient manifold, initial submanifold, and initial direction, respectively. The main result of this work is an affirmative answer to the problem above under suitable conditions. Here, the ambient manifold is assumed to admit a time function in a neighborhood of the initial submanifold. No further condition on the causal structure is required.

After fixing some notation in section 2, we perform the reduction of the membrane equation to a system of quasilinear hyperbolic equations by employing a gauge used in the context of the Einstein equations (cf. [8]). In section 4 we consider solutions of the membrane equation given by parametrized immersions. In section 5 these local solutions are glued together to obtain a solution to the geometric Cauchy problem (1). The main existence result, Theorem 5.2, shows that for non-compact regularly immersed submanifolds of any codimension in an ambient manifold of any dimension, existence and uniqueness holds. Uniform assumptions lead to a lower bound on the length of timelike curves lying in the world volume, which gives rise to a geometric notion of time of existence of a solution to the Cauchy problem (1). In Corollary 5.6 it is shown that smooth data lead to smooth solutions of the membrane equation. In Theorem 5.7 we present the main uniqueness result for the membrane equation in the case of smooth data.

The results are taken from the author's Ph.D. thesis; for further reference see [9].

2. Notation

Let N^{n+1} be an $(n+1)$-dimensional smooth manifold endowed with a Lorentzian metric h. The Levi-Civita connection with respect to h and the corresponding Christoffel symbols are denoted by $\mathbf{D}$ and $\mathbf{\Gamma}$, respectively. Covariant derivatives of order ℓ are denoted by $\mathbf{D}^\ell$. The $(1,3)$ or the $(0,4)$-version of the curvature of h is denoted by $\mathbf{Rm}$.

Suppose Σ^{m+1} is an $(m+1)$-dimensional timelike submanifold of N. The metric and connection induced on Σ are then given by

$$g := h|_\Sigma \quad \text{and} \quad \nabla_X Y := (\mathbf{D}_X Y)^\top \text{ for vector fields } X, Y \text{ tangent to } \Sigma.$$

The Christoffel symbols of ∇ are denoted by Γ. The second fundamental form and the mean curvature of Σ are given by

$$I\!I(X,Y) := (\mathbf{D}_X Y)^\perp \qquad \text{and} \qquad H = \operatorname{tr}_g I\!I. \tag{2}$$

The usual definition of the mean curvature involves a factor dependent on the dimension, which was omitted here since we are only interested in the homogeneous equation.

The following definitions introduce types of initial submanifolds. We refer to them as the *immersion type* of the initial submanifold.

Definition 2.1 *Let M^m be an m-dimensional manifold, and let $\varphi : M \to N$ be an immersion. The image of φ, $\Sigma_0 := \mathrm{im}\varphi$, is called a* regularly immersed submanifold *of N.*

Definition 2.2 *Let Σ_0 be a regularly immersed submanifold of dimension m with immersion $\varphi : M^m \to N$. The set Σ_0 is called a* locally embedded submanifold *if for every point $q \in \Sigma_0$ there exist open sets $q \in V \subset N$ and $U \subset M$ such that*

$$\varphi : U \to \varphi(U) \text{ is a diffeomorphism and } \quad \varphi^{-1}(V \cap \Sigma_0) = U. \tag{3}$$

Definition 2.3 *Let Σ_0 be a regularly immersed submanifold of dimension m with immersion $\varphi : M^m \to N$. The set Σ_0 is called a* regularly immersed submanifold with finite intersections *if for every point $q \in \Sigma_0$ there exist a neighborhood $V \subset N$ of q and finitely many open disjoint sets $U_\ell \subset M$ such that*

$$\varphi : U_\ell \to \varphi(U_\ell) \text{ is a diffeomorphism for every } \ell \text{ and } \quad \varphi^{-1}(V \cap \Sigma_0) = \bigcup U_\ell. \tag{4}$$

Let Σ_0 be a regularly immersed submanifold with immersion $\varphi : M^m \to \Sigma_0 \subset N$. We make use of the induced metric $\mathring{g} = \varphi^* h$ on M. The Levi-Civita-connection compatible with $\mathring{g}$ is denoted by $\mathring{\nabla}$. The connection on the pull-back bundle $\varphi^* TN$ is denoted by $\widehat{\nabla}$, and members of $\varphi^* TN$ are called *vector fields along* φ. Using this notation we introduce the *second fundamental form of* φ defined as the vector field

$$\mathring{I\!I}(X, Y) = (\widehat{\nabla}_X d\varphi(Y))^{\perp}_{\mathrm{im}\varphi}, \tag{5}$$

where $d\varphi$ denotes the differential of φ, and $(\,.\,)^{\perp}_{\mathrm{im}\varphi}$ denotes the part normal to the image of φ. If Σ_0 is a locally embedded submanifold then this definition is independent of the local embedding and gives the usual definition of second fundamental form of a submanifold.

Let E be a Riemannian metric defined on N. For a vector field ξ along φ we introduce the following norm

$$|\widehat{\nabla}^\ell \xi|^2_{\mathring{g},E} = \mathring{g}^{i_1 j_1} \cdots \mathring{g}^{i_\ell j_\ell} E_{AB} \widehat{\nabla}_{i_1,\dots,i_\ell} \xi^A \widehat{\nabla}_{j_1,\dots,j_\ell} \xi^B \tag{6}$$

with the abbreviation $\widehat{\nabla}_{i_1,\dots,i_\ell} = \widehat{\nabla}_{i_1} \cdots \widehat{\nabla}_{i_\ell}$. Here, we used the following set of indices for coordinates. On M and on the initial submanifold Σ_0, coordinates carry small Latin indices running from 1 to m. Capital Latin indices as $A, B, C, \dots$ will run from 0 to n, and indicate coordinates on N. Our convention for the signature of a Lorentzian metric is $(-+\cdots+)$. Partial derivatives in coordinates are abbreviated by $\partial_A = \partial/\partial y^A$. Local coordinates on the timelike submanifold Σ carry Greek indices running from 0 to m.

An assumption on the causal structure of the ambient manifold will be made by using the following terminology.

Definition 2.4 *A real-valued function τ on N is called* time function *if its gradient* $\mathbf{D}\tau$ *is everywhere timelike. Such a time function induces a* time foliation *of N by its levelsets $\mathscr{S}_\tau$, which are spacelike hypersurfaces.*

We further introduce the lapse ψ of the foliation *given by τ by*

$$\psi^{-2} = -h(\mathbf{D}\tau, \mathbf{D}\tau), \tag{7}$$

and the future-directed unit normal to the time foliation is given by

$$\widehat{T} = -\psi \mathbf{D}\tau.$$

The dual vector field ∂_τ to the differential $d\tau$ of the time function is given by

$$\partial_\tau = -\psi^2 \mathbf{D}\tau \quad \Rightarrow \quad \partial_\tau = \psi \widehat{T}.$$

The existence of such a function provides us with a possibility of introducing a Riemannian metric.

Definition 2.5 *Suppose N admits a time function τ. We consider the decomposition into tangential and normal part with respect to the slices, denoted by $(\,.\,)^\top$ and $(\,.\,)^\perp$, respectively. We introduce a Riemannian metric E associated to the time foliation by a change of the sign of the unit normal $\widehat{T}$ as follows*

$$E(v,w) := h(v^\top, w^\top) + \lambda\mu, \quad \textit{where } v^\perp = \lambda \widehat{T} \textit{ and } w^\perp = \mu \widehat{T}. \tag{8}$$

Since the slices are spacelike this metric is Riemannian.

We use the norm on tensors induced by the metric E defined as follows

$$|T|_E^2 = E^{A_1 C_1} \cdots E^{A_k C_k} \, E_{B_1 D_1} \cdots E_{B_\ell D_\ell} \, T^{B_1,\ldots,B_\ell}_{A_1,\ldots,A_k} \, T^{D_1,\ldots,D_\ell}_{C_1,\ldots,C_k}.$$

3. Reduction

In this section we present a method to obtain a strictly hyperbolic system from the membrane equation by using parametrizations and a specific gauge condition.

Let $F : W \subset \mathbb{R} \times M \to (N, h)$ denote an immersion with induced metric $g := F^*h$. We denote the Christoffel symbols with respect to g by Γ. The image of F will correspond to a solution Σ of the geometric IVP (1).

Let x^μ be coordinates on $\mathbb{R} \times M$, and let y^A be coordinates on N. We denote the covariant derivative on the pullback bundle F^*TN by $\widehat{\nabla}^F$. The membrane equation for the image of F then reads in this situation

$$g^{\mu\nu} \widehat{\nabla}^F_\mu \partial_\nu F - g^{\mu\nu} (\widehat{\nabla}^F_\mu \partial_\nu F)^\top_{\mathrm{im}F} = 0 \tag{9}$$

or, equivalently $$g^{\mu\nu} \partial_\mu \partial_\nu F^A + g^{\mu\nu} \partial_\mu F^B \partial_\nu F^C \mathbf{\Gamma}^A_{BC}(F) - \Gamma^\lambda \partial_\lambda F^A = 0, \tag{10}$$

where $(\,.\,)^\top_{\mathrm{im}F}$ denotes the part tangential to the image of F.

The hyperbolicity of (10) comes from the signature of the metric induced by F on W. The degeneracy of (10) is a consequence of the invariance under tangential diffeomorphisms of a solution. It becomes manifest in the contracted Chrisoffel symbols induced on W, which contain a term involving second-order derivatives of F cancelling the tangential part of the leading second-order term.

To overcome the problem of degeneracy it is necessary to fix a gauge. We consider the so-called harmonic map gauge taken from [8], where it is used to show well-posedness of the Einstein equations.

Recall that a map $v : (M_1, g_1) \to (M_2, g_2)$ between two pseudo-Riemannian manifolds is called *harmonic map* (or *wave map*) if it satisfies

$$g_1^{ij}\partial_i\partial_j v^a - g_1^{ij}\Gamma(g_1)_{ij}^k\partial_k v^a + g_1^{ij}\partial_i v^b\partial_j v^c\Gamma(g_2)_{bc}^a = 0. \tag{11}$$

Definition 3.1 *A solution $F : W \subset \mathbb{R} \times M \to N$ of the membrane equation (10) is in* harmonic map gauge, *if the following condition is satisfied*

$$\mathrm{id} : (W, g = F^*h) \to (W, \hat{g}) \quad \textit{is a harmonic map,} \tag{12}$$

where $\hat{g}$ is a fixed background metric. We call this condition harmonic map gauge condition.

Remark 3.2 *In coordinates, condition (12) reads*

$$g^{\mu\nu}(\Gamma_{\mu\nu}^\lambda - \hat{\Gamma}_{\mu\nu}^\lambda) = 0. \tag{13}$$

Note that this is the trace of the difference of two connections, and is therefore independent of coordinates.

By inserting the harmonic map gauge condition (13) into (10) we obtain the *reduced membrane equation*

$$g^{\mu\nu}\partial_\mu\partial_\nu F^A + g^{\mu\nu}\partial_\mu F^B\partial_\nu F^C\mathbf{\Gamma}_{BC}^A(F) - g^{\mu\nu}\hat{\Gamma}_{\mu\nu}^\lambda\partial_\lambda F^A = 0. \tag{14}$$

The reduced equation is strictly hyperbolic since the contracted Christoffel symbols $g^{\mu\nu}\Gamma_{\mu\nu}^\lambda$ are replaced by a lower-order term. We now show equivalence of the equations (14) and (10).

Lemma 3.3 *The membrane equation (10) together with condition (13) is equivalent to (14).*

Proof:
We only need to show that the reduced membrane equation (14) implies (10) and condition (13); the other direction follows immediately. Suppose the reduced membrane equation (14) holds. We compute the contracted Christoffel symbols with respect to the metric induced by a solution

$$g^{\mu\nu}\Gamma_{\mu\nu}^\lambda = g^{\kappa\lambda}g^{\mu\nu}\hat{\Gamma}_{\mu\nu}^\delta\partial_\delta F^A h_{AD}\partial_\kappa F^D = g^{\mu\nu}\hat{\Gamma}_{\mu\nu}^\lambda.$$

Using this identity together with (14) gives us the desired result. □

Remark 3.4 *This result can be derived in a somewhat more abstract fashion if we use (9) instead of (10). The reduced membrane equation (14) can be expressed*

$$g^{\mu\nu}\widehat{\nabla}_\mu^F\partial_\nu F = g^{\mu\nu}\hat{\Gamma}_{\mu\nu}^\lambda\partial_\lambda F.$$

The right hand side of the latter equation is tangential to the image of F, therefore the normal part of the left hand side has to vanish, and the tangential part has to coincide with the right hand side which yields the desired identity (13).

Remark 3.5 *Other gauges corresponding to the prescription of the contracted Christoffel symbols are possible. Therefore, we can mine the rich repertoire of gauges for the Einstein equations (cf. [8]).*

Remark 3.6 *In contrast to the case of the Einstein equations the reduction process needs no further equation.*

Based on (10) we consider the following problem involving existence and uniqueness. Let Σ_0 be a regularly immersed m-dimensional submanifold of N with immersion $\varphi : M^m \to \Sigma_0 \subset N$. Let α be a function on M called *initial lapse*, and let β be a vector field on M called *initial shift*. Suppose $\nu : M \to TN$ is an unit timelike future-directed vector field along φ normal to Σ_0.

Existence

Find a neighborhood $W \subset \mathbb{R} \times M$ of $\{0\} \times M$ and an immersion $F : W \to N$ solving the IVP

$$H(\mathrm{im}F) \equiv 0,\ F|_{t=0} = \varphi,\ \partial_t F|_{t=0} = \alpha\,\nu + d\varphi(\beta) \tag{15}$$

such that $\partial_t F$ is timelike, and the image of $F(t) : M \to N$ is spacelike. The last term consists of the differential of φ applied to the vector field β. The parameter t denotes the first component of $\mathbb{R} \times M$.

Uniqueness

Suppose Σ_0 is locally embedded, and let ν be an unit timelike future-directed vector field on Σ_0. Assume $\bar{\varphi} : M \to N$ to be an immersion with $\Sigma_0 = \mathrm{im}\bar{\varphi}$, and let $\bar{\alpha}$ and $\bar{\beta}$ be another choice of initial lapse and shift.

Show that for immersions $F, \bar{F} : W \subset \mathbb{R} \times M \to N$ solving (10), and attaining the initial values

$$\begin{aligned} F|_{t=0} &= \varphi,\ \partial_t F|_{t=0} = \alpha\,\nu \circ \varphi + d\varphi(\beta), \\ \bar{F}|_{t=0} &= \bar{\varphi},\ \partial_t \bar{F}|_{t=0} = \bar{\alpha}\,\nu \circ \bar{\varphi} + d\bar{\varphi}(\bar{\beta}), \end{aligned}$$

there exists a local diffeomorphism Ψ of W such that $F \circ \Psi^{-1} = \bar{F}$.

Remark 3.7 *Throughout the following we use a* special background metric $\hat{g}$ *defined on* $\mathbb{R} \times M$*. If the initial values of* F *are* $F|_{t=0} = \varphi$ *and* $\partial_t F|_{t=0} = \alpha\,\nu + d\varphi(\beta)$*, then we define*

$$\hat{g} := -\alpha^2 dt^2 + \mathring{g}_{ij}(\beta^i dt + dx^i)(\beta^j dt + dx^j). \tag{16}$$

Observe that $\mathring{g}_{ij}, \alpha,$ *and* β *are independent of the parameter* t*.*

4. Parametrized Immersions

This section is devoted to discuss the Cauchy problem (15). Solutions will be obtained as parametrization through spacelike immersions. First we present the main assumptions of this paper. The main results of this section are stated in Propositions 4.2 and 4.6.

4.1. Assumptions

We use the assumptions on the date presented in this section if not explicitly stated otherwise. In order to get results which are independent of the scale of the ambient manifold — multiplication of the metric with the square of a positive constant — we introduce a constant $R > 0$. The purpose of this constant is to absorb scaling such that scale-invariant constants occur in the assumptions.

Let $s > m/2 + 1$ be an integer, and let $V \subset N$ and $U \subset M$ denote open sets.

Assumptions on the ambient manifold: We suppose N to admit a time function τ in V. Assume the metric h and τ to be C^{s+3}. Let ψ denote the lapse of the time foliation induced by τ defined by (7), and let E denote the Riemannian metric introduced in Definition 2.5.

We suppose there are constants C_1, C_2, C_N, and C_τ independent of R such that the following inequalities are uniformly satisfied in V

$$C_1 \le R^{-1}\psi \le C_2, \quad \sum_{\ell=0}^{s+1} R^{2+\ell}|\mathbf{D}^\ell \mathbf{Rm}|_E \le C_N, \quad \text{and} \quad \sum_{\ell=1}^{s+2} R^{1+\ell}|\mathbf{D}^\ell(\mathbf{D}\tau)|_E \le C_\tau,$$

where $\mathbf{D}(\mathbf{D}\tau)$ denotes the $(1,1)$-tensor obtained by applying the covariant derivative to the gradient of the time function τ.

Assumptions on the initial submanifold: Assume φ to be C^{s+2}. Let N admit a time function in a neighborhood of $\varphi(U)$. We suppose there exist constants ω_1, C_φ independent of R such that the following inequalities are uniformly satisfied in U

$$\inf\{-h(\gamma, \widehat{T}) : \gamma \text{ unit timelike future-directed normal to } \Sigma_0\} \le \omega_1,$$
$$\sum_{\ell=0}^{s} R^{\ell+1}|\widehat{\nabla}^\ell \mathring{I\!I}|_{\mathring{g},E} \le C_\varphi,$$

where $\widehat{T}$ denotes the unit timelike future-directed normal to the time foliation on N, and $\mathring{I\!I}$ denotes the second fundamental form of φ (cf. (5)).

Assumptions on the initial direction: Assume ν to be C^{s+1}. We suppose there exist constants ω_2, C_ν independent of R such that the following inequalities are uniformly satisfied in U

$$-h(\nu, \widehat{T}) \le \omega_2 \qquad \text{and} \qquad \sum_{\ell=1}^{s+1} R^\ell |\widehat{\nabla}^\ell \nu|_{\mathring{g},E} \le C_\nu.$$

Remark 4.1 *If the initial submanifold Σ_0 is locally embedded, and the initial direction is defined on Σ_0 rather than on the domain of the embedding, then the assumptions on Σ_0 and ν do not depend on the local embedding.*

When considering the initial value problem (15) for immersions we have the addinitional assumptions on initial lapse and shift.

Assumptions on initial lapse and shift: Assume α and β to be C^{s+1}. We suppose there exist constants L, C_α, and C_β independent of R such that the following inequalities are uniformly satisfied in U

$$R^2(-\alpha^2 + |\beta|^2_{\mathring{g}}) \le -L,$$
$$\sum_{\ell=1}^{s+1} R^\ell |\mathring{\nabla}^\ell \beta|_{\mathring{g}} \le C_\beta, \quad \text{and} \quad \sum_{\ell=0}^{s+1} R^\ell |\mathring{\nabla}^\ell \alpha|_{\mathring{g}} \le C_\alpha.$$

4.2. Existence

The following result gives a solution to the IVP (15). For a detailed proof we refer to [9].

Proposition 4.2 *Suppose for each $p \in M$ there exist a neighborhood $U \subset M$ of p, and an open set $V \subset N$ containing $\varphi(U)$ such that the assumptions described in the previous section are satisfied in V and U.*

Then there exist a neighborhood $W \subset \mathbb{R} \times M$ of $\{0\} \times M$ and a C^2-immersion $F : W \to N$ such that $\partial_t F$ is timelike, and the image of $F(t) : M \to N$ is spacelike solving the reduced membrane equation (14) with respect to the background metric $\hat{g}$ defined by (16), and attaining the initial values $F|_{t=0} = \varphi$ and $\partial_t F|_{t=0} = \alpha\,\nu + d\varphi(\beta)$.

Remark 4.3 *The proof involves solving the reduced membrane equation (14) in a neighborhood of each point $p \in M$. Standard local uniqueness results for hyperbolic equations then show that two local solutions coincide on their common domain. Therefore, a spatially local version of Proposition 4.2 is also valid.*

Remark 4.4 *Let ℓ_0 be an integer, and suppose $s > m/2 + 1 + \ell_0$. Then the solution F is of class $C^{2+\ell_0}$.*

Remark 4.5 *If the assumptions are uniform in p, then the domain W of the solution has the form $[-RC_0, RC_0] \times M$ with a constant $C_0 > 0$ independent of the scale.*

4.3. Uniqueness

Let Σ_0 be a locally embedded submanifold of N. Suppose $\varphi, \bar{\varphi} : M \to N$ are two immersions satisfying the conditions of Definition 2.2 for Σ_0. The metrics induced on M by $\varphi, \bar{\varphi}$ we denote by $\mathring{g}, \mathring{\bar{g}}$, respectively. Let $\psi_0 : M \to M$ denote the local diffeomorphism defined by $\varphi \circ \psi_0^{-1} = \bar{\varphi}$. The differential of ψ_0 is a member of $T^*M \otimes TM$, and we use the following norm

$$|d\psi_0|^2_{\mathring{g},\mathring{\bar{g}}} = \mathring{g}^{ij} \mathring{\bar{g}}_{k\ell} (d\psi_0)^k_i (d\psi_0)^\ell_j.$$

Norms of higher-order derivatives of ψ_0 are defined analogously. Let $\alpha, \bar{\alpha} > 0$ be two functions on M, and let $\beta, \bar{\beta}$ be two vector fields on M. Assume ν to be defined on Σ_0.

We obtain the following uniqueness result.

Proposition 4.6 *Let φ, α, β and $\bar{\varphi}, \bar{\alpha}, \bar{\beta}$ satisfy the assumptions of sections 4.1. Suppose there exist constants C_1^ψ, C_2^ψ independent of R such that*

$$|d\psi_0|_{\mathring{g},\mathring{\bar{g}}} \le C_1^\psi \quad \textit{and} \quad R|d^2\psi_0|_{\mathring{g},\mathring{\bar{g}}} \le C_2^\psi. \tag{17}$$

Let $F, \bar{F} : W \subset \mathbb{R} \times M \to N$ be two C^{s+2}-solutions of the membrane equation (10) attaining the initial values

$$F|_{t=0} = \varphi,\ \partial_t F|_{t=0} = \alpha\, \nu \circ \varphi + d\varphi(\beta),$$
$$\bar{F}|_{t=0} = \bar{\varphi},\ \partial_t \bar{F}|_{t=0} = \bar{\alpha}\, \nu \circ \bar{\varphi} + d\bar{\varphi}(\bar{\beta}).$$

Let $\hat{\bar{g}}$ be the background metric defined by (16) using the initial values of $\bar{F}$. Assume $\bar{F}$ to be in harmonic map gauge with respect to $\hat{\bar{g}}$.

Then there exists a local diffeomorphism Ψ of M such that $F \circ \Psi^{-1}$ and $\bar{F}$ coincide on a neighborhood of $\{0\} \times M$.

From [9] we have the following uniqueness result for solutions to the IVP (15), which are in harmonic map gauge with respect to the background metric defined by (16).

Proposition 4.7 *Let the assumptions of section 4.1 be satisfied. Assume $F : W \to N$ and $\bar{F} : \widetilde{W} \to N$ to be two immersions of class C^2 solving the IVP (15). Suppose both solutions are in harmonic map gauge with respect to the background metric defined by (16).*

Then there exists a neighborhood of $\{0\} \times M$, on which the two solutions F and $\bar{F}$ coincide.

Remark 4.8 *If the domains W and $\widetilde{W}$ have the form $[-RC_0, RC_0] \times M$, then there exists a scale-invariant constant C such that F and $\bar{F}$ coincide for $-RC \leq t \leq RC$.*

Remark 4.9 *As is the case for the existence result stated in Proposition 4.2, a spatially local version of Proposition 4.7 holds as well. From uniqueness results for hyperbolic equations, we obtain that uniqueness holds on cones.*

The previous result relies on the fact that both solutions are in harmonic map gauge with respect to the same background metric. The following proposition gives a condition under which it is possible to align the harmonic map gauge of two solutions.

Proposition 4.10 *Suppose in the situation of Proposition 4.6 there exist open sets $W, \widetilde{W} \subset \mathbb{R} \times M$ such that for a fixed point $p \in M$ it holds $(0, p) \in W$ and $(0, \psi_0(p)) \in \widetilde{W}$. Suppose there exists a diffeomorphism $\Psi : W \to \widetilde{W}$ such that*

$$\Psi : (W, g = F^*h) \to (\widetilde{W}, \hat{g}) \text{ is a harmonic map.} \tag{18}$$

Furthermore, assume the inverse to satisfy the initial conditions

$$\Psi^{-1}|_{\tilde{t}=0} = (0, \psi_0^{-1}) \quad \text{and} \quad \partial_{\tilde{t}} \Psi^{-1}|_{\tilde{t}=0} = \hat{\lambda} \partial_t + \hat{\chi} \tag{19}$$

with $\hat{\lambda}(p) = \frac{\bar{\alpha}(p)}{\alpha(\psi_0^{-1}(p))}$ *and* $\hat{\chi}(p) = d(\psi_0^{-1})_p(\bar{\beta}) - \hat{\lambda}(p)\beta[\psi_0^{-1}(p)]$.

Then $F \circ \Psi^{-1}$ and $\bar{F}$ coincide on a neighborhood of $(0, p)$.

Proof:
From the harmonic map equation satisfied by Ψ we derive the condition for the solution $F \circ \Psi^{-1}$ of the membrane equation to be in harmonic map gauge with respect to the background metric $\hat{\bar{g}}$ (cf. (12)).

It remains to show that the initial values of Ψ^{-1} give us the appropriate initial values for $F \circ \Psi^{-1}$. From the definition of ψ_0 we derive that $F \circ \Psi^{-1}|_{t=0} = \bar{\varphi}$. To obtain the initial value for the velocity we compute

$$\begin{aligned}\partial_{\tilde{t}}(F \circ \Psi^{-1})|_{\tilde{t}=0}(p) = \hat{\lambda}(p)\alpha[\psi_0^{-1}(p)]\nu \circ \varphi[\psi_0^{-1}(p)] + \hat{\lambda}(p)d\varphi_{\psi_0^{-1}(p)}(\beta) \\ + d(\varphi \circ \psi_0^{-1})_p(\bar{\beta}) - \hat{\lambda}(p)d\varphi_{\psi_0^{-1}(p)}(\beta)\end{aligned}$$

Since the second and the last term cancel, the claim follows from the definition of the initial values of Ψ^{-1}. Therefore, $F \circ \Psi^{-1}$ and $\bar{F}$ satisfy the reduced membrane equation (14) with respect to the background metric $\hat{\bar{g}}$ attaining the initial values of $\bar{F}$. From Proposition 4.7 we obtain that both immersions coincide locally. □

To obtain Proposition 4.6 it remains to show existence of a local diffeomorphism satisfying condition (18). Since the metrics $g = F^*h$ and $\hat{g}$ are Lorentzian, the harmonic map equation for Ψ has a structure similar to the reduced membrane equation (14). The main difference is that the harmonic map equation is semilinear. Hence, the same techniques can be applied as to obtain a solution, and we have the following result.

Proposition 4.11 *In the situation of Proposition 4.6 there exist a neighborhood $\widehat{W} \subset \mathbb{R} \times M$ of $\{0\} \times M$ and a C^2-immersion $\Psi : \widehat{W} \to \mathbb{R} \times M$ satisfying (11) with $(M_1, g_1) = (W, g)$ and $(M_2, g_2) = (\mathbb{R} \times M, \hat{\bar{g}})$. Furthermore, Ψ is invertible in a neighborhood of $\{0\} \times M$, and the inverse attains the initial values (19).*

5. Geometric results

In this section we consider existence and uniqueness for the membrane equation in purely geometric terms. An existence result is obtained including a geometric notion of time of existence of a solution. The uniqueness result shows that two submanifolds solving the IVP coincide on a neighborhood of the initial submanifold.

5.1. Existence

The next definition gives a notion of "time of existence" which respects the geometric behavior of a solution; note this does not necessarily coincide with the time parameter of Proposition 4.2.

Definition 5.1 (Time of existence) *Let Σ be a solution of the IVP (1). The* time of existence τ_Σ *of Σ is given by*

$$\tau_\Sigma := \inf_{p \in \Sigma_0} \sup\{\text{length of all timelike future-directed curves in } \Sigma \text{ emanating from } p\}.$$

Our aim is an existence result for the IVP (1) which includes a lower bound on the time of existence. Uniform assumptions imposed on ambient manifold, initial submanifold, and initial direction provide us with such a lower bound.

Theorem 5.2 *Let $\rho > 0$ be a constant. Suppose for each $q \in \Sigma_0$ there exists a neighborhood $V \subset N$ of q such that $B^E_{R\rho}(q) \subset V$, and the assumptions of section 4.1 are satisfied in V and $\varphi^{-1}(V \cap \Sigma_0)$ with constants independent of q.*

Then there exists an open $(m+1)$-dimensional regularly immersed Lorentzian submanifold Σ of class C^2 satisfying the IVP (1). Furthermore, there exists a scale-invariant constant $\delta > 0$ such that

$$\tau_\Sigma \geq R\,\delta.$$

Proof:
From Proposition 4.2 and Remark 4.5 we obtain an immersion $F : [-T, T] \times M \to N$ of class C^2 solving the IVP (15) with initial lapse equal 1 and initial shift equal 0. Letting $\Sigma := \mathrm{im}F$ gives us a regularly immersed timelike submanifold solving the IVP (1).

The construction of F provides us with a lower bound on the time parameter T, and an estimate for the timelikeness of $\partial_t F$. Hence, a lower bound exists for the length of the timelike curves $t \mapsto F(t, p)$ for all $p \in M$. □

Remark 5.3 *The proof shows that the theorem applies to the situation where the assumptions are valid only locally.*

The following corollaries show that a solution to the IVP (1) for the membrane equation can be constructed in such a way that the immersion type of the initial submanifold is preserved.

Corollary 5.4 *Let Σ_0 be locally embedded; for each point $q \in \Sigma_0$ let $U_q \subset M$ and $V_q \subset N$ denote sets satisfying the conditions (3) of Definition 2.2. Suppose that for each point $q \in \Sigma_0$ the assumptions of section 4.1 are satisfied in V_q and U_q.*

Then there exists a locally embedded timelike $(m+1)$-dimensional submanifold Σ of class C^2 solving the IVP (1).

Proof:
A local solution F_q to the IVP (15) with initial lapse equal 1 and initial shift equal 0 can be obtained by a local version of Proposition 4.2 (cf. Remark 4.3). This solution is defined on a neighborhood of $(0, \varphi^{-1}(q))$ with values in V_q. From the inverse function theorem we obtain that the solution is an embedding in a neighborhood of $(0, \varphi^{-1}(q))$. By shrinking the domain W_q of F_q further we achieve that the set $W_q \cap (\{0\} \times M)$ lies in U_q. Consider the family $(W_q)_q$ of the domains of all local solutions. The intersection with $\{0\} \times M$ provides a covering of M. Choose points q such that the intersection of the family of domains of the local solutions with $\{0\} \times M$ form a locally finite covering subordinate to the above covering. Define a mapping which coincides with the local solutions on each set belonging to the locally finite subcovering above. From a local version of Proposition 4.9 we derive that local solutions coincide on their common domain. Therefore, this mapping is a well-defined immersion, whose image is a locally embedded timelike submanifold of class C^2. □

To obtain a solution for regularly immersed initial submanifolds with locally finite intersections it is necessary to change the way in which the local solutions are pieced together.

Corollary 5.5 *Let Σ_0 be regularly immersed with locally finite intersections; for each point $q \in \Sigma_0$ let $U_{q,\ell} \subset M$ and $V_q \subset N$ denote sets satisfying the conditions (4) of Definition 2.3. Suppose that for each point $q \in \Sigma_0$ the assumptions of section 4.1 are satisfied in V_q and $U_{q,\ell}$ for every ℓ.*

Then there exists a timelike $(m+1)$-dimensional regularly immersed submanifold Σ with locally finite intersections of class C^2 solving the IVP (1).

Proof:
For each $q \in \Sigma_0$ we pick the finitely many $U_{q,\ell}$ satisfying (4), and solve the membrane equation in $U_{q,\ell}$ with values in V_q. We shrink the domain of the solutions $F_{q,\ell}$ to obtain embeddings. By shrinking the set V_q to a subset $\tilde{V}_q \subset N$ we achieve that

$$(F_{q,\ell})^{-1}(\tilde{V}_q \cap \mathrm{im} F_{q,\ell}) = W_{q,\ell} \subset \mathrm{domain}(F_{q,\ell}) \subset \mathbb{R} \times U_{q,\ell} \quad \text{for all } \ell.$$

Consider the family $\mathcal{U} = (\tilde{V}_q \cap \Sigma_0)_{q \in \Sigma_0}$. Choose a locally finite covering $(\tilde{V}_{q_\lambda} \cap \Sigma_0)_{\lambda \in \Lambda}$ of Σ_0 subordinate to $\mathcal{U}$. Let $\tilde{U}_{q,\ell}$ denote the part of $W_{q,\ell}$ which belongs to $\{0\} \times M$. Consider the family $(\tilde{U}_{q,\ell})_{q \in \Sigma_0}$. Then the family $(\tilde{U}_{q_\lambda,\ell})_\lambda$ is a locally finite covering of M subordinate to $(U_{q,\ell})_{q \in \Sigma_0}$ due to the finiteness of the sets $U_{q,\ell}$ for fixed q.

Let F be a mapping defined by $F_{q,\ell}$ on $\tilde{U}_{q,\ell}$. Since the local solutions coincide on common domains, this mapping is well-defined. By construction it follows that $\Sigma := \mathrm{im} F$ is regularly immersed with locally finite intersections and is of class C^2. □

We show that smooth data lead to a smooth solution of the IVP (1) for the membrane equation again respecting the immersion type of the initial submanifold.

Corollary 5.6 *Assume (N, h) to be smooth, and suppose Σ_0 is either smoothly*

(i) regularly immersed,

(ii) locally embedded, or

(iii) regularly immersed with locally finite intersections.

Suppose N admits a smooth time function τ in a neighborhood of the initial submanifold Σ_0. Assume ν to be smooth.

Then there exists an open smooth $(m+1)$-dimensional timelike submanifold Σ which is

(i) regularly immersed,

(ii) locally embedded, or

(iii) regularly immersed with locally finite intersections,

satisfying the IVP (1).

Proof:
For each integer ℓ_0 it follows from the smoothness of h, τ, the immersion φ, and the initial direction that the assumptions of section 4.1 are satisfied for $s > m/2 + 1 + \ell_0$ in a neighborhood of each point $q \in \Sigma_0$. Depending on the immersion type of Σ_0 we apply Theorem 5.2 and Remark 5.3, or Corollaries 5.4 and 5.5. We obtain a solution Σ of class $C^{2+\ell_0}$ by taking Remark 4.4 into account. □

5.2. Uniqueness

In this section we consider the geometric uniqueness problem. In Proposition 4.6 it was shown that the construction of a solution made in Proposition 4.2 is independent of the choice of immersion of the initial submanifold, initial lapse, and initial shift. Therefore, it remains to construct an immersion of an arbitrary solution to (1) which is in harmonic map gauge with respect to the background metric defined by the initial values.

Theorem 5.7 *Assume (N, h) to be smooth, and suppose Σ_0 is smooth locally embedded. Let N admit a smooth time function τ in a neighborhood of the initial submanifold Σ_0. Suppose the initial direction ν is smooth.*

Let Σ_1 and Σ_2 be two open smooth $(m+1)$-dimensional locally embedded Lorentzian submanifolds of N solving the IVP (1).

Then there exists a neighborhood $\Sigma_0 \subset V \subset N$ of Σ_0 such that

$$V \cap \Sigma_1 = V \cap \Sigma_2.$$

Remark 5.8 *In view of the fact that the image of a solution to (15) is a solution to (1) (cf. proof of Theorem 5.2), Theorem 5.7 and Proposition 4.6 show that initial lapse and shift can be given freely.*

Our strategy to prove this theorem is to compare an arbitrary solution with the solution constructed in the previous section. To apply the uniqueness result of Proposition 4.6 we need to construct an immersion satisfying the IVP (15).

Proposition 5.9 *Let (N, h), Σ_0, and ν satisfy the assumptions of Theorem 5.7. Let Σ be a smooth locally embedded solution to the IVP (1).*

Then there exists an immersion $F : W \subset \mathbb{R} \times M \to N$ with $\Sigma \supset \mathrm{im} F$ a locally embedded submanifold. Furthermore, F has the properties that $\partial_t F$ is timelike, and $F(t) : M \to N$ has a spacelike image. The initial values of F are given by $F|_{t=0} = \varphi$ and $\partial_t F|_{t=0} = \nu \circ \varphi$.

Proof:
Let $p \in M$, and let $\gamma_{\varphi(p)}(t)$ be a geodesic in Σ attaining the initial values $\gamma_{\varphi(p)}(0) = \varphi(p)$ and $\dot{\gamma}_{\varphi(p)}(0) = \nu \circ \varphi(p)$. Set $F(t, p) = \gamma_{\varphi(p)}(t)$. Then F is an immersion, since φ is assumed to be an immersion, and ν is assumed to be unit timelike. The initial values of F follow from the initial values of the geodesics. From the smoothness of the initial data it follows that F is also smooth.

The construction above is similar to Gaussian coordinates. In an analogous way it follows that $\partial_t F$ is timelike, and $F(t) : M \to N$ has a spacelike image. From the same argument we derive that the geodesics do not cross in a neighborhood of any point in Σ_0 as long as φ is an embedding, and Σ is an embedded submanifold around that point. $\square$

We are now in the position to give a proof of the main uniqueness result.

Proof of Theorem 5.7:
Let F_0 be the solution to the IVP (15) with initial values $F_0|_{t=0} = \varphi$ and $\partial_t F_0|_{t=0} = \nu \circ \varphi$ constructed in Corollary 5.6. Let $\widehat{\Sigma} := \mathrm{im} F_0$ denote the locally embedded image of the solution F_0. We compare a smooth solution Σ to the IVP (1) with the solution $\widehat{\Sigma}$.

From Proposition 5.9 we obtain that Σ admits an immersion $F : W \subset \mathbb{R} \times M \to \Sigma \subset N$ with locally embedded image, and initial values $F|_{t=0} = \varphi$ and $\partial_t F|_{t=0} = \nu \circ \varphi$ in a neighborhood of the initial submanifold Σ_0.

Proposition 4.6 now yields that there is a local diffeomorphism Ψ such that $F \circ \Psi^{-1}$ and F_0 coincide. Hence, the desired result follows. $\square$

Acknowledgments

The author would like to thank Gerhard Huisken for interesting discussions and encouragement, and Paul Allen for helpful assistance.

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